ALTERNATING-CURRENT CIRCUITS

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PREFACE TO THE THIRD EDITION

Since the publication of the second edition of "Alternating-current Circuits" in 1931, Professor J. A. Correll has passed away. The manuscript for this third edition has been prepared with the assistance of E. W. Johnson, Associate Professor of Electrical Engineering at the University of Minnesota.

In this revision the authors have been guided by criticisms of colleagues and others and by their own experience in the use of the text in their classes. They have extended the use of the polar form of the vector in its application to mesh circuits and have added new material in the form of treatment of mutual inductance and mesh circuits. The definitions in this text have been changed and extended to conform with new standards of the American Institute of Electrical Engineers.

The problems at the end of each chapter have been changed and many new problems have been added. A book of answers is available for those who desire to have it used by their classes; however, for review and for the use of those who desire to use problems without answers, a number of problems are included in each chapter for which answers are not given.

J. M. BRYANT,
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MINNEAPOLIS, MINN.,
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PREFACE TO THE FIRST EDITION

This text is intended as an introductory course in the study of Alternating-current Circuits and Transmission Lines. The material has been used as a course for junior students in Electrical Engineering at the University of Texas for the past eleven years, being revised and brought up to date each year to keep pace with the rapid advances in methods of teaching and applications. The original notes were used by one of the authors at the University of Illinois for several years and were based on lectures given by Dr. Ernst J. Berg now of Union College.

In the first six chapters the theory of alternating-current circuits is discussed and the equations applying to the various types of circuits are developed from the fundamental physical and mathematical principles in the same manner in which they have been developed in lectures before the classes. In the remaining chapters these principles have been applied to polyphase circuits and transmission lines. Free use has been made of the vector diagram and of the mathematics of a vector in a plane as developed by Dr. Steinmetz and others.

This text is not intended to be a complete treatise on transmission lines, but rather the transmission line has been introduced as a practical illustration of a type of complex circuit. Numerous numerical examples have been given to illustrate the application of the equations and also the engineering methods of attacking problems involving intricate equations.

In this text most of the circuits are assumed to contain nonmagnetic materials and the sine wave has been introduced to simplify the development of the equations. In the last chapter a study of nonsinusoidal wave forms is made by the use of the Fourier series, and a method of solution of typical circuits is developed. A study of circuits containing magnetic materials is reserved for a future text on alternating-current machines.

A study of transients in alternating-current circuits has been omitted from this text since it is the belief of the authors that it is best for the student to become familiar with steady state conditions in circuits, both direct and alternating, before the more difficult physical and mathematical relations of transients are introduced.

This text covers material sufficient for a course requiring at least three hours per week for an entire year. For a more extended course the instructor may introduce material of his own amplifying this development or introducing equations for the transient conditions most frequently met with.

In the preparation of this text the authors have assumed a knowledge of physics, such as is usually acquired by the average undergraduate in the universities of this country, and a knowledge of mathematics which includes calculus and the solution of second-order linear differential equations.

The authors wish to acknowledge the assistance given by Professor J. W. Ramsay, of the Department of Electrical Engineering; of Professor J. W. Calhoun, of the Department of Applied Mathematics for many valuable suggestions; and of Mr. Truman S. Gray for the preparation of the drawings from which the numerous cuts have been made.

The authors will appreciate any suggestions for improvement of the text and any notations of errors which may be discovered.

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AUSTIN, TEXAS,
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ALTERNATING-CURRENT CIRCUITS

CHAPTER I

FARADAY'S LAW AND THE GENERAL RELATIONS OF ELECTROMOTIVE FORCE, CURRENT, AND FLUX IN ALTERNATING-CURRENT CIRCUITS

1. Generation of an Emf.—In a-c phenomena it is necessary to consider constantly changing conditions of current, flux, and emf, and the interrelations of these quantities in the transmission of energy. In any circuit where flux is changing, and this flux cuts across a conductor or is interlinked with one or more turns of that conductor, a relation exists between the flux and the emf induced in the conductor. This relation is expressed by Faraday's law, which states that *the emf induced in a circuit is proportional to the time rate of change in the flux and to a constant of the circuit.* Expressed mathematically this becomes

$$e_i = -k\frac{d\phi}{dt} \text{ abvolts,} \tag{1}$$

where e_i is the induced emf;

$d\phi$ is the change in the amount of flux enclosed;

dt is the amount of time required for the change; and

k is a constant, the value of which depends upon the units chosen.

When a coil of N turns is considered and the entire flux is enclosed by all these turns at any one instant, this equation becomes

$$e_i = -N\frac{d\phi}{dt} \text{ abvolts,} \tag{2}$$

where ϕ is measured in lines of force;
t is measured in seconds;
N is measured in turns; and
e_i is measured in abvolts.

As stated above, the emf in Eq. (2) is expressed in the c.g.s. system of units, and is called the "abvolt." The foregoing law may be restated as follows: *When the flux through a coil is changing at the rate of one line of force per second through one turn, the emf induced in that turn is one abvolt, or, as sometimes expressed, when a conductor cuts across a flux at the rate of one line of force per second, one abvolt is induced in that conductor.* As will be explained later, these two expressions for this law are identical. Faraday's law is fundamental to an understanding of the phenomena taking place in a-c circuits where the flux threading through the circuit changes, due to any cause either within or without that circuit.

Alternating Current.—An alternating current is a periodic current that has alternately positive and negative values.

Symmetrical Alternating Current.—A symmetrical alternating current is an alternating current all values of which separated by a half period have the same magnitude but opposite sign. Unless otherwise specified, the term alternating current refers to symmetrical alternating current.

2. Application of Faraday's Law.—As an application of Faraday's law, consider the emf induced in a coil rotating at a uniform angular velocity in a uniform magnetic field. In Fig. 1 the coil *abcd*, which is free to rotate about the axis *mn*, is represented lying in the magnetic field between the two poles *NS*, the flux density of the field being assumed uniform, and the direction of rotation of the coil being assumed to be clockwise. The reference, or starting, time is taken when the coil is in the position shown, *i.e.*, t is equal to zero when the coil *abcd* is in a vertical plane with the coil side *ab* above the coil side *cd*. In this position, the coil encloses the maximum amount of magnetic flux. As the coil starts to rotate, the flux enclosed decreases in magnitude, the amount of flux enclosed being proportional to the projection of the coil diameter *ad* upon the vertical axis. The lines enclosed decrease from the time when t is equal to zero until the coil has reached the position $a'b'c'd'$, as shown by the dotted lines, the armature having rotated through a quarter of one revolution, at which time no flux is enclosed by the coil

As the coil passes this horizontal position, the flux threading through the coil begins to increase in amount, its direction through the coil, however, being reversed. If the original direction of the flux relative to the coil, which is from right to left when the coil side *ab* is above the horizontal, is taken as positive, the reversed direction with respect to the coil, during the half revolution when coil side *ab* is below the horizontal, may be considered as negative; and the wave of flux enclosed may be plotted as shown as ϕ_{ad} in Fig. 2. The progress of the coil may be plotted in terms of time, angle swept over, or of revolutions, each quarter revolution being equal to 90 deg., the vertical dotted lines in Fig. 2, designated t_1, t_2, etc., being located at 90-deg. intervals.

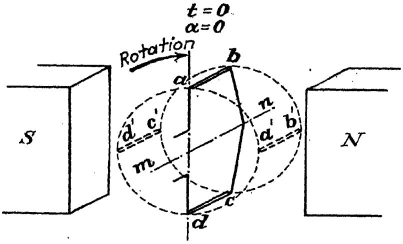

FIG. 1.—Coil rotating in uniform magnetic field.

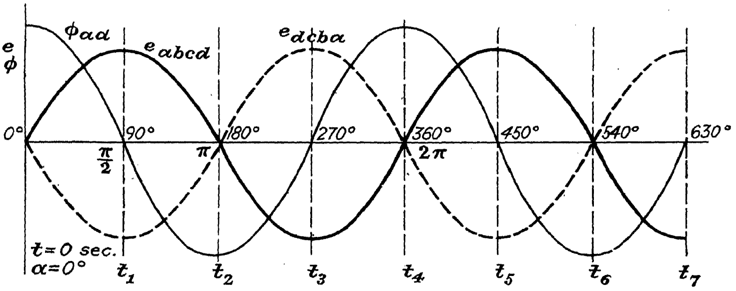

FIG. 2.—Waves of flux enclosed and emf induced in the coil of Fig. 1.

The method of derivation of the flux-enclosed wave is also shown in Fig. 3, which is a cross-sectional view of the coil shown in Fig. 1, at right angles to the axis of rotation. The flux enclosed by the coil *abcd* at any instant is proportional to the length *fg* on the vertical line *st* for any angle α, where α is the angle in radians swept through by the coil from the zero position. The length *fg* is equal to the diameter of the coil times the cosine of the angle α, *i.e.*, $fg = ad \cos \alpha$, so that the flux enclosed is a cosine wave as plotted in Fig. 2.

As stated by Faraday's law, Eq. (1), the magnitude of the induced emf depends upon the magnitude of the time rate of change of flux $d\phi/dt$, and its direction depends upon the sign of the rate of change. The rate of change of the flux $d\phi/dt$ is equal to the slope of the flux wave; so that the slope of the flux wave may be used to study the variation in the induced emf. It is seen from Fig. 2 that the flux changes from a maximum value in the positive direction, at time $t = 0$, to a maximum value in a negative direction, at the time $t = t_2$, the slope being negative during this interval of time, corresponding to 180 deg. in revolution of the coil. During this interval of 180 deg., the emf in the coil is induced in a positive direction, *i.e.*, from *a* to *d* through the coil. The emf from *a* to *d* may be expressed as e_{abcd}, the subscripts indicating the direction through which the circuit is traversed, the first letter indicating the point of view or point of entry into the circuit. When the direction of the emf is the same as the direction in which the circuit is being traversed, *i.e.*, when the potential at the exit is positive with respect to that at the point of entry, the emf is designated as positive. In this case, the potential at *d* is positive with respect to that at *a*, and the emf e_{abcd} is positive during this half revolution. It is to be noted that an increase in flux traversing the coil in a negative sense produces the same direction of emf in the

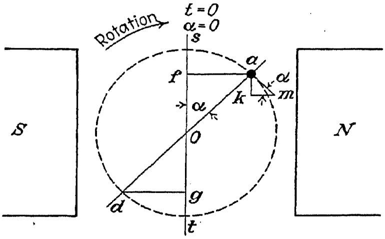

FIG. 3.—Cross section of coil shown in Fig. 1.

coil as a decrease in flux when the direction of the flux is positive, in either case the potential of *d* being above the potential of *a*. When the slope of the flux curve is positive, as in the interval from t_2 to t_4, the negative flux is decreasing from t_2 to t_3, the positive flux is increasing from t_3 to t_4, and the emf is negative. That is, the potential of *a* is above that of *d*, and e_{abcd} is negative.

If the point of viewing the coil is from *d* instead of from *a*, the direction of traversing the coil is reversed and the emf designation is e_{dcba}. In this case, during the first 90 deg., measured from $t = 0$, the potential of *a* is below that of *d*, the point of entry into the circuit, and the potential e_{dcba} is negative. The potential e_{dcba} is equal and opposite at all instants of time to the potential e_{abcd}. This wave is plotted at 180 deg. from e_{abcd}, as e_{dcba}, Fig. 2.

Some of the points of the wave e_{abcd}, Fig. 2, may quickly be obtained from the flux wave by inspection. When the flux wave is passing through a maximum, either positive or negative, the slope is zero and the emf is zero. When the slope of the flux wave is the greatest, the emf is a maximum. This condition occurs when the flux wave is passing through zero for the cosine flux wave of Fig. 2.

The emf wave may be derived also by considering Faraday's law as applying to a conductor cutting through a magnetic field, the emf induced being proportional to the rate of cutting of the lines of force. The emf induced in each of the two conductors of the coil *abcd* will be considered separately. In conductor *ab*, during the first 90 deg., the emf at *b* is above that at *a*, there being a rise in potential from *a* to *b*. The rate of cutting of the lines of force by conductor *ab*, under these assumed conditions, is a sine wave, as can be shown by referring to Fig. 3. The number of lines cut during every interval of time is proportional to the vertical distance traversed by the conductor during this time. The rate of traversing the field in a vertical direction is the vertical component of the velocity of the coil. The velocity of the coil can be drawn as *am*, Fig. 3, perpendicular to the radius *oa*. The vertical component of velocity is *ak* and is equal to *am* sin α. Since the tangential velocity *am* is constant, the vertical velocity is proportional to the sine of the angle α, and the rate of cutting the flux is a sine wave. The rate of cutting the flux for the conductor *ab* is plotted as $(d\phi/dt)_{ab}$ in Fig. 4.

The general form of the emf wave can also be seen by an inspection of Fig. 1. When the conductor *ab* is in the position shown at $t = 0$, it is moving parallel to the lines of force, there is no vertical component of velocity, and the induced emf is zero. At a quarter of a revolution later, when this conductor has reached the 90-deg. position, it is moving perpendicularly to the field; the vertical component of velocity is a maximum; and the generated emf is a maximum. During the same interval, the potential in conductor *dc* will be opposite to that in *ab*, *i.e.*, the potential at *c* is below that at *d*, and there is a potential drop from the front to the back of the conductor, since the conductor cuts the same flux while traveling in the direction opposite to that of *ab*, conductor *ab* moving downward and conductor *dc* moving upward. These relations are shown by the wave forms in Fig. 4. The potential in *dc* is always equal and opposite to that in *ab*, since the field is uniform and the coil is symmetrical. However, the direction in which the emf in *dc* is added around the circuit must be considered. Since *b* and *c* are connected together, the potential rise in *dc* is added around the circuit in the reverse direction to that of *ab*. The potential of *b* is above that of *a* whereas that of *d* is above *c*, so that when *b* and *c* are connected, the potential rise is continuous, and e_{abcd} is equal to the sum of e_{ab} and e_{cd}, being equal to twice e_{ab} in this case, since e_{cd} is equal to e_{ab}.

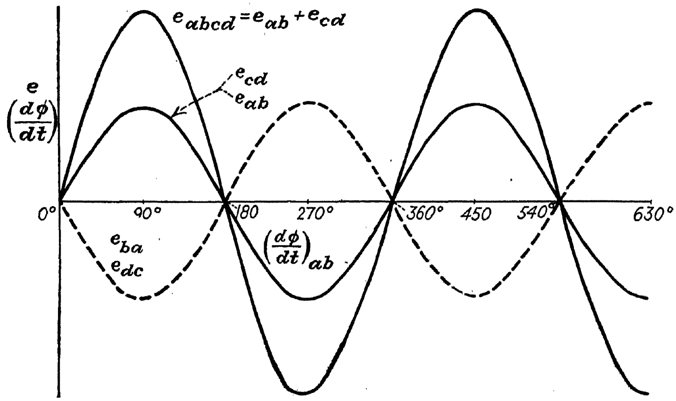

FIG. 4.—Waves of flux cut and emf induced in conductors of the coil of Fig. 1.

The potential e_{abcd} must be identical when derived by either method, *i.e.*, the rate of change of flux enclosed by the coil must be equal to the sum of the rates of flux cutting for the two conductors that form the coil sides. That this is so may be seen from Fig. 3. Taking the increment of angle in Fig. 3 as the angle α, the lines cut by conductor *ab* are proportional to the distance *sf*, and those cut by *cd* are proportional to *gt*. The change in the lines enclosed during this interval is equal to the difference between *st* and *fg*, which is equal to the sum of *sf* and *gt*. Thus the change in lines enclosed is just equal to the total lines cut. This may be demonstrated for any other increment of angle.

Wave Shape.—The wave shape, or wave form, of an alternating current is the shape of the curve obtained when the instantaneous values of the current are plotted against time in rectangular coordinates. Two alternating quantities are said to have the same wave shape when their ordinates of corresponding phase bear a constant ratio to each other. The wave shape, as thus understood, is, therefore, independent of the frequency of the current and of the scale to which the curve is plotted.

Phase.—The phase of a periodic quantity, for a particular value of the independent variable, is the fractional part of a period through which the independent variable has advanced, measured from an arbitrary origin. In the case of a simple sinusoidal quantity, the origin is usually taken as the last previous passage through zero from the negative to the positive direction.

Phase Angle.—The phase angle of a periodic quantity, for a particular value of the independent variable, is the angle obtained by multiplying the phase by 2π, *if the angle is to be expressed in radians, or by* 360 *deg., if the angle is to be expressed in degrees.*

Phase Difference.—The phase difference between two sinusoidal quantities that have the same period is the fractional part of a period (not greater than one-half) through which the independent variable must be assumed to be advanced, in order that similar values of the two quantities may coincide.

Angular Phase Difference.—The angular phase difference between two periodic quantities that have the same period is 2π *rad. (or* 360 *deg.) times the phase difference.*

Synchronism.—Synchronism expresses the phase relation between two or more periodic quantities, when the phase difference between them is zero.

Angle of Lead.—The angle of lead of one sinusoidal quantity, with respect to a second sinusoidal quantity having the same period, is the angular phase difference by which the second quantity must be advanced to coincide with the first.

Angle of Lag.—The angle of lag of one sinusoidal quantity, with respect to a second sinusoidal quantity having the same period, is the angular phase difference by which the second quantity must be assumed to be retarded to coincide with the first.

3. Study of Flux and Emf Waves by the Step-by-step Method. The preceding study of the shape of flux and emf waves in the rotating coil of Fig. 1 has been made on the basis of a uniform flux distribution, and the resultant curves, as plotted, agree with that assumption. Sometimes it becomes important to construct the emf curve for a machine having a nonuniform flux distribution. The wave of flux enclosed by the coil may be of such form that it cannot be expressed conveniently in a simple mathematical equation; and in consequence a step-by-step method must be used for the development. If the coil revolves at a uniform angular velocity, expressed in angular units as ω rad. per second, the angle passed over in t sec. will be $\omega t = \alpha$ rad. For the purpose of deriving and plotting the emf curve, increments of time are chosen that are small enough to give a reasonably accurate wave form, but that are large enough to be convenient for the purposes of plotting and calculation.

Suppose that an increment of time Δt is taken. The flux enclosed changes from ϕ_1 to ϕ_2 lines during this time or by an amount $\phi_2 - \phi_1 = \Delta\phi$ lines. The average value of the emf induced during this interval of time is

$$e_{i_{(av)}} = -k\frac{\Delta\phi}{\Delta t} \text{ abvolts.} \qquad (3)$$

Where a large number of calculations are to be made it may be more convenient to express the interval Δt in angular units rather than in time units. Then, since the corresponding angle is

$$\begin{aligned} \alpha &= \omega t \text{ rad.,} \\ \Delta\alpha &= \omega\Delta t \text{ rad., or} \\ \Delta t &= \frac{\Delta\alpha}{\omega} \text{ sec., and} \end{aligned}$$

$$e_{i_{(av)}} = -k\omega\frac{\Delta\phi}{\Delta\alpha} = -k'\frac{\Delta\phi}{\Delta\alpha} = -k'\frac{\phi_2 - \phi_1}{\alpha_2 - \alpha_1} \text{ abvolts.} \qquad (4)$$

Choosing equal increments of time or angle, $\Delta\alpha$ becomes constant, and Eq. (4) becomes

$$e_{i(av)} = -\frac{k'}{\Delta\alpha}\Delta\phi = -k''\Delta\phi = -k''(\phi_2 - \phi_1) \text{ abvolts.} \qquad (5)$$

Example 1

Calculate the average voltage for the increment of time represented by the angular increment from 35 to 45 deg. for the flux-enclosed wave shown in Fig. 5. The coil has a single turn and rotates with an angular velocity of 0.2π rad. per second.

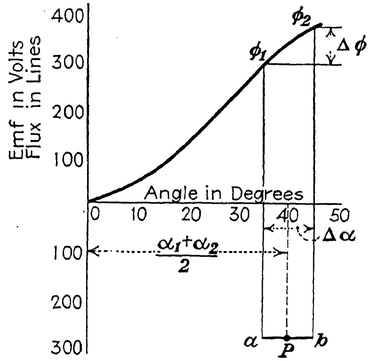

FIG. 5.—Wave of enclosed flux and average emf induced during a short interval of time.

Since the coil has one turn, k in Eq. (3) is equal to unity. The increment of time for 10 deg. or $\pi/18$ rad. is $\pi/18$ divided by 0.2π or 0.278 sec. From Fig. 5, ϕ_2 is 378 lines and ϕ_1 is 300 lines, or

$$\Delta\phi = 378 - 300 = 78 \text{ lines.}$$

The average emf generated during this interval of time, from Eq. (2), is

$$e_{i(av)} = -k\frac{\Delta\phi}{\Delta t} = -\frac{78}{0.278} = -281 \text{ abvolts.}$$

Using Eq. (5) for the derivation,

$$\begin{aligned} e_{i(av)} &= -k''\Delta\phi = -\frac{1}{0.278}\Delta\phi \\ &= -3.59 \times 78 = -281 \text{ abvolts.} \end{aligned}$$

This value of 281 abvolts, the average potential over the increment of angle $\Delta\alpha$, can be plotted as a uniform voltage

TABLE I
Example 2

α (degrees)	ϕ_{abcd}	$\Delta\phi_{abcd} = \phi_2 - \phi_1$	$e_{abcd} = -k''\Delta\phi_{abcd}$	ϕ_{fghk}	$\Delta\phi_{fghk}$	e_{fghk}	e_{khgf}	$e_{af} = e_{abcd} + e_{khgf}$
−5	996	.....	...	−87				
0	.....	0	0	.....	174	−174	174	174
5	996	.....	...	87				
10	.....	−30	30	.....	172	−172	172	202
15	966	.....	...	259				
20	.....	−60	60	.....	164	−164	164	224
25	906	.....	...	423				
30	.....	−87	87	.....	151	−151	151	238
35	819	.....	...	574				
40	.....	−112	112	.....	133	−133	133	245
45	707	.....	...	707				
50	.....	−133	133	.....	112	−112	112	245
55	574	.....	...	819				
60	.....	−151	151	.....	87	−87	87	238
65	423	.....	...	906				
70	.....	−164	164	.....	60	−60	60	224
75	259	.....	...	966				
80	.....	−172	172	.....	30	−30	30	202
85	87	.....	...	996				
90	.....	−174	174	.....	0	0	0	174
95	−87	.....	...	996				
100	.....	−172	172	.....	−30	30	−30	142
105	−259	.....	...	966				
110	.....	−164	164	.....	−60	60	−60	104
115	−423	.....	...	906				
120	.....	−151	151	.....	−87	87	−87	64
125	−574	.....	...	819				
130	.....	−133	133	.....	−112	112	−112	21
135	−707	.....	...	707				
140	.....	−112	112	.....	−133	133	−133	−21
145	−819	.....	...	574				
150	.....	−87	87	.....	−151	151	−151	−64
155	−906	.....	...	423				
160	.....	−60	60	.....	−164	164	−164	−104
165	−966	.....	...	259				
170	.....	−30	30	.....	−172	172	−172	−142
175	−996	.....	...	87				
180	.....	0	0	.....	−174	174	−174	−174

during that interval, as shown by the line *ab* in Fig. 5. By considering the other increments of angle for a full revolution of the coil, there results a wave comprising a series of steps. A smooth wave may be plotted through the centers of these intervals to approximate the actual emf wave. For this reason, it is better to plot the value of $e_{i(av)}$ as a single point P at the mid-ordinate of the interval and to draw the curve through these mid-points.

If the flux-enclosed wave is to be obtained from the cross section of the magnetic circuit showing the flux distribution, as in Fig. 1, the scale must be large enough so that the number of lines of flux enclosed for the various coil positions may be counted with accuracy. The flux-enclosed wave should be plotted from these values, as in Fig. 2. Where the complete emf wave is to be determined, the numerical work should be tabulated to aid in clearness. The tabulation for such a solution is shown in Table I, Example 2. The intervals here chosen are 10 deg., the values of flux enclosed being tabulated for each 10-deg. interval, starting with −5 deg. The difference between the flux at the beginning and at the end of each interval is tabulated at the mid-points, corresponding at each angle with the average emf induced in the same interval; the curve through these values of emf is plotted for each 10 deg., starting from zero degrees, giving the required wave form.

Example 2

Plot the flux and emf waves and their sum for two coils 90 deg. apart revolving at a uniform velocity in a uniform magnetic field.

4. Relation of Emfs in Two Separate Coils Cutting the Same Flux.—If a second coil *fghk* is placed on the same armature as *abcd*, shown in Fig. 6, its flux and emf waves may be plotted in reference to the flux and emf waves of the coil *abcd*, already plotted. The same initial starting time must be used for both sets of waves in order to keep them in their proper phase relation. By considering the emf generated from the point of view *f*, it is seen that the flux enclosed by the new coil *fghk* is zero at the starting time, when the flux enclosed by *abcd* is a maximum. A quarter of a revolution later, the flux ϕ_{abcd} has become zero and ϕ_{fghk} has reached a positive maximum. The wave of flux for the second coil ϕ_{fghk} is found to be exactly like ϕ_{abcd} in shape and magnitude, but lagging behind it by 90 deg. The same relations must hold between the emfs as between the fluxes; so e_{fghk} is an emf wave of the same shape and magnitude but displaced from e_{abcd} by 90 deg. The relation between these emfs is shown in Fig. 7, where both curves are plotted in relation to the same origin. These waves are 90 deg. in time phase and are said to be in time quadrature. It is to be noted that if the emf of the coil *fghk* is considered from the point of view of *k* instead of *f*, the flux-enclosed wave ϕ_{khgf} is just the negative of ϕ_{fghk}, *i.e.*, the two fluxes are displaced by 180 deg. Consequently, the emf waves e_{fghk} and e_{khgf} are also displaced by 180 deg., as shown in Fig. 7.

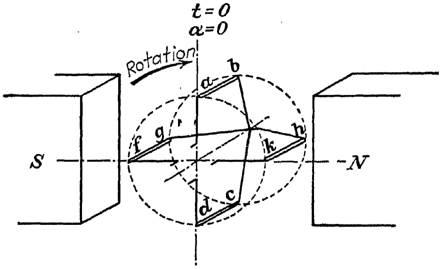

FIG. 6.—Two coils rotating in uniform magnetic field.

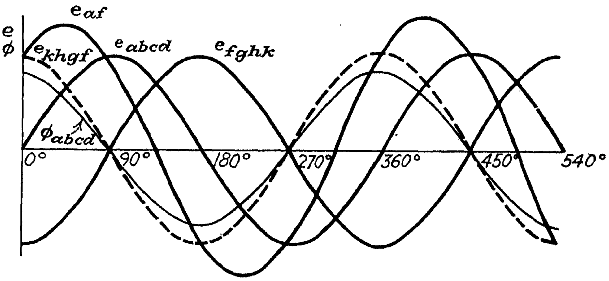

FIG. 7.—Emf waves for coils of Fig. 6.

Quadrature.—Quadrature expresses the phase relationship between two periodic quantities of the same period when the phase difference between them is one-fourth of a period.

Opposition.—Opposition expresses the phase relationship between two periodic quantities of the same period when the phase difference between them is one-half period.

5. Addition of Emf Waves.—Suppose that d is connected to k, Fig. 6, and the resulting wave of emf e_{adkf} is required. By Kirchhoff's laws there must be a dynamic balance of emfs at each instant of time, or, considering instantaneous values of the emfs at corresponding instants of time,

$$e_{af} = e_{abcd} + e_{khgf} \text{ volts.} \tag{6}$$

The emf wave e_{khgf} is 180 deg. out of phase with the wave e_{fghk}, *i.e.*, the instantaneous values are equal in magnitude but opposite in sign, so that at any instant

$$e_{khgf} = -e_{fghk} \text{ volts,}$$

and

$$e_{af} = e_{abcd} - e_{fghk} \text{ volts.} \tag{7}$$

An illustration of the addition of two emf waves is given in Example 2 and in Fig. 7. The emf e_{af}, Fig. 7, may be obtained graphically by adding e_{khgf} to e_{abcd} as is done in Table I, and the resultant curve may be plotted from these values. It is found that the resultant emf wave e_{af} is not in phase with either of the waves e_{abcd} or e_{khgf} and that its maximum value is not equal to the sum of the maxima of these waves. It is seen that Kirchhoff's law for the emfs in a-c circuits is used with instantaneous values of emfs, *i.e.*, a dynamic balance must be dealt with instead of a static balance, as is found in d-c circuits.

6. Current and Emf Waves in a Circuit Containing Resistance Only.—General solutions will first be dealt with for several simple circuits in which an alternating current of any wave form is assumed to flow, particular mathematical solutions for the same circuits being considered later. Suppose that a wave of current from an a-c generator be sent through a circuit that is composed of a resistor, the resistance of which is constant. The instantaneous value of the drop of potential across this resistor, *i.e.*, the emf that must be impressed across the circuit to force the given current through it, will be a constant times the instantaneous value of the current, or

$$e_r = ri \text{ volts,} \tag{8}$$

where e_r is the resistance drop, volts;

r is the resistance, ohms; and

i is the current flowing in the circuit, amp, at the instant of time considered.

The emf wave of Eq. (8) will be of the same shape as the current wave and it will cross the axis in phase with it, since each ordinate of the current wave is multiplied by the same constant factor r to obtain the emf wave, as in Fig. 8. The ordinates of the emf and current waves may be equal, or either may be the greater, according to the scales chosen for each of the quantities used as ordinates. It is understood that values of time, or corresponding values of angle, are to be used as abscissas for all a-c waves. The waves of resistance drop—the voltage impressed to force the current through the resistance—and the current are shown in Fig. 8.

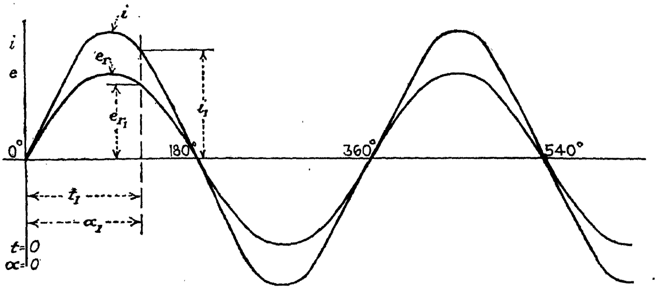

FIG. 8.—Waves of current and resistance drop.

7. Current and Emf Waves in an Electric Circuit Linked with a Magnetic Circuit.—A conductor carrying a current has a magnetic field around the conductor. The strength of the field is proportional to the current that produces it, any variation in current causing a variation in magnetic flux. The magnetic flux and the electric circuit are interlinked, so that a variation of current causes a variation in flux enclosed. The variation of the flux, in turn, causes an emf to be induced in the circuit. In a circuit in which an alternating current flows, the magnetic flux is alternating and an alternating emf is induced in the circuit.

The variation of the flux with the current will first be studied, applying the laws of the magnetic circuit; and then the emf induced in the circuit will be derived by the application of Faraday's law to the flux variation. As an illustration, consider a coil consisting of a closely wound toroid or hollow ring in which the cross section of the coil is small compared with the radius of the toroid. Suppose that the same current wave as considered in Art. 6 is sent through this coil. From the laws of the magnetic circuit, it is found that the magnetomotive force in ampere turns is

$$m = Ni \text{ ampere turns} \tag{9}$$

at each instant of time corresponding with each instantaneous value of current, the current being expressed in amperes. This magnetomotive force is, therefore, a wave of the same shape as, and in phase with, the current, as shown in Fig. 9. The magnetomotive force expressed in gilberts is

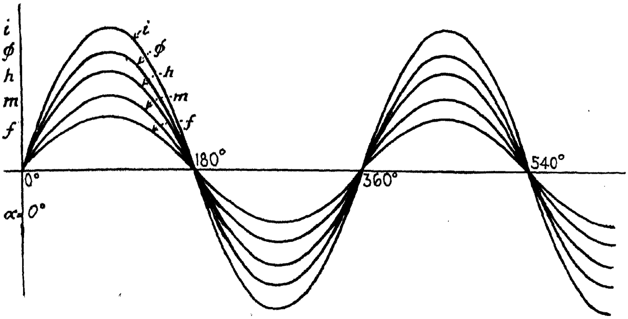

FIG. 9.—Relative shape and position of current f, m, h, and flux waves for a coil with an air core.

$$f = 0.4\pi Ni = k'i \text{ gilberts,} \tag{10}$$

which is another wave of the same shape as, and in phase with, the current, also shown in Fig. 9. The field intensity or magnetizing force at any point within the coil is then

$$h = \frac{0.4\pi Ni}{l} = k''i \text{ oersteds,} \tag{11}$$

another wave of the same shape as, and in phase with, the current wave, as shown in Fig. 9, since all the factors in the equation are constants, and may be included in the scale factor k''.

Since the flux density is B, where $B = \mu h$, the flux passing through the coil may be stated as

$$\phi = BA = \frac{0.4\pi NiA\mu}{l} \text{ maxwells,} \tag{12}$$

where A is the area of the coil, sq cm;

μ is the permeability of the core;

l is the length of the coil or ring, the circumference of a circle through the center of the circular sections of the toroid;

B is the average flux density in the toroid;

N is the number of turns in the coil; and

ϕ is the flux enclosed by the coil.

For a nonmagnetic material the value of the permeability is constant and equal to 1, for all values of current. For magnetic materials the value of the permeability is some function of the current and hence varies as a function of the time for a-c waves. The present discussion will be limited to coils with nonmagnetic and nonmetallic cores; the magnetic materials and other metals being discussed in the consideration of a-c machinery. The preceding equation, Eq. (12), with the limitation of nonmetallic cores, becomes

$$\phi = \frac{0.4\pi NiA}{l} = k'''i \text{ maxwells.} \tag{13}$$

This wave is also of the same shape as, and in phase with, the current wave, as shown in Fig. 9. Equation (13) gives the instantaneous values of the flux corresponding to instantaneous values of current flowing in the circuit, this flux varying through the coil from instant to instant with the same law as the current.

8. Electromotive Force Induced in a Coil Due to an Alternating Current Flowing in That Coil.—From Faraday's law, Eq. (2), a varying flux will induce an emf in the coil so that

$$e_i = -N\frac{d\phi}{dt} \text{ abvolts.} \tag{2}$$

If the equation of the curve of variation of the flux is known, it may be applied to Eq. (2) and the equation of the induced emf may be derived. When this equation is not known, but the shape of the enclosed flux wave is given, so that its wave form may be plotted, the step-by-step method, explained in Art. 3, may be used. In Art. 3, the variation of flux was obtained by rotating the coil in a magnetic field. In this article, the flux is produced by the current flowing in the coil, and the variation in flux enclosed is produced by the variation in the current flowing. However, the values of flux tabulated are independent of the means by which the flux is obtained, so that the step-by-step method derived in Art. 3 applies equally well to this new derivation, as is illustrated in Example 3, in which the first seven columns are used in the derivation of the induced emf wave corresponding to a given current wave.

9. Electromotive Force to Overcome the Induced Emf.—In order to send the current through the coil, there must be impressed at each instant an emf equal and in opposition to the induced emf, producing a dynamic balance, similar to the force of action in mechanics in overcoming the force of reaction exerted by a mass being accelerated in any direction. The wave of impressed emf to overcome the induced emf must be of the same shape as the induced emf, but 180 deg. out of phase with it, as shown in Fig. 10, where the current, flux, induced emf, and the emf to overcome the induced emf are shown in proper phase relation to each other. The equation of this emf is

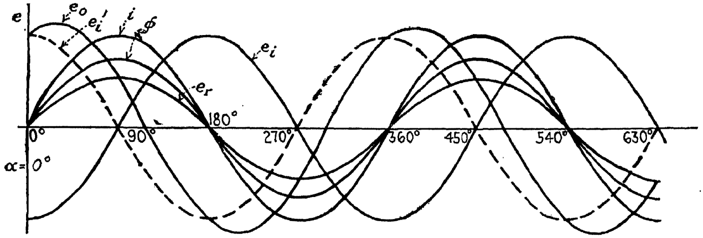

FIG. 10.—Waves of current, flux, resistance drop, induced emf, emf to overcome the induced emf, and the impressed emf for a coil with an air core.

$$e_i' = -e_i = N\frac{d\phi}{dt} \text{ abvolts.} \tag{14}$$

From Fig. 10 it is seen that the emf to overcome the induced emf leads the current and flux waves.

10. The Impressed Emf.—When an alternating current flows through a coil, there is a resistance drop, as discussed in Art. 6, and an emf drop to overcome the induced emf, as discussed in Art. 9. In order to cause a given alternating current to flow, there must be impressed an emf at each instant having a value

$$e_0 = e_r + e_i' \text{ volts,} \tag{15}$$

called the "impressed emf." It should be noted that, although the shape of each of the emf waves may be different, Eq. (15) must be satisfied at each instant.

The derivation of the e_i' wave and the addition of the waves to obtain the e_0 wave may be accomplished best, for an irregular wave of current and flux, by means of tabulation, as explained in the previous paragraphs. The new columns may be added to Table II, Example 3, below. The value of e_0 must be determined for a sufficient number of values of the angle accurately to plot the wave of e_0 to the proper scale, which scale will be the same as that already established for the other emf waves, such a set of waves being illustrated in Fig. 10. It should be noted that the wave of impressed emf is in advance of the current wave, but by a smaller angle than that for the emf to overcome the induced emf.

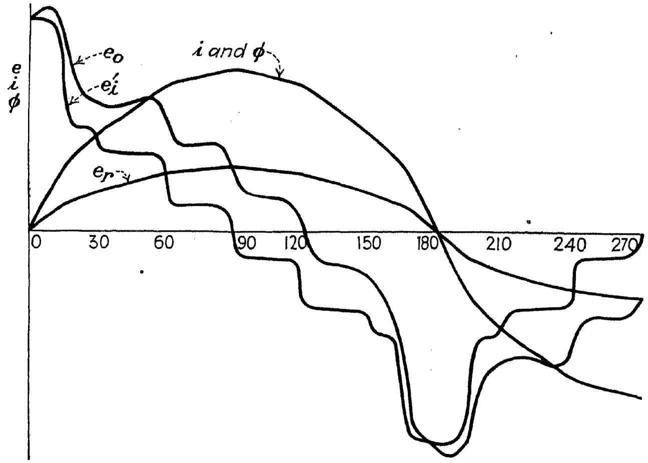

FIG. 11.—Wave forms for Example 3.

Example 3

Derive the waves for e_r, m, h, ϕ, e_i, e_i', and e_o by tabulation for the following current wave, assuming that $N = 100$, $h = 0.9m$, $\phi = 0.9h$, $e_i = -2\Delta\phi$, and $r = 20$. The waves are plotted in rectangular coordinates in Fig. 11.

α =	0,	15,	30,	45,	60,	75,	90,	105,	120,	135,	150,	165,	180 deg.
i =	0,	4,	6,	7.5,	9,	9.5,	10,	9.5,	9,	7.5,	6,	4,	0.

TABLE II
Example 3

α (degrees)	i	$m = 100i$	$h = .9m$	$\phi = .9h$	$\Delta\phi = \phi_2 - \phi_1$	$e_i = -2\Delta\phi$	$e_i' = -e_i$	$e_r = 20i$	$e_0 = e_r + e_i'$
0	0	0	0	0	....	.....	.....	0	
7.5	2	......	.....	....	324	−648	648	40	688
15	4	400	360	324	....	.....	.....	80	
22.5	5	......	.....	....	162	−324	324	100	424
30	6	600	540	486	....	.....	.....	120	
37.5	6.75	......	.....	....	121.5	−243	243	135	378
45	7.5	750	675	607.5	....	.....	.....	150	
52.5	8.25	......	.....	.....	121.5	−243	243	165	408
60	9	900	810	729	....	.....	.....	180	
67.5	9.25	......	.....	....	40.5	− 81	81	185	266
75	9.5	950	855	769.5	....	.....	.....	190	
82.5	9.75	......	.....	....	40.5	− 81	81	195	276
90	10.0	1000	900	810	....	.....	.....	200	
97.5	9.75	......	.....	....	− 40.5	81	− 81	195	114
105	9.5	950	855	769.5	....	.....	.....	190	
112.5	9.25	......	.....	....	− 40.5	81	− 81	185	104
120	9	900	810	729	....	.....	.....	180	
127.5	8.25	......	.....	....	−121.5	243	−243	165	− 78
135	7.5	750	675	607.5	....	.....	.....	150	
142.5	6.75	......	.....	....	−121.5	243	−243	135	−108
150	6	600	540	486	....	.....	.....	120	
157.5	5	......	.....	....	−162	324	−324	100	−224
165	4	400	360	324	....	.....	.....	80	
172.5	2	......	.....	....	−324	648	−648	40	−608
180	0	0	0	0	....	.....	.....	0	
187.5	− 2	......	.....	....	−324	648	−648	− 40	−688
195	− 4	− 400	−360	−324	....	.....	.....	− 80	
202.5	− 5	......	.....	....	−162	324	−324	−100	−424
210	− 6	− 600	−540	−486	....	.....	.....	−120	
217.5	− 6.75	......	.....	....	−121.5	243	−243	−135	−378
225	− 7.5	− 750	−675	−607.5	....	.....	.....	−150	
232.5	− 8.25	......	.....	....	−121.5	243	−243	−165	−408
240	− 9	− 900	−810	−729	....	.....	.....	−180	
247.5	− 9.25	......	.....	....	− 40.5	81	− 81	−185	−266
255	− 9.5	− 950	−855	−769.5	....	.....	.....	−190	
262.5	− 9.75	......	.....	....	− 40.5	81	− 81	−195	−276
270	−10.0	−1000	−900	−810	....	.....	.....	−200	

11. Generation of a Sinusoidal Wave of Emf.—*Simple Sinusoidal Quantity.*—*A simple sinusoidal quantity is the product of a constant and the sine of an angle having values varying linearly with the values of the independent variable. Thus*

$$y = A \sin (\omega x + \alpha),$$

where A, ω, and α are constants. In Art. 1 a coil was considered as rotating at a uniform velocity in a uniform bipolar magnetic field; Fig. 12 represents such a coil and field. When the coil is in such a position that its plane is perpendicular to the direction of the lines of force from the field, as in position HK, the maximum number of lines is enclosed by the coil. As the coil revolves through the angle α in the time t sec., the amount of flux enclosed is seen to follow the law,

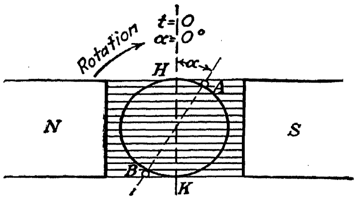

FIG. 12.—Generation of the sinusoidal wave $e = E_m \sin \alpha$ by the rotation of a coil at uniform velocity in a uniform magnetic field.

$$\phi = \Phi_m \cos \alpha = \Phi_m \cos \omega t \text{ maxwells.} \qquad (16)$$

By Faraday's law the emf induced in the coil, due to the rate of change of the flux through the coil, is

$$e_i = -N\frac{d\phi}{dt} \times 10^{-8} \text{ volts.} \qquad (2)$$

Hence, differentiating Eq. (16), and substituting this value for $d\phi/dt$ in Eq. (2), the induced emf equation becomes

$$e_i = \omega N \Phi_m \times 10^{-8} \sin \omega t \text{ volts,} \qquad (17)$$

The maximum value of the induced emf in Eq. (17) occurs when $\sin \omega t = 1.0$, since ωt is the only variable in time or angle in the equation. Hence,

$$\begin{aligned} E_m &= \omega N \Phi_m \times 10^{-8} \text{ volts,} && (18) \\ &= 2\pi f N \Phi_m \times 10^{-8} \text{ volts,} && (19) \end{aligned}$$

and Eq. (17) may be rewritten

$$e_i = E_m \sin \omega t \text{ volts.} \qquad (20)$$

Equation (17) is one of the most common equations in electrical engineering, and is the basis for the use of the sinusoidal equation in a-c calculations.

12. Commercial Wave Forms.—Owing to the mechanical limitations of commercial machines, it is impossible to meet the conditions assumed in Fig. 12. Magnetic lines of force enter and leave iron surfaces perpendicularly to those surfaces, and hence it is impossible to obtain the flux distribution shown above. The slotted armatures used in modern machine construction and the magnetic effect of the current flowing in the armature conductors under load still further disturb the uniformity of the flux distribution. It is, therefore, practically impossible to construct an alternator that will generate in each coil a true sinusoidal wave of emf at all loads. By a proper shaping of the pole faces, however, and by the distribution of the conductors over the armature surface, the wave form of the machine may be made to approach closely that of a sine wave at no load; the wave form of most commercial machines differs from a sine wave by such a small amount that no great error is introduced in simple calculations by assuming the wave to be sinusoidal.

It is evident that the graphical methods so far studied are too cumbersome for the solution of complicated problems. It will be shown later that the mathematical solution of such problems is greatly simplified by the assumption of sine waves, and also that such calculations based on sine waves may be extended to include waves of any shape. It must be remembered, however, that expressions hereafter derived upon the sine-wave assumption may produce serious errors if used when the wave form differs materially from a sine wave, the only accurate solution in such cases being either the graphical methods so far considered or the extended mathematical analysis that will be developed in Chap. XVI.

13. Relation of Frequency, Speed, and Poles for an Alternator.

Period.—The period of a periodic quantity is the smallest value of the increment of the independent variable that separates recurring values of the quantity. In case the independent variable is time, the letter T is often used to designate the period.

Frequency.—The frequency of a periodic quantity, in which time is the independent variable, is the number of periods occurring *in unit time. If the angular velocity of the fundamental is ω, then the frequency is $\omega/2\pi$.*

Cycle.—A cycle is the complete series of values of a periodic quantity that occur during a period.

Angular Velocity.—The angular velocity of a periodic quantity is the frequency multiplied by 2π.

For any machine, when a point on the armature passes from the center of a north pole to the center of the next north pole, one cycle of flux change has been passed through. In general, there will be as many cycles in each revolution of the armature or field of a multipolar machine as there are pairs of poles on the machine. It must not be forgotten, however, that what really counts in determining the number of cycles of emf is the number of complete cyclic changes of flux through the armature coils in series. These principles will be illustrated by a few examples.

Example 4

How many cycles will be generated per second in an eight-pole machine revolving at 750 r.p.m.?

$$f = \frac{p(\text{r.p.m.})}{2 \times 60},$$

or

$$f = \frac{8 \times 750}{2 \times 60} = 50 \text{ cycles.}$$

Example 5

What is the value of the flux per pole required for a two-pole machine revolving at 3,600 r.p.m. and generating a maximum emf of 2,000 volts with 500 turns in series on the armature, the flux change being assumed sinusoidal? What is the frequency?

$$f = \frac{2 \times 3{,}600}{2 \times 60} = 60 \text{ cycles.}$$

From Eq. (19),

$$\Phi_m = \frac{E_m \times 10^8}{2\pi f N} = \frac{2{,}000 \times 10^8}{2\pi \times 60 \times 500} = 1.06 \times 10^6 \text{ maxwells.}$$

14. Methods of Comparing Waves.—So far no method has been discussed for comparing the values of different waves of similar wave form, or of dissimilar waves, except by the comparison of instantaneous values at chosen angles or by the comparison of their maximum values. The method of comparison of instantaneous values is unsatisfactory, since each wave has an infinite number of such values. Maximum values of similar waves might be used for comparison (and such values will be used to a considerable extent), since all other values at corresponding instants will vary in proportion to the maxima. For waves of dissimilar shape, however, the comparison of maxima will not assist in a determination of the comparative effect of the two waves, and other methods must be studied. The following paragraphs present two of the most common and most useful of these methods of comparison.

15. Algebraic Mean or Average Value of a Wave.—One comparison of waves, which might suggest itself, is that of the areas of the different waves. This comparison is difficult to make, unless the waves have the same frequency and are plotted to the same scale of coordinates. For the same scale, waves of different frequencies will have different periods and different lengths of base. If, however, the areas of the different waves are divided by their respective bases, there is obtained the average or algebraic mean of their ordinates.

There are two average values which are regularly used for periodic quantities, that for a full wave or cycle and that for a half wave or half cycle. These values are defined as follows:

Average Value of a Periodic Quantity.—The average value of a periodic quantity is the algebraic average of the values of the quantity taken throughout one period.

Half-wave Average of an Alternating Quantity.—The half-wave average of an alternating quantity is the absolute value of the algebraic average of the quantity taken throughout a half period.

In the case of ordinary emf and current waves in a-c circuits, the average value is zero, but the half-wave average is not zero. For power waves and for emf or current waves with d-c components, the average value is not zero. In general, when the average value is referred to it is understood that it is the half-wave average that is required.

Since electrical waves may be expressed as functions of time or angle as, *e.g.*,

$$e = f(t) = h(\alpha) \text{ volts,} \tag{21}$$

the area of one half wave may be found as

$$\text{Area of half wave} = \int_0^\pi e\,d\alpha \text{ volts.} \tag{22}$$

The limits for Eq. (22) must be so chosen as to include one lobe of the wave between two crossing points 180 deg. apart on the time axis. If the instant the wave crosses the axis from negative to positive values is chosen as the starting point, a positive value will be obtained for the net area in Eq. (22). When this area is divided by the base, the half-wave average value of the wave becomes

$$E_{av} = \frac{\text{area}}{\text{base}} = \frac{1}{\pi}\int_0^{\pi} e\,d\alpha \text{ volts.} \tag{23}$$

When the equation of the wave is known, it may be substituted for e in Eq. (23), and the integration performed mathematically.

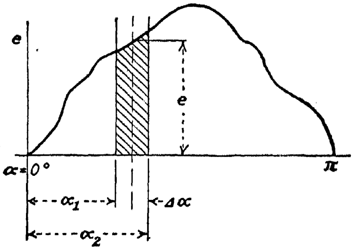

FIG. 13.—Elementary area of an emf wave.

16. Graphical Method of Determining Half-wave Average Values.—When the wave is given and its equation is not known, graphical methods may be employed to determine its area or its mean ordinate. One method of accomplishing this is to divide the area under the wave into a number of small vertical trapezoidal areas having equal bases and to obtain the sum of these areas for the total area under the curve. In the use of this method certain precautions must be taken, and, since it will be used extensively, a short review of this method will be given here.

Choosing equal values of angular increments, as $\Delta\alpha$ in Fig. 13, the area is seen to be

$$\text{area of half wave} = \sum_{\alpha=0}^{\pi}[e\Delta\alpha] = -\sum_{\alpha=\pi}^{2\pi}[e\Delta\alpha], \tag{24}$$

where e is the mean emf during each interval $\Delta\alpha$, the summation being taken over one half cycle in each case. Each small trapezoidal area is therefore $e\Delta\alpha$. From this,

$$E_{av} = \frac{\text{area}}{\text{base}} = \frac{\sum_{\alpha=0}^{\pi}[e\Delta\alpha]}{\Delta\alpha \times \text{number of } \Delta\alpha\text{'s}} \text{ volts} \tag{25}$$

$$= \frac{\sum_{\alpha=0}^{\pi}[e\text{'s}]}{\text{number of } e\text{'s}} \text{ volts,} \tag{26}$$

or the mean value of a number of equally spaced ordinates of the wave may be used as the value of E_{av}.

In using this method, however, certain precautions must be taken growing out of the above derivation of the equations. Using intervals of 10 deg., and a half cycle, or 180 deg. of the wave, should the beginning be made at 0 deg. and the last ordinate be taken at 180 deg., using 19 ordinates, or should the first ordinate be taken at 0 deg. and only 18 ordinates be used? The sum of the ordinates would be the same in either case and, in fact, would be the same if only 17 ordinates were taken, omitting both the zero values. This leads to a discussion of whether the sum of the ordinates should be divided by 18, 19, or 17, obtaining different values (all of them wrong) for the value of E_{av}. From a consideration of the method of derivation given above, the reason for this difference in value and the error may be seen. When adding ordinates at equal intervals, areas are actually being added, but to a different scale. Whatever method is used, a value proportional to the total area of the wave from 0 to 180 deg. must be obtained. When the 19 ordinates are used, a value is obtained proportional to the area from -5 to 185 deg.; when the 18 ordinates are used, beginning with the one at 0 deg., the crossing point of the wave, a value is obtained proportional to the area from -5 to 175 deg.; and when only 17 ordinates are used, beginning at 10 deg., a value is obtained proportional to the area from 5 to 175 deg. None of these methods gives a value proportional to the true area of the wave and, hence, cannot give the mean ordinate, no matter what number is used for the divisor. If, however, the start is made with the ordinate at 5 deg., and 18 ordinates, one every 10 deg., are used, ending with the ordinate at 175 deg., a value is obtained proportional to the true area, and the mean ordinate is found by dividing this value by 18, the number of ordinates taken. If the wave is plotted to some convenient scale, the ordinates may be obtained from the wave at 5 deg., 15 deg., etc.

17. Mathematical Determination of the Half-wave Average Value of a Sine Wave.—To illustrate the derivation of the half-wave average value of a wave from its mathematical equation, assume a sine wave of emf, crossing the axis when $\alpha = 0$ deg., as

$$e = E_m \sin \alpha \text{ volts.} \tag{20}$$

Substituting in Eq. (23) the value of e from Eq. (20),

$$E_{av} = \frac{1}{\pi}\int_0^\pi e\,d\alpha \text{ volts} \tag{23}$$

$$= \frac{1}{\pi}\int_0^\pi E_m \sin \alpha\,d\alpha \text{ volts}$$

$$= \frac{E_m}{\pi}\int_0^\pi \sin \alpha\,d\alpha \text{ volts,} \tag{27}$$

since E_m is a constant for any wave. By integration this becomes

$$E_{av} = \frac{E_m}{\pi}\Big| - \cos \alpha \Big|_0^\pi \text{ volts}$$

$$= \frac{2}{\pi} E_m \text{ volts} \tag{28}$$

$$= 0.636\ E_m \text{ volts.} \tag{29}$$

From Eq. (29) *it is seen that there is a constant ratio* 0.636 *between the average and the maximum values of any sine wave.*

Example 6

Find the half-wave average of a symmetrical alternating wave of current having each half cycle consisting of an isosceles triangle with a maximum value of 100 amp.

For such a triangular wave, which is symmetrical about a vertical axis at $\pi/2$ rad., it is sufficient to consider only one-fourth of a cycle instead of one-half cycle in determining the half-wave average. The equation of the wave for this one-fourth cycle in terms of its slope is

$$i = m\alpha \text{ amp,}$$

where m is the slope of the line. Since the line passes through the origin, $i = 0$ when $\alpha = 0$. When $\alpha = \pi/2$, $i = I_m = 100$ amp. Therefore,

$$100 = \frac{m\pi}{2} \text{ or } m = \frac{200}{\pi}, \quad \text{and}$$

$$i = \frac{200}{\pi}\alpha \text{ amp.}$$

The area under this curve is

$$\text{area} = \int_0^{\frac{\pi}{2}} \frac{200}{\pi}\alpha d\alpha$$

$$= \frac{200}{\pi}\left|\frac{\alpha^2}{2}\right|_0^{\frac{\pi}{2}} = 25\pi.$$

The half-wave average is this area divided by the base $\pi/2$, or

$$I_{av} = 50 \text{ amp.}$$

18. Effective Value.—In examining the reason for the generation of an emf, the conclusion must be reached that some measure of emf and current must be considered which has to do with the transmission of energy in the circuit. It has been found that there are certain relations between the current flowing in a circuit and the emf to be impressed across the circuit to make that current flow. At any instant when the current is i and the emf impressed e, the rate of flow of energy into the circuit, or the instantaneous value of the power, is

$$p = ei \text{ watts.} \tag{30}$$

When the circuit is made up of a resistance into which the energy flows, the energy being given off as heat, then

$$e = ri \text{ volts.} \tag{8}$$

Substituting the value of e from Eq. (8) into Eq. (30),

$$p = ri^2 \text{ watts.} \tag{31}$$

Hence,

$$p = \frac{e^2}{r} = \frac{1}{r}e^2 \text{ watts,} \tag{32}$$

where e and r are in practical units, e in volts, and r in ohms. The average or mean power consumed during each cycle will be the same as the average for all the cycles, and may be expressed as

$$P = \left[\text{mean } \frac{e^2}{r}\right] \text{ watts}$$

$$= \frac{1}{r}[\text{mean } e^2] \text{ watts.} \tag{33}$$

The average rate of flow of the energy, or the average power consumed by the resistance, is given by Eq. (33) for values of e and the resistance. In a similar manner the mean power is found to be

$$P = r[\text{mean } i^2] \text{ watts.} \tag{34}$$

For a d-c circuit the expression for power is

$$P = \frac{E^2}{R} \text{ watts, or} \tag{35}$$

$$P = RI^2 \text{ watts.} \tag{36}$$

If, for an alternating wave form, the emf is defined as

$$E = \sqrt{[\text{mean } e^2]} \text{ volts,} \tag{37}$$

and the current as

$$I = \sqrt{[\text{mean } i^2]} \text{ amp,} \tag{38}$$

the emf and current units can be used in the power equations in the same manner as the corresponding d-c units. For example, a unit of alternating current defined by Eq. (38) produces the same rate of heating when passing through a given resistor as does a unit of direct current. These values of E and I, as given in Eqs. (37) and (38), are called the "effective" values of the emf and current.

Effective Value of a Periodic Quantity.—The effective value of a periodic quantity is the square root of the average of the squares of the values of the quantity taken throughout one period. Thus, if y is a periodic function of t,

$$Y = \sqrt{\frac{1}{T}\int_0^T y^2dt},$$

where Y is the effective value of y and T is the period.

It is seen that for power calculations the effective values of the waves must be used instead of the average or the maximum values, and *unless otherwise specified, when emf and current are expressed in volts and amperes, effective values are intended.* Voltmeters and ammeters are calibrated to give readings in effective values.

Graphically to obtain the effective value of a wave, the wave of squared values must be plotted. Suppose that the effective value of an emf wave is desired. Instantaneous values of emf are squared, and a curve is plotted through these points as shown in Fig. 14. The area under this squared curve, the cross hatched area in Fig. 14, must now be obtained and divided by the base line in order to get the average of the squared values of emf. The methods of Art. 16 may be applied in obtaining the average squared value. When this value of emf has been obtained, the square root is taken to give the effective value of the wave expressed in volts. This may be done analytically, if the equation of the wave is known. For example, if the wave of emf is expressed as

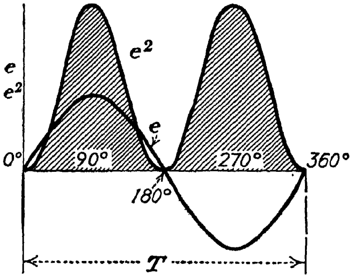

Fig. 14.—Graphical method of obtaining effective value of emf wave.

$$e = f(t) = h(\alpha) \text{ volts,}$$

then the squared wave is expressed as

$$e^2 = h^2(\alpha).$$

The area under the squared wave is

$$\text{Area} = \int_0^{2\pi} h^2(\alpha)d\alpha.$$

The mean of the squared values is

$$\text{Mean } e^2 = \frac{\text{area}}{\text{base}} = \frac{\int_0^{2\pi} h^2(\alpha)d\alpha}{2\pi}.$$

The effective value is the square root of mean e^2 and may be written

$$E = \sqrt{\frac{\int_0^{2\pi} h^2(\alpha)\,d\alpha}{2\pi}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} e^2\,d\alpha}\ \text{volts.} \tag{39}$$

The equation for effective current is

$$I = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} i^2\,d\alpha}\ \text{amp.} \tag{40}$$

Example 7

Find by a graphical method the average and effective values of the wave shown in Example 3.

TABLE III.—AVERAGE AND EFFECTIVE VALUES OF A CURRENT WAVE
Example 7

α (degrees)	i	i	i^2
0	0		
7.5	....	2	4
15.0	4		
22.5	....	5	25
30.0	6		
37.5	....	6.75	45.56
45.0	7.5		
52.5	....	8.25	68.06
60.0	9		
67.5	....	9.25	85.56
75.0	9.5		
82.5	....	9.75	95.06
90.0	10		
97.5	....	9.75	95.06
105.0	9.5		
112.5	....	9.25	85.56
120	9		
127.5	....	8.25	68.06
135	7.5		
142.5	....	6.75	45.56
150	6		
157.5	....	5	25
165	4		
172.5	....	2	4
180	0		
		82.00	646.48

$$\text{Area} = \sum i = 82.00,$$

$$I_{av} = \frac{82.00}{12} = 6.83 \text{ amp},$$

$$\text{Area} = \sum i^2 = 646.48,$$

$$I = \sqrt{\frac{646.48}{12}} = 7.33 \text{ amp}.$$

19. Duty Cycle.—The heating of an electrical conductor carrying a variable current is proportional to the integrated mean heating due to i^2r loss in that conductor. When the conductor carries current that varies with some predetermined set of events, such as a variable load on an electrical machine, the periodic variation in current taken by the machine is called a "duty cycle." The steady value of direct or alternating current, which will produce the same heating in the conductor through which the variable current flows, is used to determine the continuous rating of the machine of which the conductor is a part. Since the conductor is part of the winding of an electrical machine, the heat caused by the flow of current in this winding is proportional to the square of this equivalent value of current, whether the current is direct or alternating. In determining this equivalent current, a complete cycle of events must be taken. The square of the instantaneous values of direct current or of the effective values of alternating current should be plotted as ordinates with time as abscissas. The area between this i^2 wave and the time axis should be determined in a manner similar to that used for Example 7. The equivalent heating current is then found as the square root of this area divided by its base; *i.e.*,

$$I_{eq} = \sqrt{\frac{\text{area under the } i^2 \text{ wave}}{\text{time of one duty cycle}}} \text{ amp.} \tag{41}$$

Example 8

A d-c motor takes a current represented in a repeating duty cycle by 100 amp for 30 sec., 50 amp for 30 sec., and 10 amp for 2 min. Find the value of the equivalent continuous current for this motor that will produce the same heating.

The total cycle lasts 3 min. or 180 sec.

$$I_{eq} = \sqrt{\frac{100^2 \times 30 + 50^2 \times 30 + 10^2 \times 120}{180}} \text{ amp},$$

$$= \sqrt{\frac{378{,}000}{180}} = 46.3 \text{ amp}.$$

20. Mathematical Determination of the Effective Value of a Sine Wave.—Using the same equation for the emf wave as used in the derivation of the average value of a sine wave, Eq. (20),

$$e = E_m \sin \alpha \text{ volts}, \tag{20}$$

as representing any sine wave of emf, and by the definition of the effective value in Eq. (39), substituting the value of e^2 from Eq. (20),

$$E = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} E_m{}^2 \sin^2 \alpha d\alpha} \text{ volts,}$$

or

$$E = \sqrt{\frac{E_m{}^2}{2\pi}\int_0^{2\pi} \sin^2 \alpha d\alpha} \text{ volts,} \tag{42}$$

but, since

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2},$$

and

$$\begin{aligned} \int_0^{2\pi} \sin^2 \alpha d\alpha &= \int_0^{2\pi} \frac{1 - \cos 2\alpha}{2} d\alpha, \\ &= \frac{1}{2}\int^{2\pi} d\alpha - \frac{1}{2}\int_0^{2\pi} \cos 2\alpha d\alpha \\ &= \frac{1}{2}\bigg|\alpha\bigg|_0^{2\pi} - \frac{1}{4}\bigg|\sin 2\alpha\bigg|_0^{2\pi} \\ &= \frac{2\pi}{2} + 0 = \pi. \end{aligned}$$

Substituting this value in Eq. (42)

$$\begin{aligned} E &= E_m\sqrt{\frac{\pi}{2\pi}} \text{ volts} \\ &= \frac{1}{\sqrt{2}}E_m = 0.707E_m \text{ volts.} \end{aligned} \tag{43}$$

Hence, there is a constant ratio, 0.707, *between the effective and the maximum values of sine waves.*

Example 9

[illegible]d the effective value of the wave of Example 6 which is an isosceles [illegible]gle having an altitude of 100 amp and passing through the origin.

[illegible]om Example 6, the equation of the first quarter cycle of this wave is

$$i = \frac{200}{\pi}\alpha \text{ amp.}$$

From the definition of the effective value of a wave,

$$
\begin{aligned}
I &= \sqrt{\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{200}{\pi}\right)^2\alpha^2 d\alpha} \text{ amp} \\
&= \sqrt{\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{40{,}000}{\pi^2}\alpha^2 d\alpha} \text{ amp} \\
&= \sqrt{\frac{80{,}000}{\pi^3}\left|\frac{\alpha^3}{3}\right|_0^{\frac{\pi}{2}}} \text{ amp} \\
&= \frac{100}{\sqrt{3}} = 57.7 \text{ amp.}
\end{aligned}
$$

The i^2 wave when plotted consists of two parabolas intersecting at $\pi/2$ rad., or 90 deg., for each half wave.

21. Power Equation.—From Eq. (30) it is seen that

$$p = ei \text{ watts} \tag{30}$$

represents the instantaneous power, or the instantaneous rate of flow of the energy in a circuit, where e and i are expressed in practical units. In such a circuit, with sinusoidal waves of emf and current, and using the current wave as a reference,

$$i = I_m \sin \alpha \text{ amp} \tag{44}$$

and

$$e = E_m \sin (\alpha + \theta)\text{[1] volts} \tag{45}$$

represent any E and I waves for the same circuit. Substituting the values of e and i from Eqs. (44) and (45) into Eq. (30), the power equation becomes

$$p = E_m I_m[\sin \alpha \sin (\alpha + \theta)] \text{ watts.} \tag{46}$$

Expanding this equation,

$$
\begin{aligned}
p &= E_m I_m[(\sin \alpha \cos \theta + \cos \alpha \sin \theta) \sin \alpha] \\
&= E_m I_m[\sin^2 \alpha \cos \theta + \sin \alpha \cos \alpha \sin \theta] \\
&= E_m I_m\left[\frac{1 - \cos 2\alpha}{2} \cos \theta + \frac{\sin 2\alpha}{2} \sin \theta\right] \\
&= \frac{E_m I_m}{2}[\cos \theta - \cos 2\alpha \cos \theta + \sin 2\alpha \sin \theta] \qquad (47)
\end{aligned}
$$

---

[1] The numerical value of θ may be either positive or negative depending upon whether the voltage is ahead of, or behind, the current in phase. A negative value of θ is shown in Fig. 15, and a positive value in Fig. 16. In Fig. 17 the power wave is shown with $\theta = -90$ deg., and in Fig. 18 with $\theta = 0$ deg.

or

$$p = \frac{E_m I_m}{2}[\cos\theta - \cos(2\alpha + \theta)] \text{ watts.} \tag{48}$$

Equation (48) is the equation of a sine wave of double the frequency of the original waves since $2\alpha = 2\omega t = 2\pi(2f)t$, the wave being offset from the crossing point of the current reference wave by the angle θ, and its axis of symmetry being offset from the axis of e and i by the distance $EI \cos\theta$, as shown in Figs. 15, 16, 17, and 18.

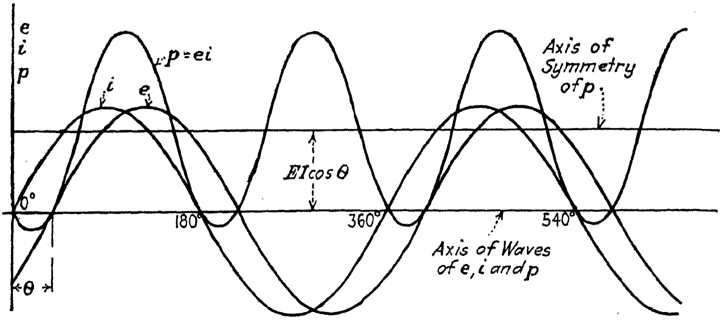

FIG. 15.—Power wave $p = ei$ with i leading e by the angle θ.

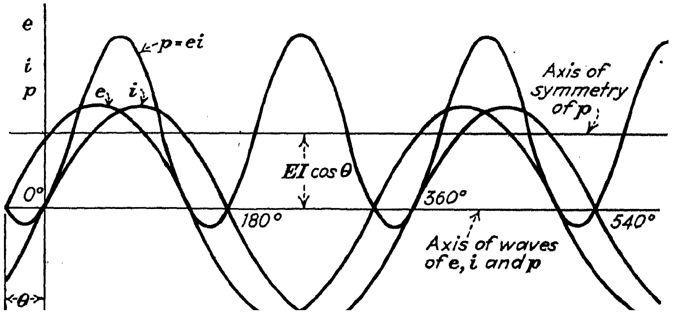

FIG. 16.—Power wave $p = ei$ with e leading i by angle θ.

Instantaneous Power.—Instantaneous power is equal to the product obtained by multiplying the instantaneous current by the instantaneous potential difference at the terminals of the circuit. Instantaneous power is given in watts when instantaneous current is in amperes and instantaneous potential difference is in volts.

22. Active Power.—*Active power is the time average of the values of the instantaneous power when the average is taken over a cycle of the alternating current.* The value of the power in which the customer is interested, or which is required for the rating of machines, is the average or algebraic mean of the power wave given in Eq. (48) and is the active power. Since, by definition, the active power is

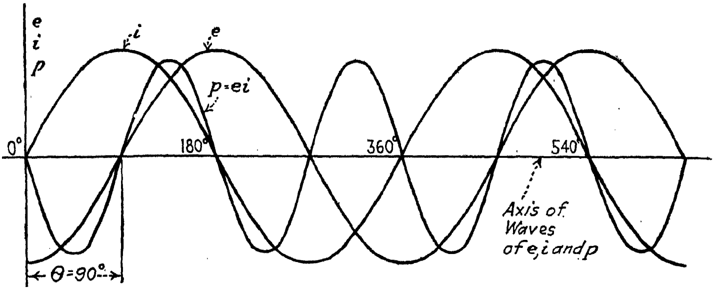

FIG. 17.—Power wave $p = ei$ with i leading e by 90 deg.

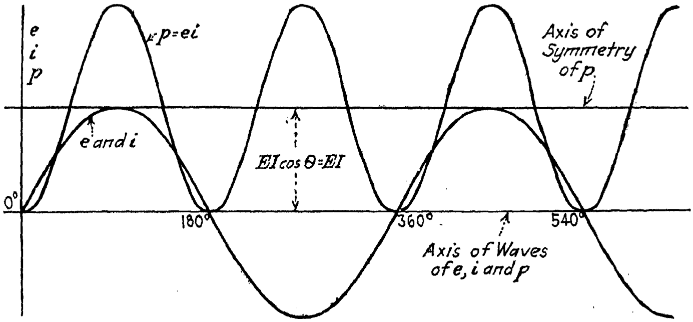

FIG. 18.—Power wave $p = ei$ with e and i in phase ($\theta = 0$).

$$P = \frac{\text{area}}{\text{base}} = \frac{1}{2\pi}\int_0^{2\pi} p\,d\alpha \text{ watts,} \tag{49}$$

and, by substituting the value of p from Eq. (48),

$$P = \frac{1}{2\pi}\int_0^{2\pi} \frac{E_m I_m}{2}[\cos\theta - \cos(2\alpha + \theta)]\,d\alpha \text{ watts,}$$

or, from Eq. (47), which is the better form for integration,

$$P = \frac{E_mI_m}{4\pi}\int_0^{2\pi}[\cos\theta d\alpha - \cos 2\alpha\cos\theta d\alpha + \sin 2\alpha\sin\theta d\alpha]$$

$$= \frac{E_mI_m}{4\pi}\Big|\alpha\cos\theta - \frac{\sin 2\alpha}{2}\cos\theta - \frac{\cos 2\alpha}{2}\sin\theta\Big|$$

$$= \frac{E_mI_m}{2}\cos\theta = \left(\frac{E_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right)\cos\theta, \tag{50}$$

or

$$P = EI\cos\theta \text{ watts.} \tag{51}$$

From Eq. (51), it is seen that the active power is the product of the voltmeter reading E by the ammeter reading I and by some factor equal to or less than 1 ($\cos\theta$ in the case of sine waves). This factor is called the "power factor" of the circuit, or, in other words, *the power factor is the factor by which the product of the voltmeter and ammeter reading must be multiplied* (in single-phase circuits) *to obtain the active power, or the reading of the wattmeter.*

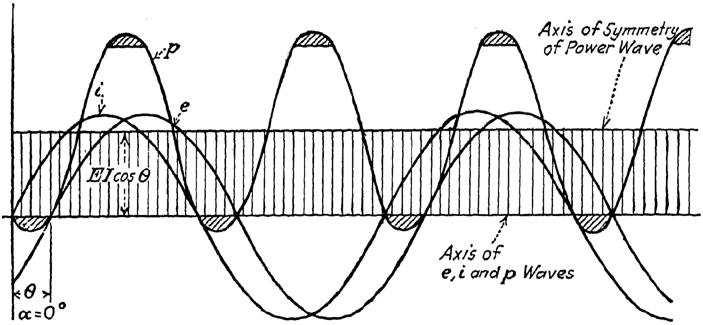

FIG. 19.—Active power P, the net area under the power wave.

The active power, as obtained in Eq. (51), is the integrated net area between the power wave and the time axis of the e and i waves divided by the base, or 2π. From an inspection of Fig. 19, this is seen to be a succession of positive areas and of negative areas. Each area represents the energy put into the circuit or received back from the circuit. If a portion be cut off from the tops of the positive loops, equal to the negative areas below the axis, by lines parallel to the axis, it will be seen that the part of the power loops remaining above the axis of symmetry of the power wave will just offset the "valleys" bounded by the power loops and the axis of symmetry. Hence, the value of the average power,

$$P = EI \cos \theta \text{ watts,} \tag{51}$$

represents the constant distance between the axis of symmetry of the emf and current waves and the axis of symmetry of the power wave. When $\theta = 90$ deg. the positive and negative areas are equal, and when $\theta = 0$ deg. the entire area is positive.

23. Measuring Instruments.—Before proceeding to a study of circuits it will be better to consider first methods of measurement of some of the quantities developed in this chapter such as current, voltage, and power. It is to be noted that in any instrument using a permanent or a constant magnetic field, *e.g.*, the D'Arsonval-type d-c meters and the astatic meters for direct currents, the torque acting on the moving elements, at any instant, is directly proportional to the current flowing in the coil at that instant. The average deflecting torque will be directly proportional to the average value of the current over any whole number of cycles. Since the average value of any a-c wave in which there is no d-c component is zero for a whole cycle or any whole multiple of cycles, it is evident that permanent magnets cannot be used in the construction of a-c meters except where the periodicity of the moving element is very low, as in the oscillograph.

Hot-wire Instruments.—From Eq. (38) and its derivation it is seen that the average heating effect produced by an alternating current is I^2R, the current being expressed in effective value. In the hot-wire type of instrument the current to be measured is sent through a wire that becomes heated owing to this current. The linear expansion of this wire is amplified and transmitted through a system of pulleys so as to cause a needle to move over a scale through an angle proportional to the expansion of the wire. Since the linear expansion of the wire is proportional to the heating, or to (mean i^2), the deflection of the needle will be proportional to the square of the effective value of the current in amperes. As the meter must indicate the first power of the current, the scale of such an instrument will not be graduated uniformly as are the scales of d-c meters. Since the heating value of either a direct or an alternating current is I^2R, this type of meter will be equally accurate for a-c or d-c circuits, and it forms one of the methods of direct comparison between the values of direct and alternating currents. This meter may be equipped with a series resistance, of the proper amount, and used also as a voltmeter for a-c circuits. The meter has zero correction for frequency and wave form, but it is difficult to keep it in accurate adjustment at different room temperatures, and it lacks a high degree of accuracy and sensitivity. It is not used at present commercially, to any great extent, except for high-frequency current measurements in wireless and telephone circuits, where the other types of instruments would have a high inductance error.

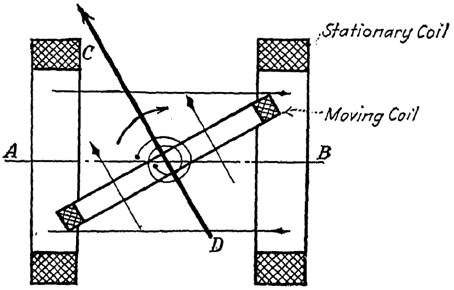

FIG. 20.—Diagram of an electrodynamometer type of instrument.

Electrodynamometer Instruments.—In its simplest form the electrodynamometer type of instrument consists of two stationary coils and one movable coil, as shown diagrammatically in Fig. 20. The three coils are connected in series, the stationary coils being connected so as to act in conjunction, producing a field along the axis *AB*, while the field due to the movable coil acts along *CD*. The relative direction of the fields at a given instant is indicated by the arrows. When a current flows in the coils, the movable element tends to set itself so that its field is parallel to that of the fixed coils and the needle moves to the right over the scale. When the current reverses its direction, both fields reverse at the same time and the torque acting on the movable element remains in the same direction as before. Spiral springs supply the necessary retarding torque and also conduct the current into and out of the movable coil. The torque, acting on the movable element, is proportional to the product of the two fields, or to the product of the currents in the fixed and movable coils, since there is no magnetic material with variable permeability in these coils.

Since the coils are all in series, the torque acting to deflect the needle at any instant is proportional to the square of the instantaneous value of the current. Owing to the inertia and the long period of the moving element, there is no vibration at commercial frequencies and the deflection is proportional to the average torque, *i.e.*, to (mean i^2) or to the square of the effective value of the current in amperes. As in the case of the hot-wire instrument, the scale is not uniformly divided. This type of instrument may also be used on d-c circuits and will give correct indications if the proper precautions are taken to eliminate the effect of stray magnetic fields. This may be done by taking reversed readings or by shielding the instrument with a soft iron shield. The coils of this instrument have some inductance that causes an error in the indication of voltmeters on circuits of very high frequency, or on circuits with higher harmonics of appreciable magnitudes in the emf wave, if calibrated at other frequencies and with sine waves.

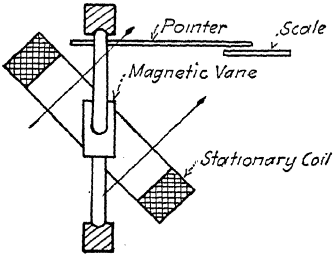

FIG. 21.—Diagram of a magnetic-vane type of instrument.

Magnetic-vane Instruments.—In this type of instrument the moving element consists of one or more vanes or thin sheets of soft iron attached to a spindle, in place of the moving coil used in the dynamometer type of instrument. Figure 21 shows the arrangement as used by one of the manufacturers. The stationary coil is inclined at an angle of about 45 deg. with the spindle, and the magnetic vane is inclined both to the spindle and to the axis of the coil. The current to be measured flows through the stationary coil, establishing a field along the axis as indicated by the arrows. The soft-iron vane tends to place itself parallel to the field, thus deflecting the needle. The retarding torque is supplied by spiral springs, but these springs carry no current.

The torque acting on the vane is proportional to the product of the field of the stationary coil and to the magnetic induction in the iron vane, which, in turn, is proportional to the magnetizing force of the coil, so long as the magnetic density in the vane is so low that the saturation of the iron may be neglected. The torque acting to deflect the needle at any instant is, therefore, proportional to the square of the current flowing at that instant, or the angle of deflection is proportional to the square of the current in amperes. These instruments may be used on d-c circuits, but their indications are not so accurate as the standard d-c instruments, or as when used on a-c circuits, owing to stray fields and the retentive power of the iron of the vane.

Ammeters.—In the hot-wire instruments the heating element for small values of current is constructed of a conductor of sufficient size to carry the current. For higher values of current a nonreactive shunt with zero temperature coefficient of resistance is used, the hot wire carrying only a fraction of the total current.

The moving element of the electrodynamometer type of instrument carries but a fraction of an ampere, because the moving coil must be as light as possible, and also because the spiral springs that conduct the current into the coil can carry only a fraction of an ampere without being materially injured. Shunts cannot conveniently be used with this type of instrument because the current will not divide between the coil and shunt in the inverse ratio of their resistances, owing to the inductance of the coils of the instrument. This desired division of current could be secured by a proper design of the shunt for a given instrument when moderate values of current are to be measured, and this is done for some laboratory standards. For instruments with high-current range, however, the skin and proximity effects in the conductor of the shunt may introduce other noncompensated errors. For this reason the electrodynamometer principle is seldom used for a-c ammeters.

In the magnetic-vane type of instrument the stationary coil can be constructed to carry the entire line current, and, since there is no current to be conducted into the moving element, it is seen that this type of instrument is most suitable for commercial a-c testing work. Even this type of instrument, however, is limited to about 100 amp as the highest current for portable laboratory instruments. For higher values of current and for current measurements in circuits of high voltage, where the instrument must be insulated from the line potential, a current transformer is used, the primary being connected into the line and the secondary short-circuited by the ammeter. The 5-amp range of ammeter is standard for use with current transformers, 1,000 amp requiring a 200:1 current transformer. The insulation of the current transformer should be able to withstand the full line potential, and the secondary, or meter circuit, should always be grounded for safety to the operator where the primary potential is dangerous to life.

Voltmeters.—Any type of instrument so far considered may be used as a voltmeter by placing a sufficient resistance in series with the meter element so that a very small current flows when the instrument is connected across the voltage to be measured. As in the case of d-c voltmeters the scale is not marked to indicate the effective value of the current in the coils, but the product of the current taken by the meter and the resistance of the meter circuit, or the effective value of the voltage impressed on the meter.

Effect of Wave Form.—It is to be noted that, in the discussion of ammeters and voltmeters, no assumption has been made as to the wave form of the current flowing through the meter. In all cases the deflection is proportional to (mean i^2), or the square of the effective current in amperes, independent of the form of the wave. The magnetic-vane instruments are the only ones of those discussed in which appreciable errors may occur due to ordinary variations in the wave form, and in these only if the magnetic density in the iron vane exceeds the saturation point of the iron.

Wattmeters.—In the derivation of the expression for active power, it has been shown that the instantaneous value of the power is the product of the instantaneous values of the current and voltage. An instrument to measure active power must, therefore, partake of the nature both of an ammeter and of a voltmeter. This can be accomplished in the electrodynamometer type of instrument by winding the stationary coils with large wire so that they can be connected in series with the line to carry the line current at all times, and by placing a high resistance in series with the moving coil so that the coil and the resistance may be connected directly across the voltage. Since the resistance drop in the circuit of the moving coil will be large in reference to the induced voltage, no appreciable error will be made, at ordinary frequencies and power factors, in assuming that the current flowing in the moving coil at any instant will be

$$i_1 = \frac{e}{r} = ke.$$

As the stationary coil carries the line current, the torque acting on the moving coil at any instant will be proportional to the product of the two currents, or,

$$t = ii_1 = kei = kp.$$

Thus, with this arrangement, the torque at any instant will be proportional to the power in the circuit at that instant. The average deflecting torque, and hence the deflection of the needle, will be proportional to the active power, and this will evidently be true whether the waves of the current and voltage are sinusoidal or not. The coils of the instrument must be so connected that the two fields will produce a torque deflecting the needle to the right. If the direction of both currents is reversed, the deflection will still be to the right or positive; but if either one is reversed without the other, the deflection of the needle will be to the left, or negative.

In the special case of sine waves it has been shown that

$$P = EI \cos \theta. \tag{51}$$

An inspection of the power wave of Fig. 15 will show that for part of the time the voltage and current are of opposite signs, or the power over that period is negative, and hence, the torque acting on the moving coil of the wattmeter is negative, or it tends to deflect the needle backward. When the power is positive, the torque is positive, or it tends to deflect the needle to the right over the scale. The resultant deflection of the needle is, therefore, proportional to the active power. It will also be seen that as θ, the angle between the two waves, increases, the magnitude of the power factor decreases, the area of the negative lobe increases, and that of the positive lobe decreases. Thus, for the same current and voltage, the negative torque on the meter coil increases and the positive torque decreases; thus the deflection of the needle is seen to be a function not only of the current and voltage but also of their phase relation. If θ is made equal to 90 deg., the positive and negative power lobes will be equal, and the active power will be zero. There will be just as much torque (integrated area) tending to deflect the needle to the left of zero as to the right, and hence the instrument will indicate zero power for this condition.

For large values of current or for dangerous potentials, current transformers are used, as explained under ammeters, with 5 amp as the current range on the meter side. For the potential-coil circuit, resistances are common if the potential is not more than 600 volts. For higher voltage measurements potential transformers are used with their primaries connected directly across the line potential and their secondaries across the potential coils of the wattmeters. These instrument transformers may introduce certain errors, which are discussed under transformers in most textbooks on electrical machinery.

Instrument Scales.—For the various types of ammeters and voltmeters discussed in the previous paragraphs, it is seen that the average torque is proportional to (mean i^2) or, to the square of the effective value of the current. If a scale be laid off with equal divisions, as shown above the line in Fig. 22, this scale will represent equal increments of (mean i^2). Since, however, $I = \sqrt{[\text{mean } i^2]}$, and it is desired to read the effective value of the current with an ammeter, or of voltage with a voltmeter, the scale must be laid off as shown below the line in Fig. 22, *i.e.*, a scale of the square roots of the equal deflections. Hence, the instrument really deflects in proportion to the square of the effective value of E or I and must be made to indicate E or I by providing it with a scale of unequal divisions. It is seen that an a-c instrument, with scale divided to indicate effective values of E or I, is more accurate at the higher part of the scale than at the lower part. Angularity in the position of the moving element in respect to the stationary coils will somewhat modify the scale, but the scale will never have uniform divisions as in a d-c D'Arsonval-type instrument.

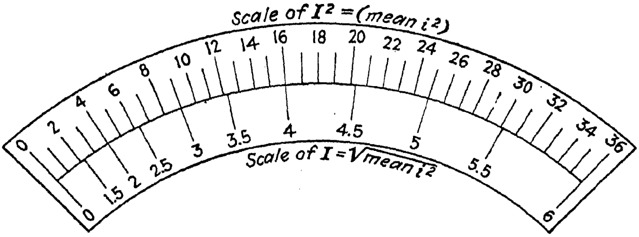

Fig. 22.—Scale of an a-c ammeter or voltmeter.

In a wattmeter the deflection, for a constant power factor, will be proportional to (mean ei) or to P. The scale for such an instrument will have uniform divisions except for the modification caused by the angularity of the moving coil in respect to the stationary coil. Since such an instrument can have no iron in its magnetic field without introducing frequency and wave-form errors, it cannot have an entirely uniform scale such as is found in the d-c instrument, the scale for most wattmeters being slightly more accurate in the center than at either end.

Problems

1-1. In an armature, similar to the one illustrated in Fig. 1, one coil is located at AB and another at CD, as shown in Fig. 1-1. Rotate this armature at a uniform velocity in the direction indicated, plotting the wave shape of the flux enclosed and the emf induced for the coil AB against α as abscissas. Plot the emf waves for the coils AB and CD in proper relation against angles as abscissas. Each line shown in Fig. 1-1 represents 10,000 maxwells. In counting the lines, the number should be estimated to the nearest 1,000 lines. Connect B to D on the outside, and bring out connections from A and C. Plot the resultant wave of emf e_{AC}.

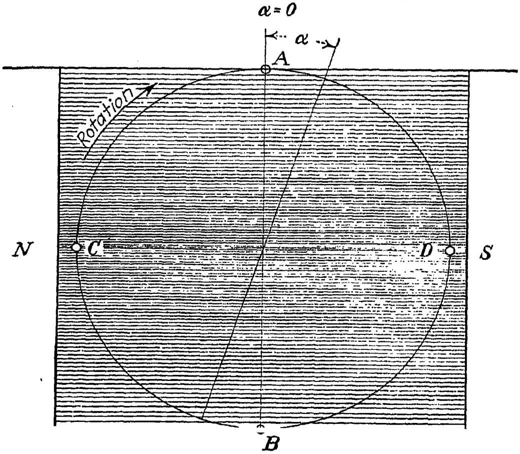

FIG. 1-1.—Figure for Prob. 1-1.

2-1. Same as Prob. 1-1, but use the flux distribution shown in Fig. 2-1.

3-1. Same as Prob. 1-1, except coil CD is 60 deg. behind coil AB.

4-1. A wave of current represented by $i = 1{,}000 \sin 1{,}000t$ is passed through a resistance of 15 ohms. Derive the equation for the emf drop across this resistance. Plot the current and emf waves.

5-1. A triangular wave of current, having an equilateral triangle with symmetrical lobes and a maximum value of 10 amp, is passed through a coil. Plot the waves of i, h, ϕ, and e_r if $r = 10$ ohms, $h = 2i$, and $\phi = 2h$.

6-1. A symmetrical flux wave, having each half wave a semicircle and with a maximum value of 10,000 maxwells, is passing through a coil. This flux wave is produced by a current in the coil. Plot the waves of i, h, and e_r if $r = 100$ ohms, $\phi = 100h$, and $h = 10i$.

7-1. Draw a sinusoidal wave of current with a maximum value of 1.00 on a sheet of rectangular coordinate paper for 360 deg., using values of ordinates from trigonometric tables. Derive and plot waves of h, ϕ, e_i, e_r, and e_o, tabulating all results, using $h = 1.4i$, $\phi = 1.1h$, $r = 2.0$, $e_i = -4\Delta\phi$. Plot these waves in proper phase relation on the same sheet of coordinate paper, and tabulate carefully the values on a separate sheet of tabular ruled paper.

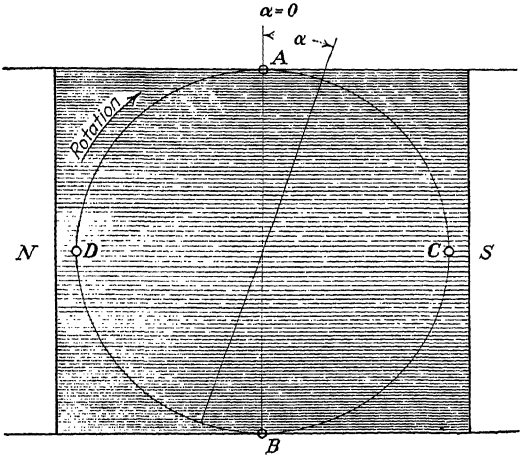

FIG. 2-1.—Figure for Prob. 2-1.

8-1. Replace the current wave of Prob. 7-1 with a symmetrical triangular wave of current, having a maximum value of 10 amp, and plot the waves asked for in Prob. 7-1, the waves derived from this current wave, using the same graphical method.

9-1. Replace the current wave of Prob. 7-1 with a series of trapezoidal waves having equal maxima and symmetrical positive and negative lobes for the different half waves, and plot as in Prob. 7-1 the waves derived from this current wave, using the same graphical method.

10-1. Take the values for a wave of current from the following table, and plot the wave for 360 deg. Derive the waves for ϕ, e_r, e_i, e_i', and e_0, assuming $N = 10$, $\phi = 0.9i$, $e_i = -0.4\Delta\phi$, and $r = 5$. Tabulate the results, and plot the waves in rectangular coordinates.

Angle	i	Angle	i	Angle	i
0°	0	60°	1.6	120°	1.5
15°	0.3	75°	2.0	135°	1.15
30°	0.7	90°	1.9	150°	0.8
45°	1.15	105°	1.7	165°	0.4
				180°	0

11-1. A two-pole turbogenerator revolves at 3,600 r.p.m.; what is its frequency?

12-1. A certain waterwheel-driven generator has 80 poles and is to generate 60 cycles. (*a*) At what speed must it be driven? (*b*) At what speed for 50 cycles?

13-1. At what speed must a generator operate to produce a frequency of 1,000 cycles if the machine has 100 poles?

14-1. A two-pole turbogenerator, rotating at 3,600 r.p.m., has a sinusoidal wave of emf of maximum value of 25,000 volts per coil, each coil having 10,000 turns; what is the value of the flux from each pole?

15-1. Plot a sinusoidal wave of emf having a maximum value of 100 volts. By tabular method, derive the half-wave average of this wave.

16-1. Plot the current wave given in Prob. 10-1, deriving the values of i for the other angles from a smooth curve through the given points. Tabulate these values, and derive the half-wave average of the current wave.

17-1. Find the half-wave average value of the wave form of current shown in Fig. 17-1, the maximum value of which is 18 amp.

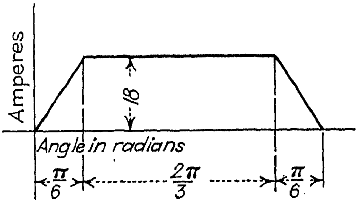

FIG. 17-1.—Figure for Prob. 17-1.

18-1. Obtain mathematically the half-wave average of a semicircular wave of emf having its diameter coinciding with the axis of abscissas and a maximum value of 50 volts.

19-1. Plot a sinusoidal wave of emf having a maximum value of 100 volts. By tabular method derive and plot the e^2 wave, and derive the effective value of the emf wave.

20-1. Plot the current wave given in Prob. 10-1. By tabular method from points obtained from the curve as drawn, derive and plot the i^2 wave, and obtain the effective value of the current wave.

21-1. Find the effective value of the wave form of current shown in Fig. 17-1.

22-1. The armature of a d-c motor is designed for an intermittent-duty cycle in which the current remains zero for 2 sec., 10 amp for 5 sec., 40 amp for 5 sec., and 100 amp for 2 sec. and this is repeated continually. What is the continuous-current rating of the armature?

CHAPTER II

SERIES CIRCUITS, SINE WAVE OF CURRENT AND ELECTROMOTIVE FORCE

In Chap. I, some of the fundamental general laws of the a-c circuit have been derived. In this chapter, these laws are applied to simple series circuits under the limitation imposed upon the equations by the use of a sine wave of current passing through the circuit. It has been demonstrated that a sine wave of emf is generated in a coil revolving at a uniform rate in a uniform magnetic field. The average and effective values of such a wave have been derived as well as the power equation and the active power, when the current and voltage are in any phase relation with each other.

24. Resistance Drop. *Resistance.—Resistance is the property of an electric circuit or of any body that may be used as part of an electric circuit that determines for a given current the rate at which electrical energy is converted into heat or radiant energy. In the general case, resistance is a function of the current, but the term is most commonly used in connection with circuits where the resistance is independent of the current. In the latter case, the rate of energy conversion is proportional to the square of the current and the resistance is then equal to the power that is converted divided by the square of the current.*

Resistor.—A resistor is a device, the primary purpose of which is to introduce resistance into an electric circuit.

Suppose that a sine wave of current,

$$i = I_m \sin \alpha = I_m \sin \omega t \text{ amp,} \qquad (44)$$

is passed through a circuit containing a constant resistance r ohms. Then, substituting the value of i from Eq. (44) into Eq. (8),

$$e_r = ri = rI_m \sin \alpha = rI_m \sin \omega t \text{ volts} \qquad (52)$$

is the equation of the resistance drop in the circuit, r being considered constant during each cycle of the current. This emf will be a maximum when sin α has the maximum value of 1, and

$$E_{m_r} = rI_m \text{ volts.} \tag{53}$$

Dividing both sides of Eq. (53) by $\sqrt{2}$, the effective value of the resistance drop becomes

$$E_r = rI \text{ volts.} \tag{54}$$

From Eqs. (52) and (53)

$$e_r = E_{m_r} \sin \alpha \text{ volts,} \tag{55}$$

which is plotted in conjunction with the current wave in Fig. 23. It is seen by an inspection of Eq. (55) and Fig. 23 that *the resistance-drop wave is in phase with, and of the same shape as, the current wave*, and that each ordinate of the current wave is multiplied by a constant r to obtain the corresponding values of the emf drop wave due to the resistance. From this a new definition of resistance follows:

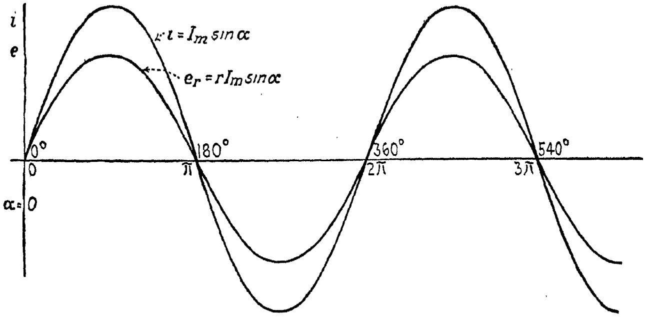

FIG. 23.—Sinusoidal waves of current and resistance drop in a series circuit.

Resistance.—Resistance may also be defined as the factor by which the current is multiplied to obtain the emf drop in phase with the current.

25. Coefficient of Self-induction.—Self-induction is a consequence of the interlinking of the magnetic flux around a conductor with the current that produces the magnetic flux. Any change in the current causes a change in the flux linkages, and a self-induced emf results. The direction of the induced emf with respect to the current was determined experimentally and is expressed by Lenz's law.

23-1. Same as for Prob. 22-1, except for the following duty cycle: 50 amp for 0.2 sec., 30 amp for 0.5 sec., and 12 amp for 0.3 sec.

24-1. Given an emf wave as shown in Fig. 24-1 having symmetrical positive and negative lobes, derive analytically the effective value for this wave.

25-1. Given two waves,

$$e = E_m \sin \alpha,\ E_m = 1.00,$$
$$i = I_m \sin (\alpha + \theta),\ I_m = 1.00,$$

plot these waves and determine graphically and plot the power wave in proper phase relation to the emf and current waves. Find graphically and analytically the average value of the power wave. Check this value against the product of the effective values of emf and current.[1]

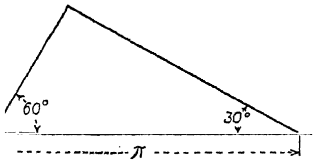

FIG. 24-1.—Figure for Prob. 24-1.

26-1. The wave of emf in Fig. 26-1 has a maximum value of 1,000 volts, and the wave of current has a maximum value of 100 amp. Plot the power wave, and find the active power and the power factor.

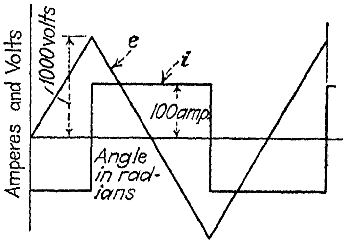

FIG. 26-1.—Figure for Prob. 26-1.

27-1. Consider the same waves of current and emf as in Prob. 26-1, except that the axis is crossed in synchronism. Plot the power wave, and find the active power and the power factor.

28-1. Same as Prob. 1-1, except that coil AB has 10 turns and coil CD has 20 turns.

29-1. Same as Prob. 2-1, except that coil AB has 10 turns and coil CD has 20 turns.

30-1. An emf represented by $e = 200 \sin (500t + 30°)$ is impressed on a lamp having a resistance of 20 ohms; what is the wave of current flowing?

31-1. Choose a semicircular wave of current having symmetrical positive and negative lobes of radius 1.00 amp and its diameter coinciding with the axis of abscissas. Derive and plot the other waves as in Prob. 7-1 for this current wave, using the same graphical method.

32-1. Same as Prob. 10-1 except use the following data:

Angle	i	Angle	i	Angle	i	Angle	i
0°	0	45°	70	90°	120	135°	183
15°	20	60°	85	105°	160	150°	130
30°	45	75°	97	120°	200	165°	62
						180°	0

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[1] NOTE TO INSTRUCTOR.—Assign to different members of the class values for θ from 0 to 180 deg. by 10- to 15-deg. intervals so as to cover the entire range of values for the angle of lead and lag.

33-1. A 10-pole generator rotates at 3,000 r.p.m. What is its frequency?

34-1. A two-pole generator has a flux of 12×10^6 lines per pole so distributed as to generate a sinusoidal wave of emf in a revolving coil. If the maximum value of E is 15,100 volts and the frequency of the induced emf is 2,000 cycles, how many turns should there be in the coil?

35-1. Plot the current wave given in Prob. 32-1. Tabulate values obtained from this curve, and derive the half-wave average for the current wave.

36-1. Plot the current wave given in Prob. 32-1. By tabular methods derive and plot the i^2 wave, and obtain the effective value of the current wave.

37-1. Obtain analytically the half-wave average and the effective values of a semicircular wave having its diameter coinciding with the axis of abscissas and having the positive and negative lobes symmetrical.

38-1. Obtain analytically the half-wave average and the effective values of a trapezoidal wave of emf having its positive and negative lobes symmetrical.

39-1. Choose emf and current waves both trapezoidal in shape with symmetrical positive and negative lobes and with the same length of base, their crossing points on the axis being 30 deg. apart. Plot the power wave, and determine the active power. Check this value against the product of the effective values of current and emf and the cosine of the angle of phase difference.

as with magnetic materials such as iron or steel, there is no constant value of inductance for the coil. For such coils Eq. (2) still holds and Eq. (57) becomes

$$e_i = -\frac{d}{dt}(Li) \text{ volts.} \tag{65}$$

Such circuits are studied in detail in the treatment of transformers and reactors with iron cores.

Example 10

a. An iron core is placed in a coil having 300 turns, and it is found that a current of 3 amp produces a flux of 1,050,000 lines; what is the inductance of the coil for this value of current?

$$L = \frac{N\phi}{I} \times 10^{-8} = \frac{300 \times 1{,}050{,}000 \times 10^{-8}}{3} = 1.05 \text{ henrys.}$$

b. If the current is reduced to 1.00 amp, the flux reduces to 780,000 lines; what is the new inductance of the coil?

$$L = \frac{N\phi}{I} \times 10^{-8} = \frac{300 \times 780{,}000 \times 10^{-8}}{1.00} = 2.34 \text{ henrys}$$

The presence of the iron not only greatly increases the inductance of the coil, but it also causes the inductance to vary greatly with variations of the current.

26. Induced Emf and Reactance.—Let the sine wave of current,

$$i = I_m \sin \alpha = I_m \sin \omega t \text{ amp,} \tag{44}$$

flow through the coil that has just been considered. The induced emf is expressed by Eq. (57) as

$$e_i = -L\frac{di}{dt} \text{ volts,} \tag{57}$$

where L is expressed in henrys and i in amperes. Differentiating Eq. (44) and substituting this value of di/dt into Eq. (57),

$$e_i = -\omega L I_m \cos \omega t = -\omega L I_m \cos \alpha \text{ volts.} \tag{66}$$

The emf to overcome this induced emf is

$$e_i' = e_x = -e_i = \omega L I_m \cos \alpha \text{ volts.} \tag{67}$$

Equation (66) gives the induced emf and Eq. (67) the emf drop caused by the inductance of the coil. The maximum value of the emf in Eq. (67) occurs when cos α is a maximum, or when cos α is equal to 1, and is expressed as

$$E_{m_x} = \omega L I_m \text{ volts.} \tag{68}$$

Dividing both sides of Eq. (68) by $\sqrt{2}$, to obtain the effective values of E_x and I,

$$E_x = \omega L I = 2\pi f L I \text{ volts.} \tag{69}$$

Replacing $\omega L = 2\pi f L$ with x, a single letter, to represent these terms,

$$E_x = xI \text{ volts,} \tag{70}$$

where

$$x = \omega L = 2\pi f L \text{ ohms.} \tag{71}$$

The drop due to the inductance, or the emf to be impressed to overcome the induced emf, is then

$$e_x = L\frac{di}{dt} \tag{67}$$

$$= xI_m \cos \alpha \text{ volts,} \tag{72}$$

which is a sine wave 90 deg. ahead of the current wave. The graphical relation of these waves to the current wave is shown in Fig. 24. It is seen, from these equations of current and emf that the induced emf wave is 90 deg. behind the current wave, but that it has the same shape as the current wave. From Eq. (72) the emf consumed by the inductance is seen to be 90 deg. ahead of the current wave and to be the same shape as the

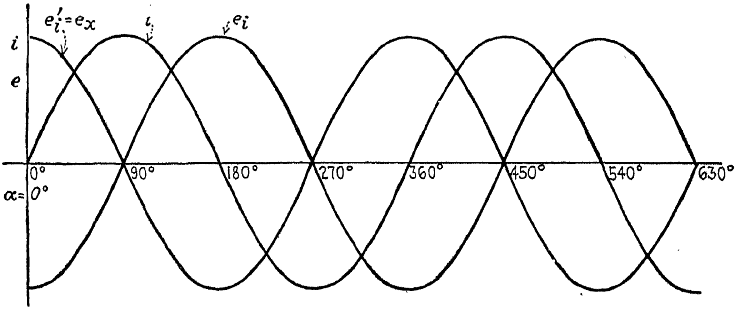

FIG. 24.—Sinusoidal waves of current, reactance drop, and induced emf for a coil with an air core.

Lenz's Law.—The induced emf in a circuit due to a change in flux linkages of the circuit tends to cause a current to flow in the circuit opposing the change in flux.

As an illustration of the effect of self-induction, consider a closely wound coil in the shape of a toroid in which the area of the coil is small compared with the radius of the toroid. When a current flows in the conductor of the coil, the flux through the coil is given by Eq. (13), derived in Art. 7, as

$$\phi = \frac{0.4\pi NiA}{l} \text{ lines,} \tag{13}$$

where the current is expressed in amperes. Since the flux is enclosed by all the turns of the coil, this value may be substituted directly in Faraday's law, Eq. (2), to obtain the value of the induced emf. Hence,

$$\begin{aligned} e_i &= -N\frac{d\phi}{dt}10^{-8} = -N\frac{d}{dt}\left(\frac{0.4\pi NiA}{l}\right)10^{-8} \text{ volts} \\ &= -\frac{0.4\pi N^2A}{l}10^{-8}\frac{di}{dt} \text{ volts.} \end{aligned} \tag{56}$$

The term $0.4\pi N^2A/l$ is a constant for any coil, being determined by the number of turns and the geometrical relations in the coil; and Eq. (56) may be written

$$e_i = -L\frac{di}{dt} \text{ volts,} \tag{57}$$

where

$$L = \frac{0.4\pi N^2A}{l}10^{-8} \text{ henrys.} \tag{58}$$

Since the flux and the flux linkages for any circuit depend upon geometrical relations, Eq. (58) must be modified for the different forms of the electric circuit, such as the various shapes of coils,[1] and for transmission lines and distribution circuits.

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[1] The value of L, as given in Eq. (58), should be corrected for the actual value, as given in "Inductance of Coils," Morgan Brooks and H. M. Turner, *Univ. Ill., Bull.* 53 or *Bull. Bureau of Standards, Reprint* 169. These formulas correct the value of L for the shape of the coil, a correction made necessary by the fact that all the flux does not interlink all the turns of the coil. For some coils the correction is small; in any case, the value of L as found by the use of the correct formula is a constant less than one times the value in Eq. (58).

In any case, the factor L is a constant for circuits of constant permeability of the magnetic flux path. It varies as the square of the turns in the circuit for coils of the same dimensions.

From Eqs. (2) and (57) it is seen that

$$e_i = -N\frac{d\phi}{dt} \times 10^{-8} = -L\frac{di}{dt} \text{ volts,}$$

or

$$N\frac{d\phi}{dt} \times 10^{-8} = L\frac{di}{dt} \text{ volts.} \tag{59}$$

Solving for L in Eq. (59)

$$L = N\frac{d\phi}{dt} \times \frac{dt}{di}10^{-8} = N\frac{d\phi}{di}10^{-8} \text{ henrys.} \tag{60}$$

$$Nd\phi \times 10^{-8} = Ldi. \tag{61}$$

Integrating Eq. (61),

$$Li = N\phi \times 10^{-8} + C, \tag{62}$$

where C is a constant of integration. To evaluate C, consider the initial condition of the circuit where $\phi = 0$ when $i = 0$, the condition of no residual flux, as in a nonmagnetic material like air; then, substituting this condition into Eq. (62), it is found that $C = 0$, and

$$Li = N\phi \times 10^{-8} \tag{63}$$

for nonmagnetic materials. Hence,

$$L = \frac{N\phi}{i} \times 10^{-8} \text{ henrys,} \tag{64}$$

or the *inductance is proportional to the flux linkages per ampere in any circuit.* Equation (64) will be found convenient in many derivations and comparisons of inductances. From Eqs. (13) and (64) it is seen that, since the flux in a coil having a core of constant permeability increases directly with the current, the inductance of such a coil must be constant. For coils in which the permeability varies over a considerable range throughout a cycle, or

$$\mu = f(t) = h(\alpha),$$

and from these same equations and Eq. (79)

$$\cos\theta = \frac{r}{z}, \tag{82}$$

and

$$\sin\theta = \frac{x}{z}. \tag{83}$$

Equations (79), (80), and (81) give definitions of E_o in terms of the constants of the coil. Rewriting Eq. (45), using these constants,

$$e_o = \sqrt{r^2 + x^2} I_m \sin(\alpha + \theta) \text{ volts}, \tag{84}$$

or

$$e_o = z I_m \sin(\alpha + \theta) \text{ volts}. \tag{85}$$

The maximum value of e_o becomes

$$E_{m_o} = z I_m \text{ volts}, \tag{86}$$

since the maximum value of $\sin(\alpha + \theta)$ is 1. The effective value of E_o is found, in terms of the effective value of the current, by dividing both sides of Eq. (86) by $\sqrt{2}$, as

$$E_o = zI \text{ volts}. \tag{87}$$

The quantity $z = \sqrt{r^2 + x^2}$, called the "impedance of the circuit," has the same dimensions as resistance and reactance.

Impedance.—Impedance is the factor by which the effective value of the current is multiplied to obtain the effective value of the impressed emf. It is of the same order and dimension as resistance and reactance, and its unit is the ohm.

From the relations in Eq. (80) it is seen that

$$z = \sqrt{r^2 + x^2} = \sqrt{r^2 + (2\pi f L)^2} \text{ ohms}, \tag{88}$$

and from Eqs. (82) and (83),

$$\cos\theta = P.F. = \frac{r}{z} = \frac{r}{\sqrt{r^2 + x^2}}, \tag{89}$$

or

$$r = z\cos\theta \text{ ohms}, \tag{90}$$

and

$$x = z\sin\theta \text{ ohms}, \tag{91}$$

From Fig. 26 it is seen that the impressed emf leads the current by the angle θ, or the current lags the impressed emf by the angle θ. That is, the impedance due to the coil causes the current to lag the impressed emf; the greater the value of the inductive reactance in relation to the resistance of the coil the greater the value of this angle of lag and the lower the power factor of the circuit.

28. Fundamental Differential Equation of an Inductive Circuit. The fundamental differential equation of an inductive circuit is obtained by substituting the values of e_r and e_x, in terms of i, and the constants from Eqs. (52) and (57) into Eq. (73) as

$$e_o = ri + L\frac{di}{dt} \text{ volts.} \tag{92}$$

This differential equation has no limitation as to sine waves, as have most of the other equations of this chapter, but it does have the limitation of a constant value of L, or of a coil with a non-magnetic core.

29. Ohm's Law of an Inductive Circuit.—The new Ohm's law becomes

$$\begin{aligned} E_o &= zI \qquad (87) \\ &= \sqrt{r^2 + x^2}I \text{ volts,} \end{aligned}$$

or

$$\begin{aligned} I &= \frac{E_o}{\sqrt{r^2 + x^2}} \\ &= \frac{E_o}{\sqrt{r^2 + (2\pi fL)^2}} \text{ amp.} \qquad (93) \end{aligned}$$

The power absorbed by such a circuit has been found to be

$$P = E_oI \cos \theta \text{ watts,} \tag{51}$$

but, from Eq. (82),

$$\cos \theta = \frac{r}{z}, \tag{82}$$

hence,

$$P = \frac{E_oIr}{z} = I^2r \text{ watts,} \tag{94}$$

as in d-c circuits. Hence, all the power is absorbed by the resistance of the circuit, and none of the emf consumed by the current wave. The scale between the maxima of the current and voltage waves is the value of x, which must have the same unit of dimension as resistance. This quantity, which is used for the combined terms in Eq. (71), or the coefficient of the current in Eq. (67), is called the "reactance."

Reactance.—Reactance is the factor by which the effective value of the current is multiplied to obtain the effective value of the emf drop 90 *deg. out of phase with the current; its unit is the ohm.*

Inductor.—An inductor is a device, the primary purpose of which is to introduce inductance into an electric circuit.

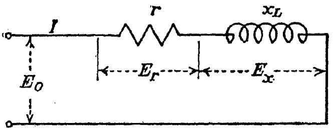

FIG. 25.—Circuit containing resistance and inductance in series.

27. Impressed Emf and Impedance.—Since both the resistance and the reactance of a coil of wire must be considered at the same time, it is necessary to know the value and the phase relation of the impressed emf in reference to the current and the other emfs in the circuit. Let Fig. 25 represent such a circuit or coil, and let the current wave,

$$i = I_m \sin \alpha \text{ amp}, \tag{44}$$

be passed through the circuit. From the previous derivation it has been found that the resistance drop will be

$$e_r = rI_m \sin \alpha \text{ volts}, \tag{52}$$

and that the reactance drop will be

$$e_x = xI_m \cos \alpha \text{ volts}. \tag{72}$$

By Kirchhoff's laws the sum of the instantaneous values of emf about any circuit must be equal to zero so that

$$e_o = e_r + e_x \text{ volts}. \tag{73}$$

Substituting the values of the emfs from Eqs. (52) and (72) into Eq. (73),

$$e_o = rI_m \sin \alpha + xI_m \cos \alpha \text{ volts}, \tag{74}$$

or

$$e_o = I_m[r \sin \alpha + x \cos \alpha] \text{ volts}. \tag{75}$$

The relation of these waves is shown in Fig. 26, where the wave for e_o is plotted from the sum of the ordinates of the e_r and the e_x

waves at corresponding values of α. The wave for e_o is seen to cross the axis θ deg. ahead of the current and to be of the form

$$e_o = E_{m_o} \sin (\alpha + \theta) \text{ volts.} \tag{45}$$

Expanding Eq. (45), to compare the terms with the identical Eq. (72),

$$e_o = \quad \sin \alpha \cos \theta + E_{m_o} \cos \alpha \sin \theta,$$

or

$$= (E_{m_o} \cos \theta) \sin \alpha + (E_{m_o} \sin \theta) \cos \alpha \text{ volts.} \tag{76}$$

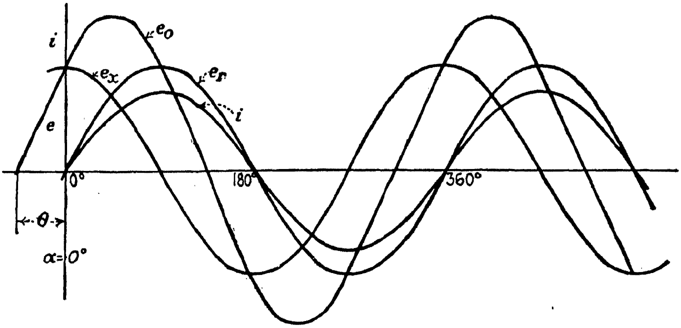

FIG. 26.—Sinusoidal wave of current, resistance drop, reactance drop, and impressed emf for the circuit of Fig. 25.

Comparing Eqs. (74) and (76) it is seen that

$$E_{m_o} \cos \theta = rI_m \text{ volts,} \tag{77}$$

and

$$E_{m_o} \sin \theta = xI_m \text{ volts.} \tag{78}$$

Combining Eqs. (77) and (78),

$$E_{m_o}{}^2 \sin^2 \theta + E_{m_o}{}^2 \cos^2 \theta = (r^2 + x^2)I_m{}^2 \text{ volts,}$$

or

$$\tag{79}$$

where

$$z = \sqrt{r^2 + x^2} \text{ ohms.} \tag{80}$$

Also, by dividing Eq. (78) by Eq. (77),

$$\tan \theta = \frac{x}{r}, \tag{81}$$

where

$$r_o = r_1 + r_2 + \ . \ . \ . \ r_n \text{ ohms,} \tag{100}$$

and

$$x_o = x_1 + x_2 + \ . \ . \ . \ + x_n \text{ ohms.} \tag{101}$$

Equation (95) may, therefore, be written

$$\begin{aligned} e_o &= r_o I_m \sin \alpha + x_o I_m \cos \alpha \text{ volts} \\ &= I_m (r_o \sin \alpha + x_o \cos \alpha) \text{ volts.} \end{aligned} \tag{102}$$

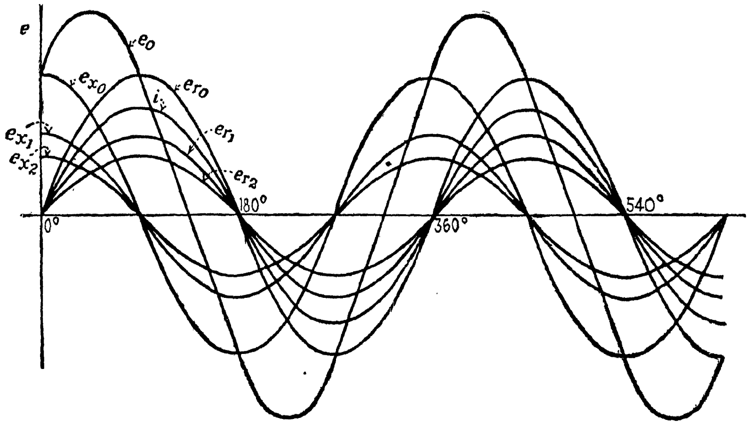

FIG. 28.—Current and emf waves for the circuit of Fig. 27.

Hence by similarity to Eqs. (75) and (84),

$$e_o = \sqrt{r_o^2 + x_o^2} I_m \sin (\alpha + \theta_o) \text{ volts} \tag{103}$$

$$= z_o I_m \sin (\alpha + \theta_o) \text{ volts,} \tag{104}$$

where

$$\begin{aligned} z_o &= \sqrt{r_o^2 + x_o^2} \\ &= \sqrt{(r_1 + r_2 + \ . \ . \ . \ + r_n)^2 + (x_1 + x_2 + \ . \ . \ . \ + x_n)^2} \text{ ohms,} \end{aligned} \tag{105}$$

and

$$\tan \theta_o = \frac{x_o}{r_o} = \frac{x_1 + x_2 + \ . \ . \ . \ + x_n}{r_1 + r_2 + \ . \ . \ . \ + r_n}. \tag{106}$$

From Eqs. (54), (70), and (87) the effective values of the several voltages may be written

$$E_{r_1} = Ir_1 \text{ volts},$$
$$E_{r_2} = Ir_2 \text{ volts},$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$E_{r_n} = Ir_n \text{ volts},$$
$$E_{r_o} = E_{r_1} + E_{r_2} + \ . \ . \ . + E_{r_n} = I(r_1 + r_2 + \ . \ . \ . + r_n)$$
$$= Ir_o \text{ volts.} \tag{107}$$
$$E_{x_1} = Ix_1 \text{ volts},$$
$$E_{x_2} = Ix_2 \text{ volts},$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$E_{x_n} = Ix_n \text{ volts},$$
$$E_{x_o} = E_{x_1} + E_{x_2} + \ . \ . \ . + E_{x_n} = I(x_1 + x_2 + \ . \ . \ . + x_n)$$
$$= Ix_o \text{ volts.} \tag{108}$$
$$E_o = Iz_o = I\sqrt{r_o^2 + x_o^2} =$$
$$I\sqrt{(r_1+r_2+ \ . \ . \ . +r_n)^2+(x_1+x_2+ \ . \ . \ . +x_n)^2} \text{ volts.} \tag{109}$$

It can be seen from these equations that the group of impedances in series may be replaced, as far as the total circuit is concerned, by a single impedance that has a resistance of r_o ohms, a reactance of x_o ohms, and an impedance of z_o ohms, the single impedance being equivalent to the group of impedances as far as terminal relations are concerned. The power factor of the entire circuit is

$$\cos \theta_o = \frac{r_o}{z_o} = \frac{r_1 + r_2 + \ . \ . \ . + r_n}{z_o}. \tag{110}$$

The total power absorbed by the circuit is

$$P_o = E_oI \cos \theta_o \text{ watts.} \tag{51}$$

Substituting the values of E_o and $\cos \theta_o$,

$$P_o = Iz_oI\frac{r_o}{z_o} \text{ watts}$$
$$= I^2r_o \text{ watts}$$
$$= I^2r_1 + I^2r_2 + \ . \ . \ . + I^2r_n \text{ watts,} \tag{111}$$

or the total power is the sum of the losses in the several parts of the circuit.

inductance goes to produce losses in the circuit. The cyclic absorption and return of energy by an inductance will be discussed later.

Example 11

A coil having a resistance of 2 ohms and an inductance of 0.002963 henry is connected across a 60-cycle a-c circuit. What is the impressed emf, the resistance drop, the reactance drop, the phase angle, the power consumed, and the power factor when 100 amp flow through the coil?

Resistance drop:

$$E_r = rI = 2 \times 100 = 200 \text{ volts.}$$

Reactance drop:

$$\begin{aligned} \omega &= 2\pi 60 = 377 \text{ rad. per second,} \\ x &= \omega L = 377 \times 0.00296 = 1.116 \text{ ohms,} \\ E_x &= 100 \times 1.116 = 111.6 \text{ volts.} \end{aligned}$$

Impressed emf:

$$\begin{aligned} E_o &= \sqrt{E_r^2 + E_x^2}. \\ &= \sqrt{\overline{200}^2 + \overline{111.6}^2} = 229 \text{ volts.} \end{aligned}$$

Impedance:

$$z = \frac{E_o}{I} = \frac{229}{100} = 2.29 \text{ ohms,}$$

or

$$z = \sqrt{r^2 + x^2} = \sqrt{2^2 + \overline{1.116}^2} = 2.29 \text{ ohms.}$$

Power consumed:

$$P = I^2 r = \overline{100}^2 \times 2 = 20{,}000 \text{ watts} = 20 \text{ kw.}$$

Power factor:

$$P.F. = \cos\theta = \frac{r}{z} = \frac{2}{2.29} = 0.875 = 87.5 \text{ per cent.}$$

30. Series Circuit.—Let several coils be connected in series, as shown in Fig. 27, and let a current,

$$i = I_m \sin\alpha \text{ amp,} \tag{44}$$

be sent through the circuit. The resistance drop in coil 1 will be

$$e_{r_1} = r_1 I_m \sin\alpha \text{ volts.}$$

Since the same current flows through all parts of the circuit, the resistance drops in the other coils will be

$$\begin{aligned} e_{r_2} &= r_2 I_m \sin\alpha \text{ volts}, \\ &\cdots\cdots\cdots \\ e_{r_n} &= r_n I_m \sin\alpha \text{ volts}. \end{aligned} \tag{52}$$

The reactance drops will be

$$\begin{aligned} e_{x_1} &= x_1 I_m \cos\alpha \text{ volts}, \\ e_{x_2} &= x_2 I_m \cos\alpha \text{ volts}, \\ &\cdots\cdots\cdots \\ e_{x_n} &= x_n I_m \cos\alpha \text{ volts}. \end{aligned} \tag{72}$$

Applying Kirchhoff's laws to the entire circuit the total impressed emf at any instant must be the sum of these instantaneous voltages, or

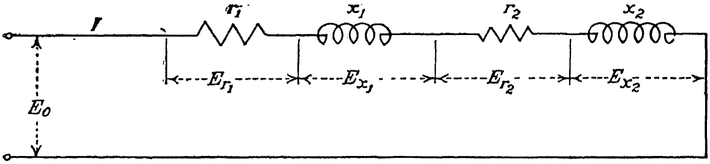

FIG. 27.—Two coils in series.

$$\begin{aligned} e_o = {} & r_1 I_m \sin\alpha + r_2 I_m \sin\alpha + \ . \ . \ . \ + r_n I_m \sin\alpha \\ & + x_1 I_m \cos\alpha + x_2 I_m \cos\alpha + \ . \ . \ . \ + x_n I_m \cos\alpha \text{ volts}. \end{aligned} \tag{95}$$

An inspection of Eq. (95) and Fig. 28 shows that $e_{r_1}, e_{r_2} \ . \ . \ . \ e_{r_n}$ are all in phase and add to form the wave

$$\begin{aligned} e_{r_o} &= e_{r_1} + e_{r_2} + \ . \ . \ . \ + e_{r_n} \\ &= (r_1 + r_2 + \ . \ . \ . \ + r_n) I_m \sin\alpha \text{ volts} \quad (96) \\ &= r_o I_m \sin\alpha \text{ volts}. \quad (97) \end{aligned}$$

Likewise, $e_{x_1}, e_{x_2} \ . \ . \ . \ e_{x_n}$ are all in phase and add to form the wave

$$\begin{aligned} e_{x_o} &= e_{x_1} + e_{x_2} + \ . \ . \ . \ + e_{x_n} \\ &= (x_1 + x_2 + \ . \ . \ . \ + x_n) I_m \cos\alpha \text{ volts} \quad (98) \\ &= x_o I_m \cos\alpha \text{ volts}, \quad (99) \end{aligned}$$

between different systems caused by changes in the charge and the incident flow of electric current, and of the consequent establishment of an electromagnetic stress. In other words, the transfer of the stored energy of the electrostatic field to the stored energy of the electromagnetic field, and the oscillations sometimes set up, which continue until the energy is finally dissipated by resistance in the form of heat, are all connected phenomena. Another explanation of some of the phenomena is attempted by the theory of the penetration of the charge into the dielectric separating the conductors. This theory seems to aid in the explanation of dielectric losses and the fact that, after a condenser has been discharged quickly, another charge may be drawn from it after it stands for a short period. These phenomena, however, may also be explained by the laws of transient phenomena with leakage-resistance paths.

32. Capacitance of Capacitors.—For the present purpose it is sufficient to consider that

$$Q = CE_C \text{ coulombs,}$$

where Q is the quantity of electricity stored in coulombs;

C is the capacitance of the capacitor (a quantity similar to L and replacing terms depending upon the material of the dielectric and upon its dimensions); and

E_C is the emf in volts impressed on the terminals of the capacitor.

When the capacitor is charging or discharging, the quantity of stored electricity will be changing by the same amount on each side of the capacitor. This will be manifested by a current flowing into one side and another current flowing out of the other side. From the definition of current as the rate of flow of the quantity of electricity,

$$i = \frac{dq}{dt} = C\frac{de_c}{dt} \text{ amp,}^1 \qquad (112)$$

from which the definition and the unit of capacitance is derived.

Capacitance.—Capacitance is the factor by which the rate of change of the emf across a condenser is multiplied to obtain the current flowing. The unit, the farad, is that value which will cause an

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[1] In Eq. (112) the positive sign is used before de_c/dt, since an increase in voltage is required to produce a positive flow of current into the capacitor.

ampere to flow when the emf across the condenser is varying at the rate of 1 *volt per second.* The farad is too large a unit for most considerations, and the *microfarad* is more common, there being 1,000,000 μf in 1 farad.

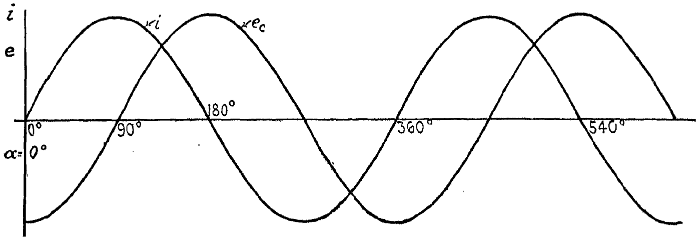

FIG. 29.—Sinusoidal waves of current and emf drop for a capacitor.

33. Capacitive Reactance.—Suppose that the emf across a capacitor varies according to a sine-wave law, or

$$e_c = E_{m_c} \cos \omega t = E_{m_c} \cos \alpha \text{ volts.} \tag{113}$$

The current flowing will be found by differentiating this equation and substituting the value of de_c/dt in Eq. (112) as

$$i = -\omega C E_{m_c} \sin \alpha \text{ amp,} \tag{114}$$

the maximum value of which is

$$I_m = -\omega C E_{m_c} \text{ amp,} \tag{115}$$

since the maximum value of $\sin \alpha$ is 1. The effective value is found as

$$I = -\omega C E_c = -2\pi f C E_c \text{ amp.} \tag{116}$$

From Eq. (116),

$$E_c = \frac{-1}{\omega C} I = x_c I \text{ volts,} \tag{117}$$

where

$$x_c = \frac{-1}{\omega C} = \frac{-1}{2\pi f C} \text{ ohms} \tag{118}$$

is the reactance of the capacitor at the frequency of f cycles per second. From the definition of reactance in Art. 26, it is seen that *the reactance drop across the condenser is* 90 *deg.* behind

Example 12

Find the value of e_0, the sum of two waves of emf represented by

$$e_1 = 100 \sin (\alpha + 30°), \text{ and}$$
$$e_2 = 200 \sin (\alpha - 60°),$$

both waves being referred to the same axis of angle or time.

The sum of these waves is given by

$$\begin{aligned} e_0 &= e_1 + e_2 \text{ volts} \\ &= 100 \sin (\alpha + 30°) + 200 \sin (\alpha - 60°) \text{ volts.} \end{aligned}$$

Expanding this equation,

$$\begin{aligned} e_0 &= 100 \sin \alpha \cos 30° + 100 \cos \alpha \sin 30° + 200 \sin \alpha \cos 60° \\ &\qquad - 200 \cos \alpha \sin 60° \\ &= 86.6 \sin \alpha + 100 \sin \alpha + 50 \cos \alpha - 173.2 \cos \alpha \\ &= 186.6 \sin \alpha - 123.2 \cos \alpha. \end{aligned}$$

Expressing e_0 as a sine wave in reference to the chosen axis,

$$\begin{aligned} e_0 &= E_{m_0} \sin (\alpha + \theta_0) \\ &= (E_{m_0} \cos \theta_0) \sin \alpha + (E_{m_0} \sin \theta_0) \cos \alpha, \text{ or} \\ E_{m_0} &= \sqrt{186.6^2 + 123.2^2} = 223.5 \text{ volts,} \\ \sin \theta_0 &= \frac{-123.2}{223.5} = -0.550, \\ \theta_0 &= -33.3°, \text{ and} \\ e_0 &= 223.5 \sin (\alpha - 33.3°) \text{ volts.} \end{aligned}$$

Example 13

Two coils are connected in series, and a sine wave of current, having an effective value of 10 amp, is sent through the circuit. The resistances and reactances of the coils are as follows: $r_1 = 4$ ohms, $x_1 = 12$ ohms, $r_2 = 18$ ohms, $x_2 = 6$ ohms. (*a*) Find the equation of the impressed voltage. (*b*) Find the value of the single impedance that may be substituted for the two series impedances. (*c*) Find the equation for the impressed voltage using the single equivalent impedance. (*d*) Solve for the effective value of the impressed voltage.

a. $$i = 10\sqrt{2} \sin \alpha = 14.14 \sin \alpha \text{ amp.}$$

From Eq. (95),

$$\begin{aligned} e_0 &= 4 \times 14.14 \sin \alpha + 18 \times 14.14 \sin \alpha + 12 \times 14.14 \cos \alpha \\ &\qquad + 6 \times 14.14 \cos \alpha \text{ volts} \\ &= 56.6 \sin \alpha + 254.5 \sin \alpha + 169.7 \cos \alpha + 84.8 \cos \alpha \text{ volts} \\ &= 311 \sin \alpha + 254.5 \cos \alpha \text{ volts} \\ &= 402 \sin (\alpha + 39° 17') \text{ volts.} \end{aligned}$$

b. $$\begin{aligned} r_0 &= r_1 + r_2 = 4 + 18 = 22 \text{ ohms,} \\ x_0 &= x_1 + x_2 = 12 + 6 = 18 \text{ ohms,} \\ z_0 &= \sqrt{r_0^2 + x_0^2} = \sqrt{22^2 + 18^2} = 28.4 \text{ ohms.} \end{aligned}$$

c. From Eq. (102),

$$\begin{aligned} e_0 &= r_0 I_m \sin \alpha + x_0 I_m \cos \alpha \text{ volts} \\ &= 22 \times 14.14 \sin \alpha + 18 \times 14.14 \cos \alpha \text{ volts} \\ &= 311 \sin \alpha + 254.5 \cos \alpha \text{ volts} \\ &= 402 \sin (\alpha + 39°17') \text{ volts.} \end{aligned}$$

d. From Eqs. (107), (108), and (109),

$$\begin{aligned} E_{r_0} &= Ir_0 = 10 \times 22 = 220 \text{ volts,} \\ E_{x_0} &= Ix_0 = 10 \times 18 = 180 \text{ volts,} \\ E_0 &= Iz_0 = 10 \times 28.4 = 284 \text{ volts.} \end{aligned}$$

31. Capacitance and Capacitors. *Capacitor.—A capacitor is a device, the primary purpose of which is to introduce capacitance into the circuit.* The earliest study of electricity had to do with what is known as "static electricity," or stored electricity, in contrast with "dynamic electricity," or the flow of electric current along a wire, which has so far been studied in this text. In other words, static electricity had to do with the potential energy of electricity stored rather than with the kinetic energy of the flow of electricity. The device in which the static electricity is stored is called a "capacitor."

For the present consideration, the "capacitor" is supposed to be composed of two or more metallic conductors separated by some kind of an insulation called the "dielectric." A commercial capacitor takes the form, in ordinary engineering practice, of sheets of tinfoil separated by sheets of paraffined paper, mica, or other good insulator. This consideration of the phenomena, however, does not bar other forms of capacitors using air for a dielectric, consideration of which will be given under the head of transmission lines, or of cables for telephones or underground transmission of energy, or of the windings of electrical machines which are, in effect, capacitors.

The early theory accounting for the phenomena of the storage of electricity was based upon the supposition of a charge residing on the surface of the conductor. This theory explained most of the phenomena investigated by early scientists. With the development of the more modern theories of electricity and magnetism, however, new developments arose that could not be explained by this theory. From further studies grew the more modern theory of magnetic and electrostatic stresses in the surrounding medium as explaining much better the interaction

Power:

$$\begin{aligned} P &= EI\cos\theta \\ &= 100 \times 7.46 \times 0.746 \\ &= 560 \text{ watts,} \end{aligned}$$

or

$$P = I^2r = \overline{7.46}^2 \times 10 = 560 \text{ watts.}$$

35. Resistance, Inductance, and Capacitance in Series.—Consider the circuit of Fig. 32 with resistance, inductance, and capacitance in series, and with a sine wave of current of single frequency, represented by the equation

$$i = I_m \sin\alpha \text{ amp}, \qquad (44)$$

flowing in the circuit. From Kirchhoff's laws,

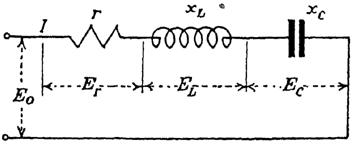

FIG. 32.—Circuit with resistance, inductance, and capacitance in series.

$$e_o = e_r + e_L + e_C \text{ volts} \qquad (124)$$

$$= ri + L\frac{di}{dt} + \frac{1}{C}\int idt \text{ volts.} \qquad (125)$$

Equation (125) *is the complete differential equation of a series circuit containing R, L, and C.* Applying the sine wave of current, Eq. (44), to Eq. (125),

$$e_o = rI_m \sin\alpha + \omega LI_m \cos\alpha - \frac{1}{\omega C}I_m \cos\alpha \qquad (126)$$

$$= rI_m \sin\alpha + (x_L + x_C)I_m \cos\alpha \qquad (127)$$

$$= I_m[r\sin\alpha + (x_L + x_C)\cos\alpha] \text{ volts,} \qquad (128)$$

or

$$e_o = zI_m \sin(\alpha + \theta) \text{ volts,} \qquad (85)$$

where

$$z = \sqrt{r^2 + (x_L + x_C)^2} \text{ ohms} \qquad (129)$$

$$= \sqrt{r^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2} \text{ ohms,} \qquad (130)$$

and

$$\tan\theta = \frac{x_L + x_C}{r}, \qquad (131)$$

$$\cos\theta = \frac{r}{z}, \qquad (82)$$

$$\sin\theta = \frac{x_L + x_C}{z}, \qquad (132)$$

where the numerical value of x_C carries a negative sign. The maximum value of E_o becomes

$$E_{m_0} = zI_m \text{ volts,} \qquad (86)$$

and the effective value is found to be

$$E_o = zI \text{ volts,} \qquad (87)$$

by dividing both sides of Eq. (86) by $\sqrt{2}$.

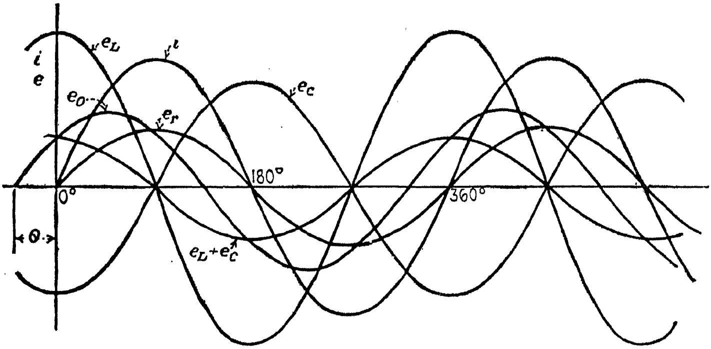

FIG. 33.—Sinusoidal waves of current, resistance drop, reactance drop, and impressed emf for the circuit of Fig. 32.

36. Ohm's Law for the Series Circuit.—The form of Ohm's law for a series circuit containing R, L, and C becomes

$$I = \frac{E_o}{z} = \frac{E_o}{\sqrt{r^2 + \left(2\pi fL - \dfrac{1}{2\pi fC}\right)^2}} \text{ amp.} \qquad (133)$$

An examination of Eq. (126) and Fig. 33 will show that the waves of e_L and e_C are 180 deg. out of phase with each other, and that these emfs tend to counteract each other in the circuit. The resultant of e_L and e_C must be combined with e_r to obtain the wave of impressed emf e_0. E_o may lead or lag or be in phase with the current wave in such a circuit, depending upon the relative values of e_L and e_C. The value of the angle θ for the entire circuit is determined from Eqs. (131), (82), or (132).

Example 15

A coil having a resistance of 10 ohms and an inductive reactance of 20 ohms is connected in series with a capacitor having a capacitive reactance of *the current, or the current is* 90 *deg. ahead of the capacitive reactance drop.* The relations of e_c and i are shown in Fig. 29. It is to be noted that the sign of the numerical value of x_c is always negative.

34. Resistance and Capacitance in Series.—Consider the circuit of Fig. 30 with r and C in series and with a sine wave of current flowing in the circuit, represented by the usual equation

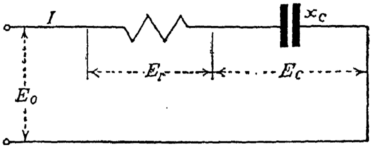

FIG. 30.—Circuit containing resistance and capacitance in series.

$$i = I_m \sin \alpha \text{ amp.} \qquad (44)$$

From Kirchhoff's laws,

$$e_o = e_r + e_c \text{ volts,} \qquad (73)$$

and from Eqs. (112) and (44) the differential equation becomes

$$e_o = ri + \frac{1}{C}\int idt \text{ volts.} \qquad (119)$$

Hence, substituting the value of i from Eq. (44),

$$e_o = rI_m \sin \alpha - \overset{1}{} \cos \alpha + K \text{ volts,} \qquad (120)$$

where K is a constant of integration, determined by the previous condition of the circuit when the switch is closed, and upon the point of closing the switch on the impressed emf wave. For this consideration, Eq. (120) reduces to the steady-state form,

$$e_o = rI_m \sin \alpha - \frac{1}{\omega C} I_m \cos \alpha \qquad (121)$$

$$= I_m(r \sin \alpha + x_c \cos \alpha) \qquad (75)$$

$$= zI_m \sin (\alpha + \theta) \text{ volts,} \qquad (85)$$

where

$$z = \sqrt{r^2 + x_c^2} \text{ ohms,} \qquad (80)$$

and

$$\tan \theta = \frac{x_c}{r} = \frac{-\frac{1}{2\pi fC}}{r} = -\frac{1}{2\pi fCr}, \qquad (122)$$

$$\cos \theta = \frac{r}{z}, \qquad (82)$$

$$\sin \theta = \frac{x_c}{z} = \frac{-\frac{1}{2\pi fC}}{z} = -\frac{1}{2\pi fCz}. \qquad (123)$$

It is seen that tan θ, sin θ, and θ are negative in respect to I as the reference wave. The waves of Eqs. (121) and (85) are shown in relation to the current wave in Fig. 31. It is seen, from an inspection of these waves and the equations, that the impressed emf e_o lags the current by the angle θ, or, in other words, the current leads the impressed emf by this same angle.

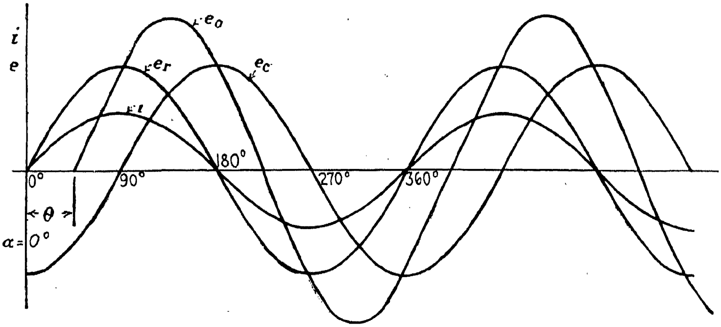

FIG. 31.—Sinusoidal waves of current, resistance drop, reactance drop, and impressed emf for the circuit of Fig. 30.

Example 14

A capacitor, having a capacitance of 300 μf, is connected in series with a resistor of 10 ohms across a circuit having an emf of 100 volts at 60 cycles. What current flows? What are the reactance and the impedance of the circuit, the power consumed, and the power factor?

Reactance:

$$x_c = \frac{-1}{2\pi fC} = \frac{-1}{2\pi \times 60 \times 300 \times 10^{-6}}$$
$$= \frac{-10^6}{377 \times 300} = -8.84 \text{ ohms.}$$

Impedance:

$$z = \sqrt{r^2 + x_c^2} = \sqrt{10^2 + \overline{8.84}^2} = 13.4 \text{ ohms.}$$

Power factor:

$$P.F. = \cos\theta = \frac{r}{z} = \frac{10}{13.4} = 0.746,$$
$$\tan\theta = \frac{-8.84}{10} = -0.884,\ \theta = -41°\,40',$$

Current:

$$I = \frac{E}{z} = \frac{100}{13.4} = 7.46 \text{ amp.}$$

At this frequency,

$$x_L = -x_C = 2\pi \times 73.3 \times 0.0531 = \frac{73.3}{60} \times 20 = 24.42 \text{ ohms.}$$

Current:

$$I = \frac{E_o}{z_o} = \frac{E_o}{r} = \frac{1{,}000}{10} = 100 \text{ amp,}$$

$$Z_{\text{coil}} = \sqrt{10^2 + 24.42^2} = 26.5 \text{ ohms.}$$

Drop across the coil:

$$E_{\text{coil}} = 100 \times 26.5 = 2{,}650 \text{ volts.}$$

Drop across the capacitor:

$$E_{\text{cap.}} = 100 \times 24.42 = 2{,}442 \text{ volts.}$$

It is seen that in this example the drop across the coil is more than twice the impressed emf, as is also that across the capacitor.

38. Variation of Current in a Series Circuit with Variation of Frequency—Constant Impressed Emf.—In Fig. 34 the current taken by a given series circuit, containing constant values of R, L, and C, is plotted against frequency with frequency varied from values below to values above the one to produce resonance. It is seen that the current is a maximum at resonance, being equal to the impressed emf divided by the resistance. The smaller the resistance in the circuit, for a given impressed emf, the greater the maximum value of this current. As resonance is approached, the emfs across the coil and the capacitor both increase owing to increase in current, but the reactance of the coil increases owing to the increase in frequency, whereas the reactance of the capacitor decreases. The emf-drop waves are shown on the same figure. At zero frequency the reactance of the capacitor is infinite, or the total emf drop, equal to the impressed emf, will appear across the capacitor, the current flowing being zero. As the frequency is increased, the reactance of the coil increases and that of the capacitor approaches zero at infinite frequency, and the coil tends to consume the entire emf impressed.

Curves are plotted in Fig. 35 showing the variation of current with frequency with constant impressed emf for three values of circuit resistance. In all three curves the same values of L and C are used, *viz.*, $L = 0.15$ henry and $C = 50$ μf. Curve A is for a resistance of 20 ohms, the same as that used in the calcu-

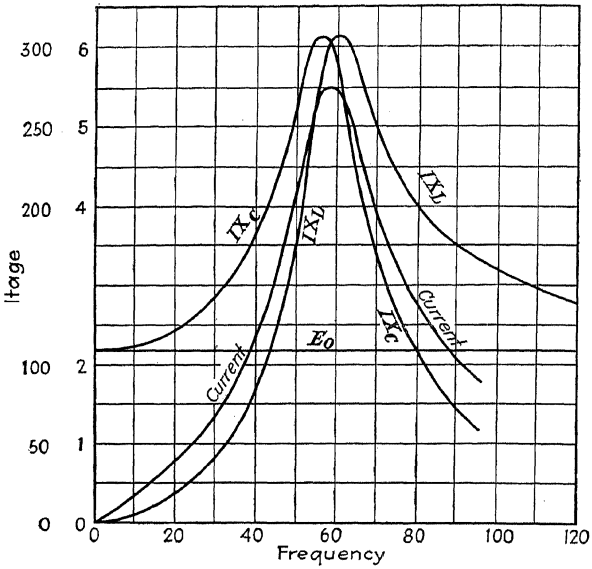

FIG. 34.—Curves showing the variation of I, E_L, and E_C with frequency for a series circuit with E_0 held constant.

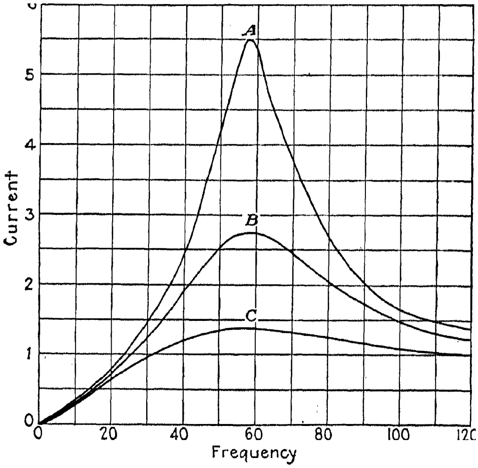

FIG. 35.—Curves showing the variation of current with frequency for different values of resistance for a series circuit, L, C, and E_0 constant.

30 ohms across a 1,000 volt a-c circuit; find the current, the drop across the coil, the drop across the capacitor, the power consumed by the circuit, the power factors of the coil, and of the whole circuit.

Total impedance:

$$z_o = \sqrt{\overline{10}^2 + (20 - 30)^2} = 14.14 \text{ ohms.}$$

Impedance of coil:

$$z_{\text{coil}} = \sqrt{\overline{10}^2 + \overline{20}^2} = 22.4 \text{ ohms.}$$

Current:

$$I = \frac{E_o}{z_o} = \frac{1{,}000}{14.14} = 70.7 \text{ amp.}$$

Drop across coil:

$$E_{\text{coil}} = 70.7 \times 22.4 = 1{,}580 \text{ volts.}$$

Drop across capacitor:

$$E_{\text{cap.}} = 70.7 \times 30 = 2{,}120 \text{ volts.}$$

Power factor of circuit:

$$P.F._o = \frac{r}{z_o} = \frac{10}{14.14} = 0.707.$$

Power factor of coil:

$$P.F._{\text{coil}} = \frac{10}{22.4} = 0.447.$$

Power consumed:

$$P = I^2r = 70.7^2 \times 10 = 50{,}000 \text{ watts} = 50 \text{ kw.}$$

37. Series-phase Resonance.—*Series-phase resonance is the steady-state condition that exists in a circuit comprising inductance and capacitance connected in series when the current in the circuit is in phase with the voltage across the circuit.* In the complete series circuit, when x_L is equal to $-x_C$, tan θ is zero and θ becomes zero. In other words, even with inductance and capacitance in the circuit, the current and emf impressed may be in phase with each other when a sine wave of current of the proper frequency flows. This condition is called "resonance." Resonance may be produced in any circuit of any character for a particular frequency by the variation of either L or C or both, or, for a given L and C, by the variation of the frequency. From the condition of resonance, for θ equal zero,

$$x_L + x_C = 0,$$
$$x_L = -x_C \text{ ohms},$$

or

$$2\pi fL = \frac{1}{2\pi fC},$$

and, solving for f,

$$f = \frac{1}{2\pi\sqrt{LC}} \text{ cycles per second.} \tag{134}$$

Equation (134) becomes the solution for the frequency that will give resonance in any series circuit. For current waves that are not simple sine waves some of the harmonics may produce resonance in a series circuit. For a fixed value of frequency the product of the capacitance in farads and of the inductance in henrys is the important consideration for the production of resonance. The method of variation of L and C is used in wireless operation for tuning the circuit for a given frequency of transmission.

Example 16

If the values of reactance in Example 15 are given at 60 cycles, what frequency will produce series-phase resonance in the circuit? With 1,000 volts impressed at resonant frequency what current will flow and what will be the drop across the capacitor and across the coil?

Inductance:

$$L = \frac{\omega L}{\omega} = \frac{x_L}{\omega} = \frac{20}{377} = 0.0531 \text{ henry.}$$

Capacitance:

$$C = \frac{-1}{\omega x_c} = \frac{-10^6}{377 \times (-30)} = 88.4\ \mu\text{f.}$$

Frequency for resonance:

$$f = \frac{1}{2\pi\sqrt{LC}} \text{ cycles per second,} \tag{134}$$

$$LC = 0.0531 \times 88.4 \times 10^{-6} = 4.70 \times 10^{-6},$$
$$\sqrt{LC} = 2.172 \times 10^{-3},$$

and

$$f = \frac{1}{2\pi \times 2.172 \times 10^{-3}} = \frac{10^3}{13.63} = 73.3 \text{ cycles per second,}$$

Since the resonant frequency is determined by the product LC, there are an infinite number of values of L and C that will produce resonance at a given frequency. Three values are illustrated in Fig. 36 where curve A is again the same as shown in Fig. 34. For curve C the resistance is kept constant at 20 ohms, the inductance is doubled, and the capacitance is halved. For curve B the inductance is divided by 3 and the capacitance is multiplied by 3. It is seen that resonance occurs at 58.2 cycles in each case and that, with constant resistance, an increase in inductance and a corresponding decrease in capacitance increases the sharpness of the tuning of the circuit. This may be explained by considering that, in a circuit having small inductance and large capacitance, the resistance is the predominant factor in the impedance over a considerable range of frequencies on either side of resonance.

40. Energy in a Series Circuit Containing Resistance Only.—From Eq. (30) it is seen that the power, or rate of transfer of energy in any circuit, is given by

$$p = ei \text{ watts,} \tag{30}$$

where p is the instantaneous power;
e is the instantaneous emf consumed; and
i is the instantaneous current flowing in the circuit.

For a circuit containing resistance only,

$$e = e_r = ri \text{ volts.} \tag{52}$$

Substituting the value of e_r from Eq. (52) into Eq. (30) and multiplying both sides of the equation by dt,

$$dw = pdt = e_r idt = ri^2dt \text{ joules,} \tag{135}$$

where dw is the instantaneous change of energy flowing in the circuit. Integrating Eq. (135) between any limits of time t_1 and t_2

$$W = \int_{t_1}^{t_2} pdt = \int_{t_1}^{t_2} e_r idt = \int_{t_1}^{t_2} ri^2dt \text{ joules.} \tag{136}$$

When $i = I$, a steady value of current, Eq. (136) becomes

$$W = r\int_{t_1}^{t_2} I^2dt = rI^2(t_2 - t_1) \text{ joules.} \tag{137}$$

When t is in seconds, W is in joules, and when t is in hours, W is in *watt-hours.* The more common unit for the sale of energy is the *kilowatt-hour*, which is 1,000 watt-hr.

For sine waves of emf and current,

$$i = I_m \sin \omega t \text{ amp,} \tag{44}$$

$$e_r = rI_m \sin \omega t \text{ volts,} \tag{52}$$

and

$$W = \int_{t_1}^{t_2} rI_m^2 \sin^2 \omega t dt \text{ joules} \tag{138}$$

$$= rI_m^2 \int_{t_1}^{t_2} \sin^2 \omega t dt \text{ joules.} \tag{139}$$

Taking one complete cycle from $t_1 = 0$ to $t_2 = T$ sec., where T is the period of the waves,

$$W = rI_m^2 \int_0^T \sin^2 \omega t dt \text{ joules} \tag{140}$$

$$= \frac{rI_m^2}{2} \int_0^T (1 - \cos 2\omega t) dt \text{ joules}$$

$$= \frac{rI_m^2}{2} \left| t - \frac{\sin 2\omega t}{2\omega} \right|_0^T \text{ joules}$$

$$= T\frac{rI_m^2}{2} \text{ joules} \tag{141}$$

$$= TrI^2 \text{ joules.} \tag{142}$$

This value of energy is seen to be the average power multiplied by the interval of time chosen, which is true for any interval. This energy is dissipated in heat since, no matter what interval of time is chosen, there is no negative value of W, *i.e.*, no electrical energy returned to the circuit.

41. Energy in a Series Circuit Containing Inductance Only.—In an inductive circuit, from Eq. (72),

$$e_x = L\frac{di}{dt} \text{ volts,} \tag{72}$$

$$dw = pdt = e_x i dt,$$

$$= Lidi \text{ joules,} \tag{143}$$

at each instant. The total energy supplied to the circuit during any interval of time $(t_2 - t_1)$ is then

$$W = \int_{t_1}^{t_2} pdt = \int_{I_1}^{I_2} Lidi \text{ joules,} \tag{144}$$

where

$$i = I_1 \text{ when } t = t_1,$$

and

$$i = I_2 \text{ when } t = t_2.$$

lations of the quantities in Fig. 34. Curve B shows the variation of current with the resistance increased to 40 ohms and curve C for a resistance of 80 ohms. Since the product LC is constant, resonance occurs at the same frequency in all three cases. The curves are, however, quite different in form. Near the point of resonance, a small change in frequency produces a much greater variation in current in circuit A than in circuits B or C. Circuit A is said to be more sharply tuned than circuits B or C, and it is seen that a decrease in resistance increases the sharpness of tuning. This is important in radio circuits where sharpness in tuning is usually desired.

39. Variation of the Voltage in a Series Circuit with Variation of Frequency—Current Constant.—For constant values of current, the resistance drop from Eq. (54),

$$E_r = rI \text{ volts,} \tag{54}$$

may be plotted as ordinates against frequency, as shown in Fig. 37. From Eq. (54) and the curve, this is seen to be a straight line parallel to the frequency axis. In a similar manner, the drop across the inductance of the circuit becomes, from Eq. (69),

$$E_L = 2\pi fLI \text{ volts,} \tag{69}$$

which is a straight line through the origin, as shown in Fig. 37. On the other hand, the drop across the capacitance is given by Eq. (117) as

$$E_c = \frac{-1}{2\pi fC}I \text{ volts,} \tag{117}$$

which, is an equilateral hyperbola in the fourth quadrant with emf and frequency as asymptotes, as shown in Fig. 37.

From these values of E_r, E_L, and E_c the impressed voltage may be determined by the use of Eq. (126). This is shown in Fig. 37 to the same scale as the other emf curves. At resonance E_o has its minimum value, or E_r. Since the current is constant for all the curves in Fig. 37, they also represent the variation of the circuit constants r, x_L, x_c, and z, with varying frequency of the sinusoidal current. A scale of these constants in ohms is shown on the vertical axis in Fig. 37. These scales are found by dividing the scale of volts by the value of the current.

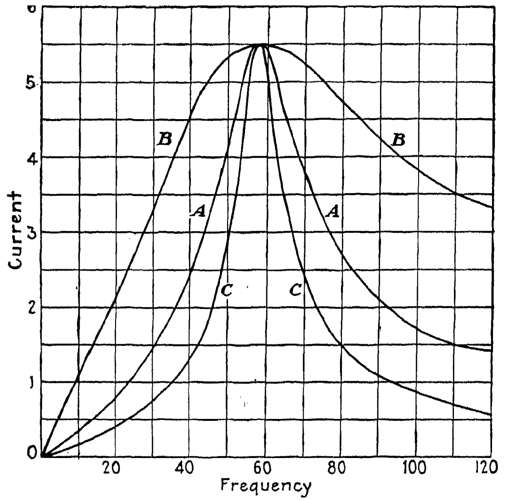

FIG. 36.—Curves showing the variation of current with frequency for different values of inductance and capacitance in a series circuit passing through resonance, E_0 constant.

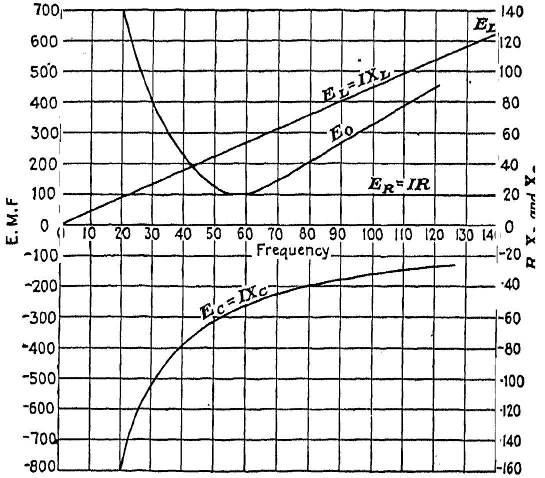

FIG. 37.—Curves showing the variation of the emf drop across the resistance, the inductance, and the capacitance and the impressed emf for a constant-current series circuit containing R, L, and C, frequency being varied.

Hence, the total energy change becomes

$$W = \tfrac{1}{2}L[I_2^2 - I_1^2] \text{ joules,} \tag{145}$$

during the interval of time $(t_2 - t_1)$ sec. If $I_1 = 0$ when $t = t_1$, or at the start, then

$$W = \tfrac{1}{2}LI_2^2 \text{ joules,} \tag{146}$$

where I_2 is the final value of the current. Hence, the value,

$$W = \tfrac{1}{2}LI_1^2 \text{ joules,}$$

in Eq. (145) must represent the energy contained in the circuit at the instant t_1 due to the current I_1. In practical units the energy change in the circuit in joules is seen to be one-half the inductance in henrys multiplied by the difference of the squares of the current in amperes. If I_2 is greater than I_1 the energy is positive or is added to the circuit; if I_2 is less than I_1 the energy is received from the circuit. *The above total change in energy is independent of the path of change, or of the intermediate values of I during the interval of time chosen, but depends only upon the terminal values of the current.*

For sine waves of current and voltage some interesting conclusions may be drawn. Let

$$i = I_m \sin \omega t \text{ amp,} \tag{44}$$

and

$$e_x = 2\pi f L I_m \cos \omega t \text{ volts.} \tag{67}$$

Substituting these values of e_x and i into Eq. (143),

$$\begin{aligned} dw &= 2\pi f L I_m^2 \sin \omega t \cos \omega t dt \text{ joules} \\ &= \pi f L I_m^2 \sin 2\omega t dt \text{ joules.} \end{aligned} \tag{147}$$

The energy change during any interval of time from t_1 to t_2 is

$$W = \int_{t_1}^{t_2} p dt = \int_{t_1}^{t_2} \pi f L I_m^2 \sin 2\omega t dt \text{ joules.} \tag{148}$$

For one complete cycle, from $t_1 = 0$ to $t_2 = T = 1/f$,

$$W = \pi f L I_m^2 \int_0^T \sin 2\omega t dt \text{ joules,}$$

or

$$W = \pi f L I_m^2 \left| -\frac{\cos 2\omega t}{2\omega} \right|_0^T = 0 \text{ joules,} \tag{149}$$

or there is no total energy change. The same result will be obtained for one half cycle, or from $t_1 = 0$ to $t_2 = T/2$. For one quarter cycle, from $t_1 = 0$ to $t_2 = T/4$,

$$W = \frac{\pi f L I_m^2}{2\omega} \Big| -\cos 2\omega t \Big|_0^{\frac{T}{4}} = \frac{L I_m^2}{2} \text{ joules,} \tag{150}$$

or this amount of energy is stored in the magnetic field of the circuit.

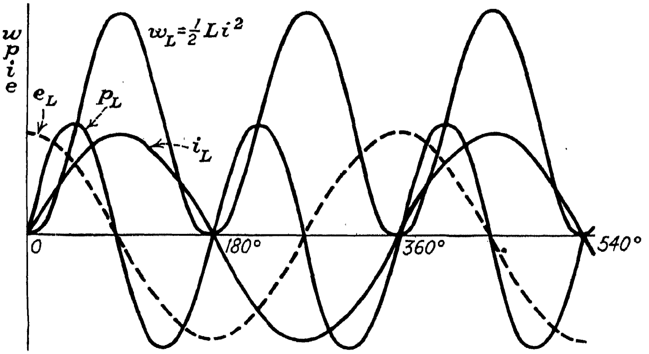

FIG. 38.—Curves of current, inductive drop, power, and stored energy for a circuit containing only inductance.

During the next quarter cycle, from $t_1 = T/4$ to $t_2 = T/2$,

$$W = \frac{\pi f L I_m^2}{2\omega} \Big| -\cos 2\omega t \Big|_{\frac{T}{4}}^{\frac{T}{2}} = -\frac{L I_m^2}{2} \text{ joules,} \tag{151}$$

or the same amount of energy is returned as was stored during the preceding quarter cycle. During the next quarter cycle, from $t_1 = T/2$ to $t_2 = 3T/4$,

$$W = \frac{L I_m^2}{2} \text{ joules}$$

is again stored. This amount is again returned during the fourth quarter cycle, or from $t_1 = 3T/4$ to $t_2 = T$. Hence, although the net power consumed by an inductance is zero, there is considerable surge of energy into and out of the circuit containing inductance. The energy is stored in and released from the magnetic field surrounding and linking with the circuit. The relations between current, inductive reactance drop, and power and energy stored in the magnetic field for the inductive circuit are shown in Fig. 38.

42. Energy in a Series Circuit Containing Capacitance Only.— In a circuit containing a capacitor, since

$$i = C\frac{de_C}{dt} \text{ amp} \tag{112}$$

and

$$dw = pdt = e_C i dt = Ce_C de_C \text{ joules} \tag{152}$$

at each instant, the total energy supplied to the circuit during any interval of time $(t_2 - t_1)$ is

$$W = \int_{t_1}^{t_2} pdt = \int_{E_1}^{E_2} Ce_C de_C \text{ joules,} \tag{153}$$

where

$$e_C = E_1 \text{ when } t = t_1,$$

and

$$e_C = E_2 \text{ when } t = t_2.$$

Hence,

$$W = \frac{C}{2}[E_2^2 - E_1^2] \text{ joules,} \tag{154}$$

where

$$W = \tfrac{1}{2}CE_2^2 \text{ joules}$$

is the energy stored in the circuit at the end of the interval, or at $t = t_2$ (see Eq. 146). If $E_1 = 0$, then

$$W = \tfrac{1}{2}CE_2^2 \text{ joules} \tag{155}$$

is the energy added to the circuit containing an uncharged capacitor. If E_2 is greater than E_1, energy is added to the circuit; but if E_2 is less than E_1, energy is taken from the circuit.

For sine waves of current and voltage some interesting conclusions may be drawn. Let

$$i = I_m \sin \omega t \text{ amp,} \tag{44}$$

and

$$e_C = -\frac{1}{2\pi fC} I_m \cos \omega t \text{ volts,} \tag{121}$$

when $r = 0$. Hence,

$$\begin{aligned} dw &= e_c i\,dt \text{ joules} \\ &= -\frac{1}{\omega C} I_m^2 \sin \omega t \cos \omega t\,dt \\ &= -\frac{1}{2\omega C} I_m^2 \sin 2\omega t\,dt. \end{aligned} \tag{156}$$

For one cycle, or from $t_1 = 0$ to $t_2 = T$,

$$W = \int_0^T -\frac{I_m^2}{2\omega C} \sin 2\omega t\,dt \text{ joules,}$$

or

$$W = -\frac{I_m^2}{2\omega C}\left| -\frac{\cos 2\omega t}{2\omega}\right|_0^T = 0 \text{ joules.} \tag{157}$$

For the first quarter cycle, from $t_1 = 0$ to $t_2 = T/4$,

$$\begin{aligned} W &= -\frac{I_m^2}{2\omega C}\left| -\frac{\cos 2\omega t}{2\omega}\right|_0^{\frac{T}{4}} \text{ joules} \\ &= -\frac{I_m^2}{2\omega^2 C} = -\frac{CE_m^2}{2} \text{ joules,} \end{aligned} \tag{158}$$

or energy is given up by the circuit. During the next quarter cycle the same amount of energy is again stored in the circuit.

The relations between current, capacitive reactance drop, and power and energy stored in the dielectric field of the capacitor for a capacitive circuit are shown in Fig. 39.

43. Energy in a Series Circuit Containing an Inductance and a Capacitance.—In an a-c series circuit containing an inductance and a capacitance in series, and with the current wave used as the reference, the energy stored in the inductance is zero when the current is zero ($\alpha = 0$), but it is a maximum in the capacitor at this same instant, since e_c is a maximum. During the next quarter cycle the energy stored in the inductance increases while that in the capacitor decreases, so that when $\alpha = 90$ deg., the energy stored in the capacitor is zero and that in the coil is a maximum, since $i = I_m$. For a resonant series circuit, the energy needed by the coil is supplied by the capacitor, and this is again restored to the coil periodically each half cycle; the only energy supplied from the external circuit being that for the I^2R losses in the circuit resistance. For any other series circuit the excess energy required by the coil or by the capacitor must be supplied temporarily from the external circuit during one quarter cycle, and this is again returned during the succeeding quarter cycle. The relations for a given current between the energy

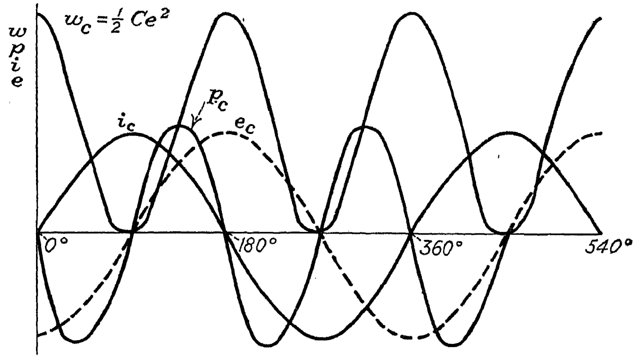

FIG. 39.—Curves of current, capacitive drop, power, and stored energy for a circuit containing only capacitance.

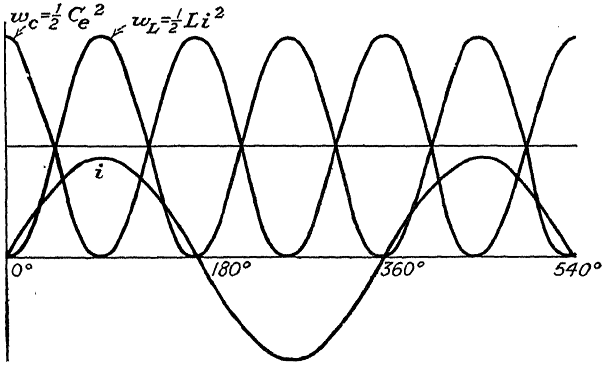

FIG. 40.—Curves of current, energy stored in inductance, and capacitance for a resonant series circuit.

stored in the magnetic field and the dielectric field for a resonant circuit are shown in Fig. 40.

In the integrations illustrated in Arts. 40 to 42, energy is represented by the area between the instantaneous-power wave and the time or angle axis. Since at any instant in a series circuit

$$W_L = \frac{LI_2^2}{2} \text{ joules,} \qquad (146)$$

and

$$W_c = \frac{CE_2^2}{2} \text{ joules,} \tag{155}$$

where I_2 is the value of i when $\alpha = \alpha_2$, any instant; and
E_2 is the value of e when $\alpha = \alpha_2$, the same instant.
These values of energy cannot be equal for all values of α or t even in a resonant circuit. However, in any resonant circuit, the coil absorbs the energy in the exact amount that the capacitor gives it up, and vice versa. The process of the interchange of energy is for the energy of the electrostatic field to transfer into the energy of the electromagnetic field by the flow of current, and for the electrostatic field again to be built up by the emf that is induced in the circuit when the current (or the electromagnetic field) changes. This is represented mathematically by saying

$$i = C\frac{de_c}{dt} \text{ amp.} \tag{112}$$

and

$$e = L\frac{di}{dt} \text{ volts.} \tag{72}$$

Indeed, these two equations are fundamentally based on the energy-change equations.

Problems

1-2. A resistor used for a heating element in a flatiron has a resistance of 18.3 ohms. A current $i = 8.5 \sin 377t$ amp flows through the resistor. What is the equation of the emf impressed across the resistor? What is the effective value of the current and emf? What is the frequency of the system to which the resistor is connected? Draw the wave forms of current and resistance drop.

2-2. A resistor of 1,840 ohms has a 760-cycle current of 0.38 amp, effective value, flowing through it. What is the equation of the current and the impressed emf? Draw the wave forms of current and emf.

3-2. In a coil wound in the form of a toroid, the diameter of each turn is 5 in., and the diameter of the circle through the centers of the turns is 25 in. The coil is wound with 1,000 turns of copper wire having a resistance of 4 ohms per 1,000 feet. (*a*) What is the resistance of the coil? (*b*) What is the coefficient of self-induction of the coil?

4-2. A circuit has an inductance of 0.21 millihenry. Calculate the reactance when the frequency of the circuit to which it is connected is 60 cycles, 3,500 cycles.

5-2. A coil with a very low resistance has a coefficient of self-induction of 6.8×10^{-4} henry. A sine wave of current having a maximum value of 0.356 amp and a frequency of 1,680 cycles flows through it. What is the equation of the emf impressed across the coil? Draw the wave forms of current and emf.

6-2. A certain circuit has a resistance of 20 ohms and a coefficient of self-induction of 2.1×10^{-3} henry. Find the impedance of the circuit at frequencies of 0, 15, 60, 760, and 5,500 cycles.

7-2. A coil has a resistance of 2 ohms and a coefficient of self-induction of 0.015 henry. A current $i = 28.28 \sin 157t$ amp is passed through the coil. Write the equations for e_r, e_x, and e_o, and draw the wave forms for i, e_r, e_x, and e_o.

8-2. A coil has a resistance of 3.2 ohms and an inductance of 7.3×10^{-4} henry. (*a*) A current of 1.68 amp at 25 cycles is passed through the coil. Write the equations for i e_r, e_x, and e_o, and draw these wave forms. (*b*) A current of 1.68 amp at 2,260 cycles is passed through the coil. Write the same equations, and draw the curves asked for in (*a*) and compare these results.

9-2. What is the angle of lag in a circuit having a resistance of 40 ohms and a reactance of 30 ohms?

10-2. An inductor has a resistance of 5 ohms and an inductance of 0.04 henry; find the equation of voltage necessary to send a current of $i = 28.28 \sin 377t$ through the inductor.

11-2. A coil has a resistance of 100 ohms and a coefficient of self-induction of 0.2 henry; what impressed emf is required to send 200 amp at 60 cycles through this coil? What is the resistance drop? What power is consumed? What is the value of the induced emf? What is the power factor?

12-2. The same coil as used in Prob. 11-2 is connected across a 11,000-volt 25-cycle circuit. What current flows? What power is consumed? What is the power factor?

13-2. A coil having a resistance of 60 ohms and a coefficient of self-induction of 0.2 henry is connected across a 10,000-volt 60-cycle generator. What is the value of the current taken from the circuit? Sketch in the waves of i, e_r, e_x, and e_o.

14-2. A coil having a resistance of 150 ohms is connected to a 22,000-volt 60-cycle a-c circuit and takes a current of 100 amp. What is the inductance of this coil?

15-2. A circuit that has been operating at 60 cycles with a lagging current having a power factor of 0.866 and drawing 200 amp from a 1,200-volt circuit is to be changed to 25 cycles with the same voltage. What must be the new inductance to maintain the same current and power factor?

16-2. A certain coil has a resistance of 50 ohms and a reactance of 50 ohms at 25 cycles. What effective emf must be impressed across this coil to send the current $i = 28.2 \sin 377t$ through this coil?

17-2. A circuit having a lagging current of 200 amp at 2,400 volts and having a power factor of 0.866 at 50 cycles is to be changed to 60 cycles. What change in resistance must be made so as to maintain the current at 200 amp at this new frequency with the same impressed emf? What is the new power factor?

18-2. A coil takes 15 amp when connected across a 220-volt 50-cycle circuit and 40 amp when connected across a 500-volt 25-cycle circuit. What are the circuit constants, the power consumed at each frequency, and the power factor for each case?

19-2. A coil takes a current of 25 amp when connected across a 125-volt d-c line and a current of 15 amp when connected across a 220-volt 60-cycle a-c line. What current will flow when it is connected across a 125-volt 25-cycle line?

20-2. A circuit contains a resistor in series with a coil. The constants of the circuit may be found from the following: resistance of resistor 25 ohms; resistance of coil 100 ohms; self-inductance of coil 1.2 henrys. A current $i = 141.4 \sin 377t$ flows through the circuit. Determine the following: (*a*) voltage drop across the resistor, (*b*) voltage drop across the coil, (*c*) total impressed voltage, (*d*) power factor of the coil, (*e*) power factor of the total circuit.

21-2. Two coils are connected in series and a sine wave of current having an effective value of 10 amp at 50 cycles is sent through the circuit. The first coil has a resistance of 4 ohms and an inductance of 0.008 henry, and the second coil has a resistance of 6 ohms and an inductance of 0.02 henry. Find the effective value of each of the several voltage drops and of the impressed voltage. What are the equivalent resistance, reactance, and impedance of the total circuit?

22-2. Add by trigonometry the following waves of emf and give the equation of the combined wave: $e_1 = 100 \sin \alpha$, $e_2 = 125 \sin (\alpha + 40°)$. What is the effective value of each component voltage and of the combined voltage?

23-2. A capacitor has a capacitance of 6.8 μf. Find the capacitive reactance for frequencies of 0, 60, 1,350, and 15,000 cycles.

24-2. Find the values of capacitance and impedance in a 60-cycle circuit in which the resistance is 80 ohms and the capacitive reactance is 60 ohms.

25-2. A capacitor having a capacitance of 734 μf is connected in series with a resistance of 3.8 ohms across a circuit having an emf $e = 141.4 \sin 377t$. Find the equation of the current flowing in the circuit.

26-2. A capacitor with a capacitive reactance of 3 ohms is connected in series with a resistance of 4 ohms across a circuit having an emf of 200 volts and a frequency of 60 cycles. What current flows? Draw the waves of e_r, i, e_c, and e_o.

27-2. An impedance containing only R and C takes 2.5 amp when a 220-volt 60-cycle emf is impressed and a current of 1.88 amp when a 220-volt 30-cycle emf is impressed. What are the constants of the circuit?

28-2. A coil, with a resistance of 20 ohms and an inductive reactance of 126 ohms, and a capacitor, with a capacitive reactance of 60 ohms at 120 cycles, are connected in series. A current of 12 amp at 120 cycles is passed through the circuit. Write the equations for the current, the emf drops across the resistance, inductive reactance, and capacitive reactance, and the emf impressed on the circuit. Draw the wave forms for these quantities in relation to the current.

29-2. A series circuit containing R, L, and C has a resistance of 2 ohms, an inductance of 1.2×10^{-4} henry, and a capacitance of 26 μf. A voltage of 578 volts at a frequency of 2,160 cycles is applied to the circuit; find the current, the drop across the resistance, the drop across the inductance, and the drop across the capacitance.

30-2. A coil having a resistance of 10 ohms and an inductance of 0.5 henry is connected in series with a capacitor having a capacitance of 10μf across a 1,100-volt a-c circuit. (*a*) If the frequency of the circuit is 60 cycles, find the current, the drop across the inductive reactance, the drop across the capacitor, the power consumed in the circuit, and the power factor of the whole circuit. (*b*) The frequency of the circuit is changed to 25 cycles; find the same values as required in (*a*).

31-2. A coil having a resistance of 10 ohms and an inductance of 0.2 henry is placed in series with a capacitor having a capacitance of 50 μf across an a-c circuit. At what frequency will resonance occur? What current will flow at this frequency if 160 volts be impressed? What is the drop across the coil? Across the capacitor?

32-2. A coil having a resistance of 10 ohms and an inductance of 0.3 henry is connected in series with a capacitor having a capacitance of 30μf. (*a*) What frequency will produce series-phase resonance in this circuit? (*b*) What current will flow at this frequency with 2,000 volts impressed? (*c*) What will be the drop across the capacitor at this time? (*d*) Draw the wave forms of i, e_r, e_L, e_c, and e_o for the resonant frequency.

33-2. A capacitor having $C = 0.05$ μf is connected in series with a coil having $R = 20$ ohms and $L = 0.05$ henry. At what frequency will series-phase resonance occur? If 200 volts be impressed, what current will flow at resonance?

34-2. A circuit having R, L, and C in series is in resonance at 50 cycles. If $R = 20$ ohms and $L = 0.04$ henry, what is the value of C?

35-2. A coil having a resistance of 2 ohms and an inductance of 0.1 henry is placed in series with a capacitor having a capacitance of 70.4 μf across a variable-frequency 40-volt a-c source. Calculate the current for various frequencies, and plot a curve showing the variation of current with frequency for the circuit, the curve passing through resonant frequency.

36-2. Plot the values of the voltage drops across the resistance, inductance, and capacitance for the circuit of Prob. 35-2 for the variable frequency, the impressed emf being constant at 40 volts.

37-2. Plot (*a*) the values of voltage drop across the resistance, inductance, and capacitance and (*b*) the impressed emf for the circuit of Prob. 35-2, the current remaining constant at 20 amp.

38-2. A circuit of very small resistance has an inductance of 1.73 henrys. A current $i = 70.7 \sin 377t$ amp passes through the inductance. Plot the waves of current, voltage drop, power, and energy stored in the inductance for 1 cycle of the current wave.

39-2. A capacitor has a capacitance of 364 μf. An emf $e = 141.4 \sin 377t$ volts is impressed on the capacitor. Draw the waves of emf, current, power, and energy stored in the capacitor for 1 cycle of the emf wave.

40-2. A series circuit contains a resistance of 10 ohms, an inductance of 0.0531 henry, and a capacitance of 88.4 μf. The resonant frequency is 73.3 cycles. An emf of 125 volts is impressed on the circuit; plot the waves of the energy stored in the magnetic field and in the dielectric field on the same graph for a period of 1 cycle.

41-2. Calculate the energy stored in the magnetic and the dielectric fields for Prob. 32-2 at the resonant frequency for a point of time 30 deg. after the applied voltage has passed through zero going from negative to positive values.

42-2. A certain coil has a resistance of 8 ohms and a reactance of 2.5 ohms on a 25-cycle circuit. What effective emf must be impressed to send the current $i = 28.2 \sin 377t$ through this coil?

43-2. If 500 volts at 25 cycles is impressed on a circuit having a resistance of 80 ohms and an inductance of 0.4 henry, what are the values of the current and the power factor? What power is consumed?

44-2. A coil has a reactance of 10 ohms at 60 cycles and a resistance equal to the reactance. A 60-cycle current having the wave form shown in Fig. 44-2 is sent through the coil. Draw the approximate wave form of the impressed emf.

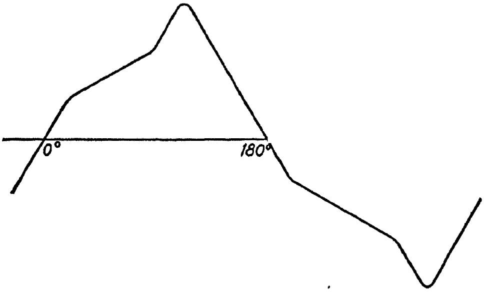

FIG. 44-2.—Figure for Prob. 44-2.

45-2. Given a wave of current flowing through a coil expressed by $i = 20 + 20 \sin 100t$. Plot the waves of current e_r, e_i, and e_o, assuming that $R = 10$ ohms and $L = 0.01$ henry.

46-2. A current having the wave $i = 20 + 200t$ is passing through a coil having a resistance of 10 ohms and an inductance of 0.1 henry. Plot the waves of e_r, i, e_L, and e_o for the first 0.1 sec.

47-2. A resistor of 25 ohms is connected in series with a coil having a resistance of 10 ohms and an inductance of 0.025 henry. A current $i = 14.1 \sin 377t$ flows through this circuit. Determine: (*a*) the voltage drop across the resistor, (*b*) the voltage drop across the coil, (*c*) the total impressed voltage, (*d*) the power factor of the coil, and (*e*) the power factor of the total circuit.

48-2. Two coils are connected in series, and a sine wave of current $i = 16.5 \sin 476t$ is sent through the circuit. The first coil has a resistance of 6 ohms and an inductance of 0.012 henry, and the second coil has a resistance of

6 ohms and an inductance of 0.009 henry. Find the equations and the effective values of the several voltage drops and of the impressed voltage.

49-2. Add by trigonometry the following waves of emf, and give the equation of the combined wave: $e_1 = 50 \sin (\alpha + 30°)$, $e_2 = 100 \sin (\alpha - 30°)$, $e_3 = 200 \cos \alpha$. What is the effective value of each voltage and of the total voltage?

50-2. A coil takes 8 amp when connected across a 110-volt d-c line and 4.2 amp when connected across a 110-volt 60-cycle line; what current will it take when connected across a 110-volt 25-cycle line?

51-2. A coil connected to a 110-volt 60-cycle lighting circuit consumes 2,000 watts when 20 amp flows. What are the values of the resistance and of the inductance of this coil?

52-2. What power is consumed by a circuit having a resistance of 100 ohms and a reactance of 200 ohms when an emf of 10,000 volts is impressed?

53-2. A coil having a resistance of 10 ohms and an inductance of 0.15 henry is connected in series with a capacitor having a capacitance of 60 μf across a 110-volt a-c circuit. (*a*) The frequency of the circuit is 60 cycles; find the current, the drop across the coil, the drop across the capacitor, the power consumed by the circuit, the power factor of the coil, and the power factor of the whole circuit. (*b*) The frequency of the circuit is changed to 25 cycles; find the same values as in (*a*).

54-2. A coil having $R = 100$ ohms and $L = 0.28$ henry is connected in series with a capacitor having $C = 100$ μf. What will be the drop across the capacitor at resonant frequency with 1,000 volts impressed? If the resistance of the coil be reduced to 10 ohms, what will be the drop across the capacitor at this same frequency?

55-2. A coil having a resistance of 10 ohms and an inductance of 0.03 henry is connected in series with a capacitor having a capacitance of 300 μf. (*a*) What frequency will produce resonance in this circuit? (*b*) What current will flow at this frequency with 200 volts impressed? (*c*) What will be the drop across the condenser at this time?

56-2. A resistance of 3.6 ohms and a capacitance of 64 μf are connected in series with an inductance, and an emf of 25.6 volts is applied to the circuit. What value of inductance is required to make the circuit resonant at 1,860 cycles? What are the current and the drop across the capacitor at resonant frequency?

57-2. Calculate the energy in the magnetic and dielectric fields for Prob. 55-2 at the resonant frequency for the point of time 45 deg. after the applied emf has passed through zero going from negative to positive values.

CHAPTER III

VECTOR REPRESENTATION OF WAVES AND EQUATIONS

So far in this text, the only methods used for representation of relations between emfs and currents or between different emfs have been by means of equations and waves. Already, in the simple series circuits considered, the expressions and diagrams have become somewhat complicated for clearness in representation. It is necessary, before proceeding to a further discussion of circuits and machines, to develop a simpler form of analysis. Such a method grows out of the vector diagram and its mathematical solution.

Vector Quantity.—A vector quantity is a physical quantity that has both magnitude and direction. A vector quantity is completely described by stating its direction and magnitude. The most common methods of describing a vector quantity are by means of the projections on a system of rectangular coordinates or by stating the magnitude and direction by means of polar coordinates.

Magnitude of a Physical Quantity.—The magnitude of a physical quantity is the absolute value of the number (either real or complex) used to represent the relationship between the physical quantity and the unit used in measuring it.

44. Generation of a Sine Wave by a Vector.—In order to understand the determination of a vector from a given wave or a set of vectors from a set of related waves, consider the generation of a sine wave by means of a rotating vector. Suppose that a straight line (Fig. 41) revolves at a uniform velocity in a counterclockwise direction about a center O. This line will pass over equal angles α in equal intervals of time t. Consider the position A, at some time when this vector crosses the axis OA, as the starting point for counting time and angles, and let a similar point on the axis of rectangular coordinates to the right mark the same instant of time. Plot the projection of the end of the vector OA on the vertical axis as ordinates against angular travel as abscissas, obtaining a sine wave, since the length of the projection of OA on the vertical axis varies as the sine of the angle passed over. This sine wave is plotted as follows: When the vector has revolved through the angle α_1 to position 1, time will have progressed an amount $\omega t_1 = \alpha_1$ along the axis of the wave, its vertical projection at that time being equal to a_1. In a similar manner a_2, a_3, a_4, etc., are determined, and a wave is drawn through these points. This wave will be a sine wave with origin at $\alpha = 0$ and symmetrical about the axis OA produced. It should be noted in this construction that

a. The ordinate of the wave becomes zero each time the vector crosses the horizontal axis, *i.e.*, at 0, 180, 360 deg., etc.

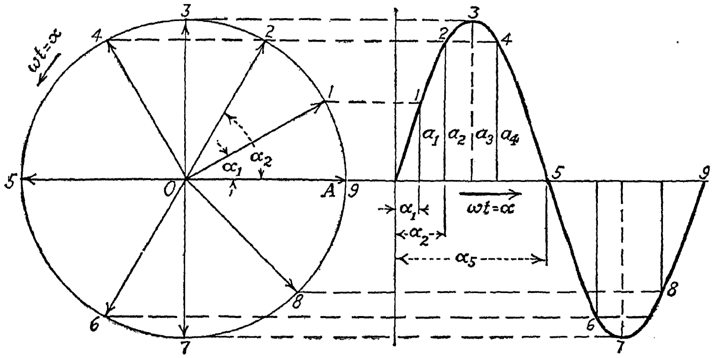

FIG. 41.—The generation of a sine wave by a rotating vector.

b. The maximum value of the sine wave comes at 90 deg. and is equal to the radius vector; *i.e.*, the height of the wave is determined by the chosen radius of the circle.

c. Conversely, positions of the vector corresponding to points on the wave may be determined by projecting these points by lines parallel to the horizontal axis until the projections cut the circle in the corresponding quadrant, the points of cutting the circle giving the positions of the end of the vector.

d. For a given sine wave there must be a corresponding circle, and for any position on the sine wave there must be a definite position of the vector in respect to the axis.

e. Hence, to construct the circle and to locate a given vector, taking the sine wave of Fig. 41 as the given wave, first select a convenient point O on the axis extended for the center of the circle; next, draw a vertical axis through this point; the projection of the maximum value of the sine wave on this vertical axis, as at $O3$, will give the radius of the required circle with center at O. Other points or positions of the vector for any given angle may then be located by first choosing an angle along the horizontal scale of the wave, and projecting the end of the corresponding ordinate of the sine wave on to the circle, as points 2, 4, 6, etc.

45. System of Vectors for Waves of the Same Frequency.—In the analysis of the series circuit, in Chap. II, it has been found that there is a continual association of waves at the same frequency but at different phase positions and with different maxima. Suppose that a study is to be made of such a set of waves to discover the relation between the vectors representing these waves at different periods of time or angles of phase. Let Fig. 42 represent any set of waves such that they have the same frequency and that one wave is the locus of the sum of the ordinates of the other two waves at corresponding angles. These waves may be designated by three equations, using each wave as a reference, but associating the waves so that

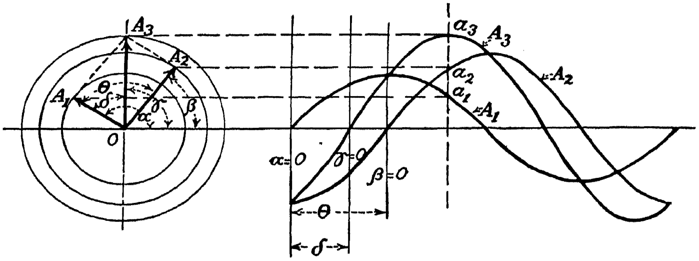

FIG. 42.—Vectors corresponding to a system of sine waves at a given instant of time.

$$a_3 = a_1 + a_2, \tag{159}$$

at each instant of time. The waves may be designated by

$$a_1 = A_1 \sin \alpha, \tag{160}$$
$$a_2 = A_2 \sin \beta. \tag{161}$$

and

$$a_3 = A_3 \sin \gamma. \tag{162}$$

From the condition of equal frequencies, or equal periods, for the three waves,

$$\beta = \alpha - \theta, \tag{163}$$

and

$$\gamma = \alpha - \delta, \tag{164}$$

where θ and δ are constant angles, or

$$a_1 = A_1 \sin \alpha, \tag{160}$$
$$a_2 = A_2 \sin (\alpha - \theta), \tag{165}$$
$$a_3 = A_3 \sin (\alpha - \delta). \tag{166}$$

Combining Eq. (159) with Eqs. (160), (165), and (166),

$$A_3 \sin (\alpha - \delta) = A_1 \sin \alpha + A_2 \sin (\alpha - \theta). \tag{167}$$

Expanding these terms,

$$\begin{aligned} A_3 \sin \alpha \cos \delta - A_3 \cos \alpha \sin \delta &= A_1 \sin \alpha + A_2 \sin \alpha \cos \theta \\ &\qquad - A_2 \cos \alpha \sin \theta \\ &= (A_1 + A_2 \cos \theta) \sin \alpha - (A_2 \sin \theta) \cos \alpha. \end{aligned} \tag{168}$$

Hence, from Eq. (168), placing coefficients of sines equal to each other and coefficients of cosines equal to each other,

$$\cos \delta = A_1 + A_2 \cos \theta,$$

or

$$\cos \delta \qquad \overset{+ A_2 \cos \theta}{} \tag{169}$$

and

$$\sin \delta = A_2 \sin \theta,$$

or

$$\sin \delta = \qquad \overset{\sin \theta}{} \tag{170}$$

Combining Eqs. (169) and (170),

$$\cos^2 \delta + \sin^2 \delta = 1.00 = \frac{(A_1 + A_2 \cos \theta)^2}{A_3^2} + \frac{A_2^2 \sin^2 \theta}{A_3^2},$$

or

$$\begin{aligned} A_3^2 &= (A_1 + A_2 \cos \theta)^2 + A_2^2 \sin^2 \theta \\ &= A_1^2 + 2A_1A_2 \cos \theta + A_2^2 \cos^2 \theta + A_2^2 \sin^2 \theta \\ &= A_1^2 + A_2^2 + 2A_1A_2 \cos \theta. \end{aligned} \tag{171}$$

Equation (171) is the well-known relation of the diagonal of a parallelogram to the two sides and the angle between them. In other words, if, as shown in Fig. 42, concentric circles are drawn with radii equal to the maximum ordinates of the three waves, the three vectors, located at the same instant of time and corresponding to the three waves, will form two sides and the diagonal of a parallelogram for any instant of time.

From Fig. 42 and Eq. (165), it is seen that θ is the constant angle between the waves for A_1 and A_2; it is also the angle between the sides of the parallelogram at the junction of the vectors A_1 and A_2 at O. This angle θ is the angle by which the second wave and its vector A_2 differ in phase from the reference wave and its vector A_1, this being the fundamental relation for the use of vectors for the representation of sine waves. In most a-c phenomena the relation between waves to be combined is of 90-deg. phase displacement, giving special forms for the analysis, as will be explained later.

46. Vectors for Resistance and Inductance in Series.—In Fig. 43, a set of waves similar to Fig. 25 and their circles are drawn for a resistance and an inductance in series, a given sine wave of current flowing through them, showing the emf wave e_r impressed across the resistor, the emf wave e_x impressed across the inductor, and the resultant impressed emf wave e_o. At a given instant of time, as t_1, and angle corresponding α_1, the vectors are located in reference to each other by a definite angle, as shown by the vector diagram at the left of Fig. 43, which shows the position of the vectors at the time t_1 corresponding to the angular position α_1. In Fig. 43 it is seen that:

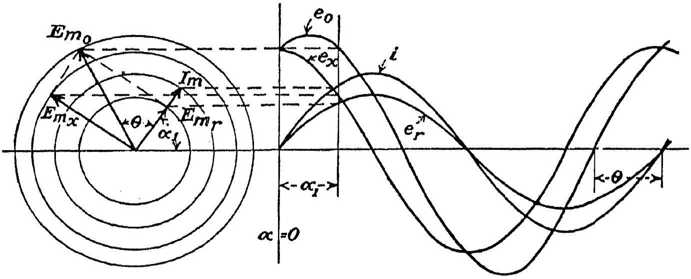

FIG. 43.—Vectors corresponding to the current and emf waves in a simple series circuit at a given instant.

a. The resistance drop vector is in phase with the current vector.

b. The reactance drop vector, due to the inductance, is 90 deg. ahead of the current vector.

c. The impedance drop, or impressed emf vector, is the resultant of the resistance drop and the reactance-drop vectors and is θ deg. ahead of the current vector.

These relations of the vectors are the same as were found for the component waves of current, resistance drop, reactance drop, and impedance drop, or impressed emf, in Chap. II. Hence, the vector diagram becomes a new means of expressing relations in a-c circuits with sine waves of the same frequency. In order to trace out the waves, all of the vectors have been considered as revolving at a uniform angular velocity $\omega = 2\pi f$ rad. per second or as making a complete revolution for each cycle of emf or current. As all of the quantities in a circuit have the same frequency, the various vectors will retain their relative length and phase positions throughout a revolution. *Since only the magnitudes and phase differences, or relative positions, of the vectors are to be considered, these relations may be shown by the system of rotating vectors in any position.*

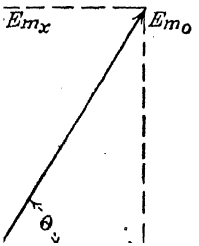

FIG. 44.—Vector diagram for a coil using maximum values.

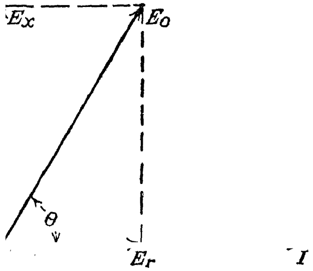

FIG. 45.—Vector diagram for a coil using effective values.

In the wave form of Fig. 43, the current wave is the reference since it passes through zero at the time the angle α is equal to zero, and its equation is $i = I_m \sin \alpha$. The vector diagram corresponding to the time when α is equal to zero, *i.e.*, for the instant when the current vector is the reference, is shown in Fig. 44. The vector relations in Fig. 44 are the same as in Fig. 43, except that they are shown for the time $\alpha = 0$, instead of when $\alpha = \alpha_1$, as in Fig. 43, but to a different scale. The vectors are identical in magnitude and position relative to each other in the two cases, but they have different positions in space owing to different reference axes. Since there is always a constant ratio between the maximum and the effective value of any sine wave, the effective values of emf and current may be used for the vector diagram, as shown in Fig. 45.

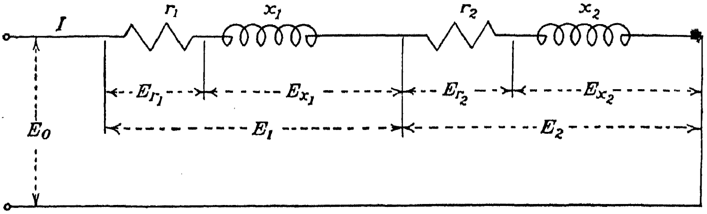

FIG. 46.—Two coils in series.

47. Two Coils in Series.—The relations derived in Art. 30 for a circuit such as shown in Fig. 46 may be represented by vectors as shown in Fig. 47, using current as the reference vector, since the same current flows in all parts of the circuit. The resistance drops are all in phase with the current, and the reactance drops are all 90 deg. ahead of the current. The following relations, derived mathematically for such a circuit as in Art. 30, are now evident from the geometry of the vector diagram in Fig. 47, as

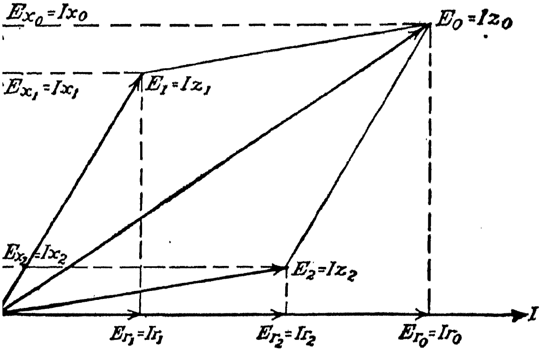

FIG. 47.—Vector diagram for two coils in series.

$$Ir_0 = E_{r_0} = E_{r_1} + E_{r_2} = Ir_1 + Ir_2 \text{ volts,} \tag{172}$$

and

$$r_0 = r_1 + r_2 \text{ ohms,} \tag{100}$$

$$Ix_0 = E_{x_0} = E_{x_1} + E_{x_2} = Ix_1 + Ix_2 \text{ volts,} \tag{173}$$

and

$$x_0 = x_1 + x_2 \text{ ohms,} \tag{101}$$

$$Iz_0 = E_0 = \sqrt{E_{r_0}^2 + E_{x_0}^2} = I\sqrt{r_0^2 + x_0^2} \text{ volts,} \tag{174}$$

where

$$z_0 = \sqrt{r_0^2 + x_0^2} \text{ ohms.} \tag{105}$$

It is seen that the voltage drop across coil 1 is

$$E_1 = \sqrt{E_{r_1}^2 + E_{x_1}^2} = I\sqrt{r_1^2 + x_1^2} = Iz_1 \text{ volts,} \tag{175}$$

and the voltage across coil 2 is

$$^2 + x_2^2 = Iz_2 \text{ volts.} \tag{176}$$

E_0 is the vector sum of E_1 and E_2 and is less than the arithmetic sum of the two voltages. Hence, *although the total resistance of the circuit is the sum of the several resistances and the total reactance is the sum of the several reactances, the total impedance of the circuit is neither the arithmetic nor the algebraic sum of the several impedances.*

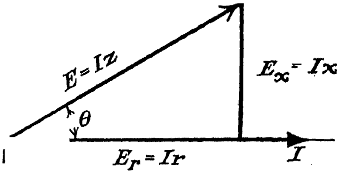

FIG. 48.—Emf diagram for circuit containing resistance and inductance in series.

48. Electromotive Force and Impedance Triangles for Circuits Containing Resistance and Inductance in Series.—From the trigonometry of the figure formed by the emf vectors, Fig. 48, it is seen that the resistance drop, the reactance drop, and the impedance drop form three sides of a right triangle with the resistance drop in phase with the current. From previous equations, using effective values,

$$\begin{aligned} E &= Iz \text{ volts} && (87) \\ &= I\sqrt{r^2 + x^2} \text{ volts} \\ &= \sqrt{I^2r^2 + I^2x^2} \text{ volts} \\ &= \sqrt{E_r^2 + E_x^2} \text{ volts,} && (174) \end{aligned}$$

where E_r and E_x are the components of E projected on the current axis and on an axis at right angles with the current vector. Also, since

$$Iz = I\sqrt{r^2 + x^2} \text{ volts},$$

or

(80)

$$\tan\theta = \frac{E_x}{E_r} = \frac{Ix}{Ir} = \frac{x}{r}. \tag{177}$$

When the values in Eq. (174) are divided by the current, the relations are those of resistance, reactance, and impedance; *i.e.*, an impedance triangle is formed similar to the emf triangle, but to a different scale, as shown in Fig. 49. The quantities r, x, and z may be expressed as vectors, but it is to be noted that they do not represent quantities of the frequency of E and I, since the values of r, x, and z are not periodic.

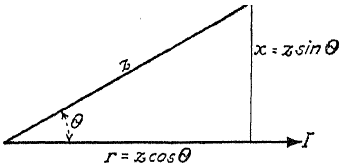

FIG. 49.—The impedance triangle.

49. Electromotive Force and Impedance Triangles for Circuits Containing Resistance and Capacitance in Series.—In Chap. II, Art. 34, it was shown that the voltage wave across a capacitive reactance lags the current wave in the circuit by 90 deg., the capacitive-reactance drop being in opposition, or 180 deg., with the inductive reactance drop. A circuit containing resistance and capacitance in series is shown in Fig. 50. The emf equations for this circuit are

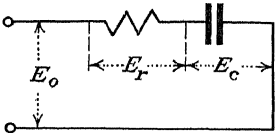

FIG. 50.—Circuit containing resistance and capacitance in series.

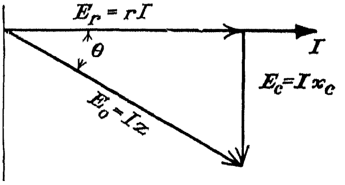

FIG. 51.—Emf diagram for the circuit of Fig. 50.

$$= \sqrt{E_r^2 + E_c^2} \text{ volts},$$

the vector diagram being as shown in Fig. 51. The corresponding impedance diagram is shown in Fig. 52, the number representing the capacitive reactance being negative.

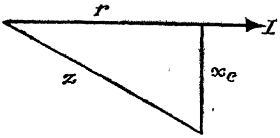

FIG. 52.—Impedance diagram for the circuit of Fig. 50.

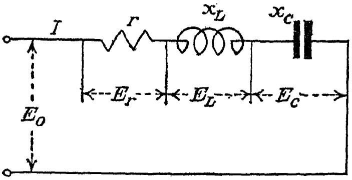

FIG. 53.—Circuit containing resistance, inductance, and capacitance in series.

50. Vectors for Circuits Having Resistance, Inductance, and Capacitance in Series.—A further investigation of the above construction will show that, for a circuit having a combination of resistance, inductance, and capacitance in series, with a sine wave of current flowing, the same method may be used and a set of vectors may be drawn representing the relative phase relations of the emfs and currents at any instant of time on the waves and circles. Such a set of resulting vectors is shown in Fig. 54 for the circuit of Fig. 53. It is seen that the vectors for E_L and E_C are 180 deg. out of phase with each other and that each is 90 deg. out of phase with the current, the inductive reactance drop leading the current by 90 deg. and the capacitive reactance drop lagging 90 deg. behind the current. An impedance diagram may be drawn for this circuit corresponding with Fig. 54.

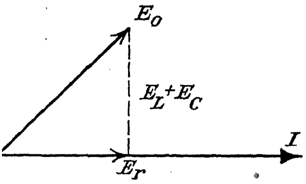

FIG. 54.—Vector diagram for the circuit of Fig. 53.

Example 17

In a circuit, Fig. 53, containing a resistance of 10 ohms, an inductive reactance of 18 ohms, and a capacitive reactance of 8 ohms, a current of 12 amp is flowing. Find the emf drop caused by the resistance, the inductance, and the capacitance, and the impressed emf.

The drop across the resistance is

$$E_r = Ir = 12 \times 10 = 120 \text{ volts,}$$

that across the inductance is

$$E_L = Ix_L = 12 \times 18 = 216 \text{ volts},$$

and that across the capacitance is

$$E_c = Ix_c = 12 \times (-8) = -96 \text{ volts}.$$

The resultant horizontal component is

$$E_r = 120 \text{ volts},$$

and the resultant vertical component is

$$E_x = E_L + E_c = 216 - 96 = 120 \text{ volts}.$$

The impressed emf becomes

$$E_0 = \sqrt{120^2 + 120^2} = 169.7 \text{ volts}.$$

The angle between the current and the impressed emf is

$$\theta = \tan^{-1}\frac{E_L + E_c}{E_r} = \tan^{-1}\frac{120}{120} = 45°.$$

51. Complex Notation, Rectangular Coordinates.—From algebra, the mathematics of the vector in a plane is obtained by the use of the complex quantity. In Fig. 55 the components of vector A are a and a', written

$$\dot{A} = a + ja', \qquad (178)$$

where

$$a = A \cos \theta, \qquad (179)$$

and

$$a' = A \sin \theta, \qquad (180)$$

or, in the form,

$$\begin{aligned}\dot{A} &= A \cos \theta + jA \sin \theta \\ &= A(\cos \theta + j \sin \theta). \qquad (181)\end{aligned}$$

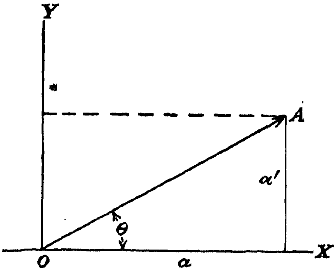

FIG. 55.—Components of a vector in a plane, rectangular coordinates.

The dot above the letter is used to designate a vector quantity. The operator $j = \sqrt{-1}$ turns the vector through 90 deg. in a counterclockwise direction each time it is used as an operator, Fig. 56. Hence, multiplying by j twice, or by j^2, where

$$j \times j = j^2 = -1, \qquad (182)$$

turns the vector through 180 deg. in a counterclockwise direction from any position. Multiplying by j three times or by j^3, as

$$j \times j \times j = j^3 = -j, \tag{183}$$

turns the vector through 270 deg. in a counterclockwise direction or through 90 deg. in a clockwise direction. The operator, $\cos\theta + j\sin\theta$, turns the vector through θ deg. in a counterclockwise direction from any position. The quantities with the j prefixed are called the "imaginary or quadrature" components and those without the j the "real" or "in-phase" components. Both components have real numerical values, the designation being purely arbitrary, their numerical relations depending upon the magnitude of the vector and its angular position in respect to the reference axis.

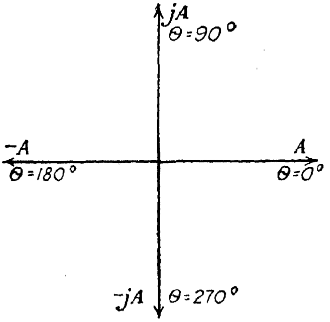

FIG. of a

52. Addition and Subtraction of Vectors.—Addition of two vectors is performed as follows: let the vector

$$A = a + ja', \tag{178}$$

be added to the vector

$$\dot{B} = b + jb', \tag{184}$$

so that

$$\dot{C} = \dot{A} + \dot{B} = (a + b) + j(a' + b') = c + jc', \tag{185}$$

or the *projection of the sum of two vectors upon a given set of axes is equal to the sum of the projections of the vectors on the same axes,* since

$$c = a + b, \tag{186}$$

and

$$c' = a' + b'. \tag{187}$$

Also, by subtraction,

$$\dot{D} = \dot{A} - \dot{B} = (a - b) + j(a' - b') = d + jd', \tag{188}$$

where

$$d = a - b, \tag{189}$$

and

$$d' = a' - b', \tag{190}$$

or the projection of the difference of two vectors on a set of axes is the difference of the projections of these vectors on the same axes.

53. Product and Division of Complex Numbers in Rectangular Coordinates.—The product of two complex numbers is found as follows:

$$\dot{G} = \dot{A}\dot{B} = (a + ja')(b + jb'), \tag{191}$$

or

$$= ab + jab' + ja'b + j^2a'b',$$

and

$$\dot{G} = (ab - a'b') + j(a'b + ab') = g + jg', \tag{192}$$

where

$$g = ab - a'b', \tag{193}$$

and

$$g' = a'b + ab'. \tag{194}$$

Division is accomplished as follows:

$$\dot{H} = \frac{\dot{A}}{\dot{B}} = \frac{a + ja'}{b + jb'}$$

$$= \frac{(a + ja')(b - jb')}{(b + jb')(b - jb')}, \tag{195}$$

or

$$\dot{H} = \frac{(ab + a'b') + j(a'b - ab')}{b^2 + b'^2} = h + jh', \tag{196}$$

where

$$h = \frac{ab + a'b'}{b^2 + b'^2}, \tag{197}$$

and

$$h' = \frac{a'b - ab'}{b^2 + b'^2}. \tag{198}$$

54. Use of Complex Notation in Vector Expressions for Resistance, Inductance, and Capacitance in Series Circuits, Current as the Reference.—The use of current as the reference is expressed in the mathematics of complex numbers as

$$\dot{I} = i + j0 \text{ amp.} \tag{199}$$

In a circuit consisting of resistance and inductance in series, the impedance of the circuit is expressed as

$$\dot{z} = r + jx \text{ ohms.} \tag{200}$$

Then the voltage drop caused by the current in Eq. (199) flowing through the impedance of Eq. (200) is

$$\dot{E} = I\dot{z} = I(r + jx) \text{ volts} \tag{201}$$
$$= Ir + jIx \text{ volts} \tag{202}$$
$$= e + je' \text{ volts.} \tag{203}$$

The vector diagram for this circuit is shown in Fig. 57, in which the triangle of emfs is built on current as the reference axis.

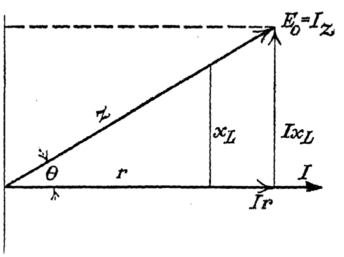

FIG. 57.—Vector and impedance diagram for an inductive circuit.

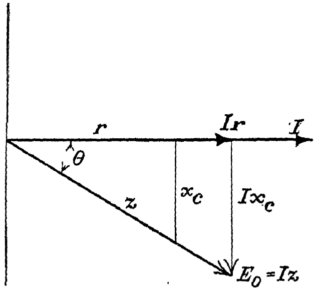

FIG. 58.—Vector and impedance diagram for a capacitive circuit.

When each side of the triangle of emfs in Fig. 57 is divided by the current, there results an impedance triangle that is similar to the emf triangle, but to a different scale. For the inductive circuit, the sign of the numerical value of x is positive, so that both Ix and x are 90 deg. ahead of the current.

For a circuit with resistance and capacitance in series, Eqs. (199) to (203) are also used, but the numerical value of x is now negative. The combined vector and impedance diagram for such a circuit is shown in Fig. 58.

When resistance, inductance, and capacitance are connected in series, the equation for the voltage drop across the combination becomes

$$
\begin{aligned}
\dot{E} &= Ir + jIx_L + jIx_c \text{ volts} && (202)\\
&= Ir + j(Ix_L + Ix_c) \text{ volts} && (204)\\
&= Ir + jI(x_L + x_c) \text{ volts.} && (205)
\end{aligned}
$$

The vector diagram is given in Fig. 59, the emf Ix_L being 90 deg. ahead of the current vector and Ix_c being 90 deg. behind it. If Ix_L is greater than Ix_c, the impressed emf for the circuit leads the current; if Ix_L is less than Ix_c, the impressed emf lags the current. If Ix_L is equal to Ix_c the impressed emf is in phase with the current, and the condition of series-phase resonance exists.

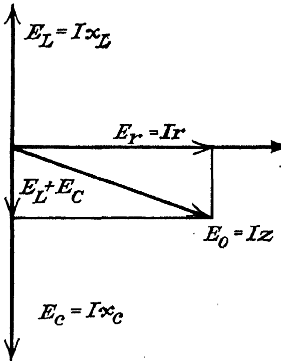

FIG. 59.—Vector diagram for circuit containing R, L, and C in series.

For the condition of several impedances in series, in which group both inductive and capacitive impedances may be included, the voltage drop across the different impedances will be expressed as

$$
\begin{aligned}
\dot{E}_1 &= Ir_1 + jIx_1 = e_1 + je_1' \text{ volts,}\\
\dot{E}_2 &= Ir_2 + jIx_2 = e_2 + je_2' \text{ volts,}\\
\\
\dot{E}_n &= Ir_n + jIx_n = e_n + je_n' \text{ volts.} && (206)
\end{aligned}
$$

The voltage impressed across these series impedances will be

$$
\begin{aligned}
\dot{E}_0 &= \dot{E}_1 + \dot{E}_2 + \ . \ . \ . + \dot{E}_n \text{ volts} && (207)\\
&= I(\dot{z}_1 + \dot{z}_2 + \ . \ . \ . + \dot{z}_n) = I\dot{z}_0 \text{ volts} && (208)\\
&= I[(r_1 + r_2 + \ . \ . \ . + r_n) + j(x_1 + x_2 + \ . \ . \ . + x_n)]\\
& \text{volts} && (209)\\
&= (e_1 + e_2 + \ . \ . \ . + e_n) + j(e_1' + e_2' + \ . \ . \ . + e_n')\\
& \text{volts} && (210)\\
&= e_0 + \qquad \text{volts.} && (211)
\end{aligned}
$$

Hence, for a series circuit, the components of the sum of the emfs is equal to the sum of the components of the emfs. This is illustrated for two impedances in series in Figs. 60 and 61.

If the sum of the resistances and the sum of the reactances are obtained in Eq. (209), it may be written as

$$
\dot{E}_0 = I(r_0 + jx_0) \text{ volts,} \qquad (212)
$$

where

$$r_0 = r_1 + r_2 + \ . \ . \ . \ + r_n \text{ ohms,} \tag{213}$$
$$x_0 = x_1 + x_2 + \ . \ . \ . \ + x_n \text{ ohms.} \tag{214}$$

Eq. (212) may be written in the form

$$\dot{E}_0 = I\dot{z}_0 \text{ volts,} \tag{215}$$

where z_0 is a single impedance that produces an effect equivalent to the group of impedances in series and may be substituted for the group in calculations involving only the total effect of the group. This is illustrated for the case of two coils in series in Fig. 62.

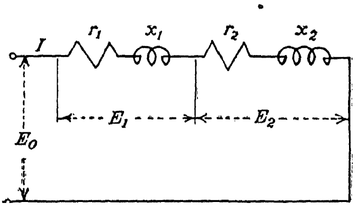

FIG. 60.—Circuit diagram for two coils in series.

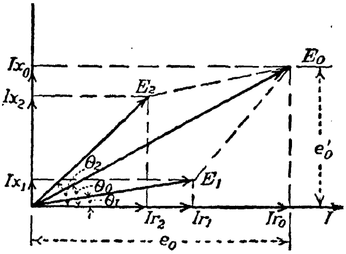

FIG. 61.—Vector diagram for two coils in series.

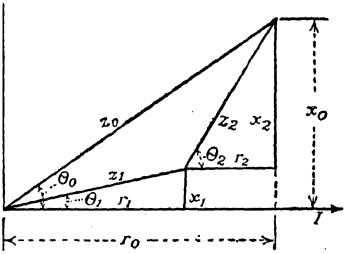

FIG. 62.—Impedance diagram for tw coils in series.

Example 18

An impedance z_1 having a resistance of 6 ohms and an inductive reactance of 10 ohms, an impedance z_2 having a resistance of 2 ohms and an inductive reactance of 8 ohms, and an impedance z_3 having a capacitive reactance of 6 ohms are connected in series, a current of 10 amp flowing through the circuit. Calculate the drop across each impedance and the total impressed emf.

Using current as the reference,

$$\dot{I} = 10 + j0 \text{ amp.}$$

Expressing the impedances in complex notation,

$$\dot{z}_1 = 6 + j10 \text{ ohms,}$$
$$\dot{z}_2 = 2 + j8 \text{ ohms,}$$
$$\dot{z}_3 = 0 - j6 \text{ ohms.}$$

Then

$$\dot{E}_1 = 10(6 + j10) = 60 + j100 \text{ volts},$$

having a magnitude,

$$E_1 = \sqrt{60^2 + 100^2} = 117 \text{ volts}.$$
$$\dot{E}_2 = 10(2 + j8) = 20 + j80 \text{ volts},$$

having a magnitude,

$$E_2 = \sqrt{20^2 + 80^2} = 82.5 \text{ volts}.$$
$$\dot{E}_3 = 10(0 - j6) = 0 - j60 \text{ volts},$$

having a magnitude,

$$E_3 = 60 \text{ volts}.$$

From Eq. (200) the impressed emf is

$$\begin{aligned}\dot{E}_0 &= 10[(6 + 2 + 0) + j(10 + 8 - 6)] \text{ volts}\\ &= 10(10 + j12) = 80 + j120 \text{ volts},\end{aligned}$$

or

$$\begin{aligned}&= (60 + 20 + 0) + j(100 + 80 - 60) \text{ volts}\\ &= 80 + j120 \text{ volts},\end{aligned}$$

having a magnitude,

$$E_0 = \sqrt{80^2 + 120^2} = 144.5 \text{ volts}.$$

55. Multiplication of a Vector Current by an Impedance.—To make a study of the multiplication of a current, which is not in phase with the axis of reference, by an impedance, let Fig. 63 represent a vector diagram of any circuit with the current leading the reference axis, and expressed by

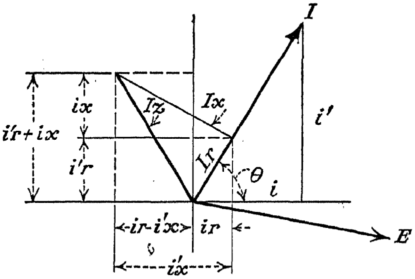

FIG. 63.—The vector product IZ and its components, when I leads the reference axis.

$$\dot{I} = i + ji' \text{ amp.} \tag{216}$$

This is to be multiplied by the impedance,

$$\dot{z} = r + jx \text{ ohms.} \tag{200}$$

In the definitions of r, x, and z in Chap. II, the common current is taken as the reference vector; hence, Ir must be in phase with I, Ix 90 deg. ahead of I, and Iz must be the vector sum of Ir and Ix, the three forming a right triangle with Iz as the hypotenuse. These emf drops must now be resolved into components in phase with, and at right angles with, the new axis in order to be able to add these emfs to other emfs referred to the same axes.

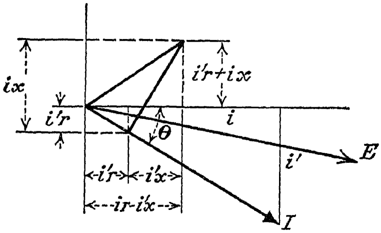

FIG. 64.—The vector product IZ and its components, when I lags the reference axis.

The component of Ir in phase with the reference axis is

$$Ir \cos\theta = (I \cos\theta)r = ir \text{ volts,} \tag{217}$$

and at right angles with the axis it is

$$Ir \sin\theta = (I \sin\theta)r = i'r \text{ volts.} \tag{218}$$

Hence, the components of the Ir drop, referred to the new axes, are the resistance drops of the components of I (i and i'), referred to the same axes. In a similar manner, the component of Ix in phase with the axis is

$$Ix \cos(90° + \theta) = -(I \sin\theta)x = -i'x \text{ volts,} \tag{219}$$

and at right angles with the axis it is

$$Ix \sin(90° + \theta) = (I \cos\theta)x = ix \text{ volts.} \tag{220}$$

Hence, the components of the Ix drop, referred to the new axes, are the reactive drops of the components of I (i and i'), referred to the new axes. An analysis of Eqs. (217) to (220) shows that each component of current referred to any set of axes produces its ir, ix, $i'r$, and $i'x$ drops in the proper phase relation with each current component, and that the total impedance drop is the vector sum of the separate resistance and reactance drops of the individual components of current. The operation of multiplying the current by $\dot{z} = r + jx$ is that of determining the magnitude of Iz and of rotating this impedance drop from the current axis as a reference to the new reference axis. The impedance drop may then be added to any other emf, referred to the same set of axes, by adding like projections of this emf and of Iz. The construction for the case in which I lags the reference is shown in Fig. 64.

This analysis follows also from the mathematics of the complex number. With the equations for the voltage impressed across an impedance where the current is not the reference,

$$\dot{E}_0 = \dot{I}\dot{z} \text{ volts,} \tag{221}$$

Expanding, using Eqs. (216) and (200),

$$\begin{aligned} \dot{E}_0 &= (i + ji')(r + jx) \text{ volts} && (222) \\ &= ir - i'x + j(i'r + ix) \text{ volts} && (223) \\ &= e_0 + je_0' \text{ volts,} && (224) \end{aligned}$$

where

$$\begin{aligned} e_0 &= ir - i'x \text{ volts,} && (225) \\ e_0' &= i'r + ix \text{ volts.} && (226) \end{aligned}$$

The current may be found from Eq. (221) if the emf and impedance are given by solving for I, *i.e.*,

$$\begin{aligned} \dot{I} &= \frac{\dot{E}}{\dot{z}} = \frac{e + je'}{r + jx} \text{ amp} && (227) \\ &= i + ji' \text{ amp.} && (228) \end{aligned}$$

If the emf is chosen as the reference, Eqs. (227) and (228) become

$$\begin{aligned} \dot{I} &= \frac{E}{r + jx} \text{ amp} && (229) \\ &= i + ji' \text{ amp.} && (216) \end{aligned}$$

Example 19

An emf of 120 volts is impressed on a circuit consisting of three impedances in series, impedance z_1 having a resistance of 6 ohms and an inductive reactance of 10 ohms, impedance z_2 having a resistance or 4 ohms and an inductive reactance of 6 ohms, and impedance z_3 having a capacitive reactance of 9 ohms. (*a*) Find the current using the emf as a reference. (*b*) Find the current using the emf 60 deg. ahead of the axis.

From the statement of the problem,

$$\begin{aligned}\dot{Z}_1 &= 6 + j10 \text{ ohms},\\ \dot{Z}_2 &= 4 + j6 \text{ ohms},\\ \dot{Z}_3 &= 0 - j9 \text{ ohms},\\ \dot{Z}_0 &= \dot{Z}_1 + \dot{Z}_2 + \dot{Z}_3 = 10 + j7 \text{ ohms}.\end{aligned}$$

a. If the emf is taken as the reference, then

$$\dot{E} = E + j0 = 120 + j0 \text{ volts},$$

$$\dot{I} = \frac{E}{\dot{Z}} = \frac{120}{10 + j7} = 8.06 - j5.63 \text{ amp},$$

having the magnitude,

$$I = \sqrt{8.06^2 + 5.63^2} = 9.83 \text{ amp}.$$

b. If the emf is 60 deg. ahead of the reference axis it is expressed as

$$\dot{E} = 60 + j104 \text{ volts},$$

and

$$\dot{I} = \frac{60 + j104}{10 + j7} = 8.92 + j4.17 \text{ amp},$$

having the magnitude,

$$I = \sqrt{8.92^2 + 4.17^2} = 9.83 \text{ amp}.$$

56. Exponential Form of Vectors.—For addition and subtraction of vectors, the rectangular forms,

$$\dot{A} = a + ja', \tag{178}$$

or

$$\dot{A} = A(\cos\theta \pm j\sin\theta), \tag{181}$$

of the vectors are the most convenient for use. For most other mathematical operations, either the exponential or the polar form is more easily used. In Eq. (181) the quantity $(\cos\theta + j\sin\theta)$ is the operator, and A is the magnitude of the vector. The angle θ is the phase of the vector in reference to some arbitrary axis. From calculus, the values of $\cos\theta$ and $j\sin\theta$ may be expanded by Maclaurin's theorem as an infinite series such that

$$\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \cdots \tag{230}$$

and

$$\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \cdots, \tag{231}$$

where θ is measured in radians. Hence,

$$\cos\theta \pm j\sin\theta = 1 \pm j\theta - \frac{\theta^2}{\underline{|2}} \mp j\frac{\theta^3}{\underline{|3}} + \frac{\theta^4}{\underline{|4}} \pm j\frac{\theta^5}{\underline{|5}} - \cdot\ \cdot\ \cdot\ \cdot \tag{232}$$

But, from the same theorem the expansion of ϵ^θ gives

$$\epsilon^\theta = 1 + \theta + \frac{\theta^2}{\underline{|2}} + \frac{\theta^3}{\underline{|3}} + \frac{\theta^4}{\underline{|4}} + \cdot\ \cdot\ \cdot\ , \tag{233}$$

and

$$\epsilon^{j\theta} = 1 + j\theta - \frac{\theta^2}{\underline{|2}} - j\frac{\theta^3}{\underline{|3}} + \frac{\theta^4}{\underline{|4}} + j\frac{\theta^5}{\underline{|5}} - - + + \cdot\ \cdot\ \cdot \tag{234}$$

$$= \cos\theta + j\sin\theta, \tag{235}$$

and in a similiar manner

$$\epsilon^{-j\theta} = \cos\theta - j\sin\theta. \tag{236}$$

Combining Eqs. (235) and (236),

$$\cos\theta = \frac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2}, \tag{237}$$

and

$$\sin\theta = \frac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{2j}. \tag{238}$$

A similar combination of expressions holds for the hyperbolic functions, which will be used later in the development of the equations of the long transmission line, as

$$\cosh\theta = \frac{\epsilon^{\theta} + \epsilon^{-\theta}}{2}, \tag{239}$$

and

$$\sinh\theta = \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2}. \tag{240}$$

In the expansion of a quantity representing a vector or a complex number, as $x + jy$, it is necessary to obtain another group of expressions similar to Eqs. (237) to (240) for the imaginary quantities, as $\cos jx$, $\sin jx$, $\cosh jx$, and $\sinh jx$. These values may be derived from Eqs. (237) to (240) by substituting jx for θ, as

$$\cos jx = \frac{\epsilon^{j(jx)} + \epsilon^{-j(jx)}}{2} = \frac{\epsilon^{-x} + \epsilon^{x}}{2} = \cosh x, \tag{241}$$

$$\sin jx = \frac{\epsilon^{j(jx)} - \epsilon^{-j(jx)}}{2j} = \frac{\epsilon^{-x} - \epsilon^{x}}{2j} = j \sinh x, \tag{242}$$

$$\cosh jx = \frac{\epsilon^{jx} + \epsilon^{-jx}}{2} = \cos x, \tag{243}$$

and

$$\sinh jx = \frac{\epsilon^{jx} - \epsilon^{-jx}}{2} = j \sin x. \tag{244}$$

From Eqs. (237) to (244) inclusive it is possible to expand functions of a complex variable, as

$$\begin{aligned}
\sin (x + jy) &= \sin x \cos jy + \cos x \sin jy \\
&= \sin x \cosh y + j \cos x \sinh y. && (245) \\
\cos (x + jy) &= \cos x \cos jy - \sin x \sin jy \\
&= \cos x \cosh y - j \sin x \sinh y. && (246) \\
\sinh (x + jy) &= \frac{\epsilon^{x+jy} - \epsilon^{-(x+jy)}}{2} = \frac{\epsilon^{x}\epsilon^{jy} - \epsilon^{-x}\epsilon^{-jy}}{2} \\
&= \frac{\epsilon^{x} (\cos y + j \sin y) - \epsilon^{-x} (\cos y - j \sin y)}{2} \\
&= \cos y \left(\frac{\epsilon^{x} - \epsilon^{-x}}{2}\right) + j \sin y \left(\frac{\epsilon^{x} + \epsilon^{-x}}{2}\right) \\
&= \sinh x \cos y + j \cosh x \sin y. && (247) \\
\cosh (x + jy) &= \frac{\epsilon^{x+jy} + \epsilon^{-(x+jy)}}{2} = \frac{\epsilon^{x}\epsilon^{jy} + \epsilon^{-x}\epsilon^{-jy}}{2} \\
&= \frac{\epsilon^{x} (\cos y + j \sin y) + \epsilon^{-x} (\cos y - j \sin y)}{2} \\
&= \cos y \left(\frac{\epsilon^{x} + \epsilon^{-x}}{2}\right) + j \sin y \left(\frac{\epsilon^{x} - \epsilon^{-x}}{2}\right) \\
&= \cosh x \cos y + j \sinh x \sin y. && (248)
\end{aligned}$$

From Eqs. (235) and (236), it is seen that the vector may be expressed also in the form

$$\dot{A} = A\epsilon^{\pm j\theta}, \tag{249}$$

where ϵ is the base of the Naperian system of logarithms (2.71828 . . .) and θ may be any number real, imaginary, or complex, the angle being expressed in radians.

57. Polar Form of a Vector.—Another method of expressing a vector is by means of the magnitude and the angle the vector makes with some predetermined reference axis, the polar form of the vector, as

$$\dot{A} = A\underline{/\theta}, \tag{250}$$

where A is the magnitude of the vector and θ is its angle in radians with the assumed reference axis. The angle may also be expressed in degrees by multiplying its value in radians by $360/2\pi$ or by $180/\pi$. In this form, it is usually written as

$$\dot{A} = A\underline{/\theta^\circ}. \tag{251}$$

Negative angles may be expressed as

$$\dot{A} = A/-\ \theta^\circ. \tag{252}$$

58. Interchangeability of Forms of the Vector.—It is often necessary to convert from one form of the vector to another. This may be done by remembering that

$$\begin{aligned} A &= a + ja', && (178)\\ &= A\ (\cos\theta + j\sin\theta) && (181)\\ &= A\epsilon^{j\theta} && (249) \end{aligned}$$

and

$$= A\underline{/\theta} \tag{250}$$

express the same vector. The conversion is made by use of the quantities,

$$\begin{aligned} A &= \\ a &= A\cos\theta, && (179)\\ a' &= A\sin\theta; && (180)\\ \tan\theta &= \frac{a'}{a}, \end{aligned}$$

or

$$\theta = \tan^{-1}\frac{a'}{a}. \tag{253}$$

Hence,

$$\dot{A} = A\underline{/\theta} = \sqrt{a^2 + a'^2}\underline{/\tan^{-1}\frac{a'}{a}} \tag{254}$$

and

$$\dot{A} = a + ja' = A\ (\cos\theta + j\sin$$

or

$$A =$$

are all expressions for the same vector.

59. Multiplication and Division of Vectors.—It has been seen that vectors expressed in rectangular coordinates may be multiplied or divided as binomials. It is, however, easier to perform the operation by first expressing the vectors in either the exponential or the polar form. The product of

$$\dot{A} = A\,(\cos\theta + j\sin\theta) = A\underline{/\theta} = A\epsilon^{j\theta} \tag{255}$$

and

$$\dot{B} \tag{256}$$

is

$$\begin{aligned}\dot{C} = \dot{A}\dot{B} &= AB(\cos\theta\cos\beta - \sin\theta\sin\beta) + \\ &\qquad jAB(\cos\theta\sin\beta + \sin\theta\cos\beta) \\ &= AB[\cos(\theta+\beta) + j\sin(\theta+\beta)],\end{aligned} \tag{257}$$

or

$$\dot{C} = \tag{258}$$

The product of the vectors has for its new magnitude the product of the magnitudes of the individual vectors and for its angle in reference to any axis the sum of the angles of the individual vectors in respect to the same axis.

Vectors may be divided by the same method as is used for multiplication, as

$$\dot{D} = \frac{\dot{A}}{\dot{B}} = \frac{A(\cos\theta + j\sin\theta)}{B\,(\cos\beta + j\sin\beta)} \tag{259}$$

$$= \frac{A\,(\cos\theta + j\sin\theta)(\cos\beta - j\sin\beta)}{B}$$

$$= \frac{A}{B}[\cos(\theta-\beta) + j\sin(\theta-\beta)] \tag{260}$$

$$= \frac{A}{B}\underline{/(\theta-\beta)} = D\underline{/(\theta-\beta)} \tag{261}$$

$$\tag{262}$$

The division of vectors in the polar form is performed by finding the quotient of their magnitudes and the difference of their angles with respect to the reference axis. The reciprocal of a vector is found by taking the reciprocal of the magnitude and by changing the sign of the angle with the reference axis, as

$$\dot{D} = \frac{1}{A\underline{/\theta}} = \frac{1}{A}\,\underline{/-\theta}. \tag{263}$$

60. Involution and Evolution of Vectors.—To perform involution of vectors it is only necessary to remember that any power of a number is the product of that number multiplied by itself as many times as the power; or

$$\dot{C} = (\dot{A})^n = (A\epsilon^{j\theta})^n = A^n\epsilon^{jn\theta},$$

or

$$\dot{C} = (A\underline{/\theta})^n = A^n\underline{/n\theta}. \tag{264}$$

The nth power of any vector is found by taking the nth power of the magnitude and multiplying the number of degrees or radians in its angle by n.

In a similar manner, the nth root of a vector is found as

$$\begin{aligned} \dot{D} &= (\dot{A})^{\frac{1}{n}} = (A\epsilon^{j\theta})^{\frac{1}{n}} = \sqrt[n]{A}\,\epsilon^{j\frac{\theta}{n}} \\ &= (A\underline{/\theta})^{\frac{1}{n}} = \sqrt[n]{A}\,\underline{/\frac{\theta}{n}}, \end{aligned} \tag{265}$$

which should be investigated for the number of roots. *In general, the nth root of a vector is a vector having for its magnitude the nth root of the magnitude of the original vector, and for its direction the angle of the original vector in degrees or radians divided by n, or the original angle plus 2π divided by n.*

61. Logarithms of Vectors.—To obtain the logarithm of a vector the exponential form is the most convenient to use. Let

$$\dot{B} = \log_\epsilon \dot{A} = \log_\epsilon (A\epsilon^{j\theta}) = \log_\epsilon (A\underline{/\theta}); \tag{266}$$

find the value of B. Expand Eq. (266),

$$\begin{aligned} \dot{B} &= \log_\epsilon A + \log_\epsilon \epsilon^{j\theta} \\ &= \log_\epsilon A + j\theta \log_\epsilon \epsilon \\ &= \log_\epsilon A + j(\theta + 2\pi n) \\ &= b + jb', \end{aligned} \tag{267}$$

where

$$b = \log_\epsilon A, \tag{268}$$

and

$$b' = \theta + 2\pi n, \tag{269}$$

θ being measured in circular radians.

The Naperian logarithm of a vector may be found in the rectangular form as a new vector having as its real part the logarithm *of the magnitude of the vector and as its imaginary part the angle of the original vector expressed in circular radians.*

62. Application of the Polar Form of Vector to the Solution of Series Circuits.—The polar form of notation for vectors affords a very convenient method for the multiplication and division of vector quantities and is particularly advantageous where several quantities are involved in the process of multiplication and division. However, in order to add the vector quantities they must be reduced to the rectangular form. Both forms will be used in the illustrative examples throughout the text, and it is recommended that the student acquire command of both methods of computation.

Example 20

Solve the circuit of Example 19 using the polar form, *i.e.*, find the current in the series circuit composed of the three impedances, $\dot{Z}_1 = 6 + j10$, $\dot{Z}_2 = 4 + j6$, and $\dot{Z}_3 = 0 - j9$, (*a*) using the applied emf of 120 volts as the reference and (*b*) with this same emf 60 deg. ahead of the reference axis.

From Example 19,

$$\dot{Z}_0 = 10 + j7 = 12.2\underline{/34^\circ} \text{ ohms.}$$

a. Solving for the current,

$$\dot{I} = \frac{120\underline{/0^\circ}}{12.2\underline{/34^\circ}} = 9.83\underline{/-34^\circ} \text{ amp}$$

having the magnitude 9.83 amp.

b. Solving for current with E taken 60 deg. ahead of axis,

$$\dot{I} = \frac{120\underline{/60^\circ}}{12.2\underline{/34^\circ}} = 9.83\underline{/60^\circ - 34^\circ} \text{ amp}$$
$$= 9.83\underline{/26^\circ} = 8.92 + j4.17 \text{ amp.}$$

Problems

1-3. Two vectors rotate at the same velocity and with the second vector 45 deg. behind the first. If the first vector is 100 units long and the second is 50 units, rotate these vectors and plot the corresponding waves by projection.

2-3. Two vectors 90 deg. apart rotate at a uniform velocity. The first vector is 200 volts and the second is 100 volts. (*a*) Plot the corresponding waves by projection. (*b*) Add these vectors, and plot the corresponding wave.

3-3. A coil having a resistance of 8 ohms and an inductance of 0.02 henry has a current of 100 amp. at 60 cycles flowing through it. Find the values of E_r, E_x, E_o, cos θ, and the value of θ in degrees. Draw the vector diagram with current as the reference.

4-3. A coil having a resistance of 8 ohms and an inductance of 0.02 henry has a current of 100 amp at 40 cycles flowing through it. Find the values of E_r, E_x, E_o, cos θ, and the value of θ in degrees. Draw the vector diagram with current as the reference vector and also with E_o as the reference vector.

5-3. Two coils having the constants $R_1 = 90$ ohms, $L_1 = 0.3$ henry, $R_2 = 40$ ohms, $L_2 = 0.2$ henry are connected in series, and a current of 100 amp at 25 cycles is sent through the circuit. Find the various resistance and reactance drops and the impressed emf and draw the complete vector diagram with current as the reference.

6-3. Use the same coils as in Prob. 5-3 but with 100 amp at 60 cycles flowing.

7-3. A resistor having a resistance of 6.3 ohms and a capacitor having a capacitance of 110 μf are connected in series, and a current of 120 amp at 60 cycles is passed through the circuit. Find the values of E_r, E_x, E_o, cos θ, and the value of θ in degrees. Draw the complete vector diagram with current as the reference.

8-3. A capacitor having a reactance of 10 ohms is connected in series with a resistance of 20 ohms. A 60-cycle alternating current of 7.07 amp is sent through the circuit. With current as reference, draw the wave forms of i, e_r, e_c, and e_o. Draw the circles for these wave forms, and locate the vectors I, E_r, E_c, and E_o for the angles 0, 135, and 240 deg. from the reference.

9-3. A coil having a resistance of 8 ohms and an inductance of 0.2 henry is connected in series with a capacitor having a capacitance of 500 μf to a 60-cycle a-c circuit. Find the values of E_r, E_L, E_c, E_o, cos θ, and the angle θ in degrees when 200 amp flows through the circuit. Draw the complete vector diagram with current as the reference vector.

10-3. Same as Prob. 9-3 but with 25 cycles as the frequency of the current flowing through the circuit.

11-3. A coil having a resistance of 10 ohms and an inductance of 0.02 henry is connected in series with a capacitor having a capacitance of 30 μf to a 50-cycle circuit, the current flowing being 100 amp. Find the values of E_r, E_L, E_c, E_o, cos θ, and the value of θ in degrees, and draw the complete vector diagram with current as the reference.

12-3. Add the vectors of emf $\dot{E}_1 = 100 + j100$, $\dot{E}_2 = 60 + j80$, and $\dot{E}_3 = 80 - j60$, and show the result in a vector diagram.

13-3. Subtract the vector $\dot{E}_1 = 100 + j100$ from the vector $\dot{E}_2 = 80 - j60$, and show the result in a vector diagram.

14-3. Multiply $\dot{A} = 40 - j50$ by $\dot{B} = -20 - j20$ by the rectangular-coordinate method.

15-3. Divide $\dot{A} = 800 + j600$ by $\dot{B} = 40 - j30$, using the rectangular-coordinate method.

16-3. Multiply $\dot{I} = 12 + j20$ by $\cos 30° + j \sin 30°$. Draw the two vectors, and show that multiplying by this operator turns the vector through 30 deg. in a counterclockwise direction.

17-3. When an emf $\dot{E} = -86.6 - j50$ is applied to a certain circuit, a current $\dot{I} = -17.3 + j10$ flows. Rotate these vectors so that E coincides with the reference axis, and find the new rectangular expressions for E, I, and Z.

18-3. A current $\dot{I} = 100 + j0$ amp is sent through an inductor, the impedance of which is $\dot{Z} = 40 + j40$ ohms. Find E_r, E_x, E_o in vector quantities and in magnitude. Draw the vector diagram with current as the reference. Use complex quantities in the calculations.

19-3. A current $\dot{I} = 12 + j20$ amp is passed through a circuit containing a resistance of 8 ohms and a capacitive reactance of 16 ohms. Find the values of E_r, E_c, and E_o in vector values and in magnitudes, and draw the vector diagram. Use complex quantities in the calculations.

20-3. Given two impedances $\dot{Z}_1 = 10 + j20$ and $\dot{Z}_2 = 6 - j8$ connected in series with 240 volts impressed, determine the current, the emf drop across each impedance, and the power factor of the total circuit and of each impedance, and draw the complete vector diagram using E as the reference. Use complex quantities in the calculations.

21-3. A coil having a resistance of 2.5 ohms and an inductance of 0.07 henry is connected in series with a capacitor having a capacitance of 100 μf across an a-c source. Find the frequency at resonance. A current of 700 amp is sent through the coil at resonant frequency; determine the various resistance and reactance drops and the impressed emf in vector value and in magnitude, and draw the complete vector diagram with current as the reference. Use complex quantities in the calculations.

22-3. A coil having a resistance of 10 ohms and an inductance of 0.3 henry is placed in series with a capacitor, having such a capacitance as to produce resonance at 50 cycles, across an a-c circuit. Calculate the capacitance of the capacitor. An emf of 500 volts at 50 cycles is applied to this circuit. Determine the values of the current and various emf drops, and draw the complete vector diagram with impressed emf as the reference.

23-3. Two impedances Z_1 and Z_2 are connected in series across an emf $\dot{E} = 80 - j60$, and a current $\dot{I} = 40 + j30$ amp flows. The drop across Z_2 is $\dot{E}_2 = 30 - j25$; what are the constants of the impedances?

24-3. Two impedances $\dot{Z}_1 = 8 - j10$ and $\dot{Z}_2 = 10 + j20$ are connected in series, and the voltage drop over Z_2 is found to be $\dot{E}_2 = 300 + j500$ volts. Find the voltage drop across the other impedance, the impressed emf, the total power, and the power factor.

25-3. Two impedances Z_1 and Z_2 are connected in series across an a-c circuit. The emf applied to the circuit is $\dot{E}_0 = 86.6 - j50$ volts, and the current is $\dot{I}_0 = 17.3 + j10$ amp. If the impedance $\dot{Z}_1 = 2 - j2$, find the resistance and reactance of the second impedance.

26-3. A coil having an impedance of $\dot{Z}_1 = 5 + j12$ ohms is to be connected in series with a second coil across a 2,300-volt circuit. The second coil is to be so designed that the total current will be 100 amp, and the power factor of the total circuit will be 0.707, current lagging. Find the impedance of the second coil, and draw the vector diagram showing the current, voltage across each coil, and the total voltage, using current as the reference.

27-3. Express the vector $\dot{E} = 55 + j95.6$ in polar form. Express the operator $\cos 60° - j \sin 60°$ in the polar form, and multiply the vector E by the polar operator. Change the resultant vector into rectangular coordinates, and compare with the components of the original vector rotated 60 deg.

in a clockwise direction that are obtained by using the complex form of the operator in the calculations.

28-3. When an emf $\dot{E} = 150\underline{/-70^\circ}$ is applied to a circuit, a current $\dot{I} = 12\underline{/-105^\circ}$ flows. Rotate these vectors so that I coincides with the reference axis, and find the new polar and rectangular expressions for E, I, and Z.

29-3. Change the vector $\dot{E} = 700 + j200$ into the polar and exponential forms.

30-3. Change the vector $\dot{I} = 1{,}000\epsilon^{-j7.6}$ to the polar and rectangular forms.

31-3. Change the vector $\dot{E} = 500\underline{/-162^\circ}$ to the rectangular and exponential forms.

32-3. Multiply $\dot{I} = 200\underline{/60^\circ}$ by $\dot{Z} = 10\epsilon^{j10}$, and express the result in polar and rectangular forms.

33-3. A voltage $\dot{E} = 190.5 - j110$ produces a current $\dot{I} = 9.53 + j5.5$ amp. Express the two vectors in polar form, solve for the impedance Z in polar form, and change to the rectangular form. What are the resistance and the reactance of the circuit?

34-3. A current $\dot{I} = 100\underline{/37^\circ}$ flows through a circuit whose impedance is $\dot{Z} = 25\underline{/-53^\circ}$. What is the impressed emf? Draw the vector diagram.

35-3. A voltage $\dot{E} = 5{,}000\underline{/-45^\circ}$ is impressed on a circuit, and a current $\dot{I} = 25\underline{/15^\circ}$ flows. What is the impedance of the circuit expressed in polar form? What are the resistance and reactance of the circuit? Draw the vector diagram.

36-3. Two impedances $\dot{Z}_1 = 10\underline{/45^\circ}$ and $\dot{Z}_2 = 10\underline{/-90^\circ}$ are connected in series across an a-c circuit. The voltage E_1 across the first impedance is found to be $\dot{E}_1 = 1{,}000\underline{/60^\circ}$; find the current and the impressed emf in polar and rectangular forms. Draw the vector diagram.

37-3. Draw the vectors representing the sine waves of Prob. 3-2: (*a*) when the current wave is passing through zero from negative to positive values; (*b*) when the impressed emf is passing through zero.

38-3. A coil having a resistance of 5 ohms and an inductance of 0.03 henry has a current of 10 amp at 60 cycles flowing through it. Find the values of E_r, E_x, E_0, $\cos\theta$, and the value of θ in degrees. Draw the vector diagram with E_o as the reference vector. Draw the wave forms of i, e_r, e_x, and e_o.

39-3. Two inductors are connected in series across a 1,000-volt 60-cycle a-c circuit. The resistances and reactances of the inductors are as follows: $R_1 = 8$ ohms, $X_1 = 8$ ohms, $R_2 = 12$ ohms, $X_2 = 7$ ohms. Find the vector values and the magnitudes of I and all the emf drops in the circuit and find the various phase angles of the vectors. Draw the vector diagram with emf impressed as the reference vector. Draw the impedance diagram.

40-3. An inductor having a resistance of 10 ohms and an inductance of 0.03 henry is connected in series with a capacitor having a capacitance of 100 μf across a 2,000-volt 60-cycle a-c circuit. Find the vector values and magnitudes of E_r, E_c, E_L, and I, and draw the vector diagram using impressed emf as the reference vector.

41-3. Multiply the quantity $\dot{I} = 100 + j100$ by the impedance $\dot{Z} = 5 - j8.66$. Show the results in a vector diagram and by the use of rectangular coordinates.

42-3. In the circuit shown in Fig. 42-3, assume that the resistance R_1 is the load on a transmission circuit and that E_1 is kept constant. Find the value of each emf and its phase relation to the current and also the power factor at each point and draw the vector diagram of the circuit.

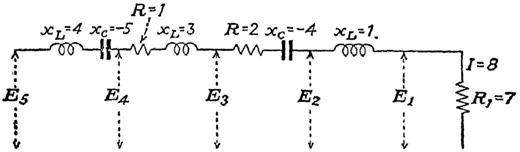

FIG. 42-3.—Figure for Prob. 42-3.

43-3. Two impedances $\dot{Z}_1 = 5 + j7$ and $\dot{Z}_2 = 3 - j4$ are connected in series across an a-c circuit. The emf applied is $\dot{E}_o = 100 + j0$ volts. Find the vector values and the magnitudes of the current and the numerous emf drops, and draw the vector diagram with E_o as the reference vector.

44-3. Two impedances $\dot{Z}_1 = 10 + j20$ and $\dot{Z}_2 = 0 - j10$ are connected in series, and the voltage drop across the second impedance is found to be $\dot{E}_2 = 120 + j120$. Find the current and the impressed emf.

45-3. Multiply $\dot{I} = 400\epsilon^{-j6}$ by $\dot{Z} = 5\epsilon^{j10}$, and express the result in polar and rectangular forms.

46-3. Divide $\dot{A} = 750\underline{/45^\circ}$ by $\dot{B} = 200 - j100$, and express the results in both polar and rectangular forms.

47-3. Two impedances $\dot{Z}_1 = 0.5\underline{/30^\circ}$ and $\dot{Z}_2 = 0.7\underline{/-60^\circ}$ are connected in series. An emf $\dot{E} = 78\underline{/150^\circ}$ is impressed. Find the current flowing in the circuit in polar form. Draw the vector diagram.

48-3. Cube the vector $\dot{E} = 40 + j40$, and express the result in both the polar and rectangular forms. Plot the original and the final vectors.

49-3. Take the square root of $\dot{E} = 40 + j40$, and express the result in polar and rectangular forms.

CHAPTER IV

PARALLEL CIRCUITS

In a-c as well as in d-c distribution, the loads are connected across parallel wires between which the emf is maintained at approximately the same value over the whole length of line. In a-c distribution, however, owing to the extensive use of the transformer, it is possible to use and control much higher voltages, standard voltages ranging from 110 volts for the incandescent lamp and small-motor circuits to 285,000 volts for some of the longest high-power transmission systems. In parallel systems of a-c distribution, the values of current taken by the different loads must be added in the proper phase relation to obtain the current flowing in the common line wires. In this chapter, the effect of the constants of the line wires will be neglected, the emf across all branches being assumed constant in value and phase relation, the effect of the line wire constants being considered in subsequent chapters.

63. Addition of Currents.—It has been found in Chap. II that when a current,

$$i = I_m \sin \alpha \text{ amp}, \tag{44}$$

flows through a series circuit the impressed emf may be expressed as

$$e_0 = zI_m \sin (\alpha + \theta) \text{ volts}, \tag{45}$$

where

$$\tan \theta = \frac{x}{r} \tag{81}$$

of the circuit, θ being the angle by which the emf differs in phase from the current. In parallel or branched circuits, each branch has the same impressed emf, and it is convenient to use this emf as the reference quantity. Writing this as

$$e = E_m \sin \alpha \text{ volts},$$

the currents in the several branches may be expressed as

$$
\begin{aligned}
i_1 &= \frac{E_m}{z_1} \sin (\alpha + \theta_1) \text{ amp} \\
&= I_{m_1} \sin (\alpha + \theta_1) \text{ amp}, \\
i_2 &= \frac{E_m}{z_2} \sin (\alpha + \theta_2) \text{ amp} \\
&= I_{m_2} \sin (\alpha + \theta_2) \text{ amp}, \\
&\cdots\cdots\cdots \\
i_n &= \frac{E_m}{z_n} \sin (\alpha + \theta_n) \text{ amp} \\
&= I_{m_n} \sin (\alpha + \theta_n) \text{ amp}.
\end{aligned}
\tag{270}
$$

By Kirchhoff's laws, the instantaneous value of the total current taken by the several branches must be

$$
i_0 = i_1 + i_2 + \cdots + i_n \text{ amp} \tag{271}
$$

$$
\begin{aligned}
&= I_{m_1} \sin (\alpha + \theta_1) + I_{m_2} \sin (\alpha + (\\
&+ \cdots + I_{m_n} \sin (\alpha + \theta_n) \text{ amp}.
\end{aligned}
\tag{272}
$$

Expanding the terms in Eq. (272),

$$
\begin{aligned}
i_0 = {} & I_{m_1} \sin \alpha \cos \theta_1 + I_{m_1} \cos \alpha \sin \theta_1 + I_{m_2} \sin \alpha \cos \theta_2 \\
& + I_{m_2} \cos \alpha \sin \theta_2 + \cdots + I_{m_n} \sin \alpha \cos \theta_n + \\
& I_{m_n} \cos \alpha \sin \theta_n \text{ amp}.
\end{aligned}
$$

Collecting terms,

$$
\begin{aligned}
i_0 = {} & [I_{m_1} \cos \theta_1 + I_{m_2} \cos \theta_2 + \cdots + I_{m_n} \cos \theta_n] \sin \alpha \\
& + [I_{m_1} \sin \theta_1 + I_{m_2} \sin \theta_2 + \cdots + I_{m_n} \sin \theta_n] \cos \alpha \\
& \text{amp}
\end{aligned}
\tag{273}
$$

$$
\begin{aligned}
&= I_{m_0} \cos \theta_0 \sin \alpha + I_{m_0} \sin \theta_0 \cos \alpha \text{ amp} \\
&= I_{m_0} \sin (\alpha + \theta_0) \text{ amp},
\end{aligned}
\tag{274}
$$

where

$$
I_{m_0} \cos \theta_0 = I_{m_1} \cos \theta_1 + I_{m_2} \cos \theta_2 + \cdots + I_{m_n} \cos \theta_n \text{ amp}, \tag{275}
$$

and

$$
I_{m_0} \sin \theta_0 = I_{m_1} \sin \theta_1 + I_{m_2} \sin \theta_2 + \cdots + I_{m_n} \sin \theta_n \text{ amp}. \tag{276}
$$

Dividing both sides of Eqs. (275) and (276) by $\sqrt{2}$, to reduce from maximum to effective values of current,

$$I_0 \cos \theta_0 = I_1 \cos \theta_1 + I_2 \cos \theta_2 + \ . \ . \ . \ + I_n \cos \theta_n \text{ amp} \quad (277)$$

$$I_0 \sin \theta_0 = I_1 \sin \theta_1 + I_2 \sin \theta_2 + \ . \ . \ . \ + I_n \sin \theta_n \text{ amp.} \quad (278)$$

From Eq. (277), if vectors are used to represent the waves, it is seen that the projection of the total current on the reference axis is equal to the sum of the projections of the several branch current vectors on the same reference axis. In the same manner, from Eq. (278), the projection of the total line current on the axis at right angles with the reference axis is equal to the sum of the projections of the several branch current vectors on the same axis. Hence, Eqs. (277) and (278) may be rewritten

$$i_0 = i_1 + i_2 + i_3 + \ . \ . \ . \ + i_n \text{ amp} \quad (279)$$

and

$$i_0' = i_1' + i_2' + i_3' + \ . \ . \ . \ + i_n' \text{ amp,} \quad (280)$$

where the small letters here designate components of the several currents instead of their instantaneous values. Using vector notation, the branch currents may be written as

$$\begin{aligned} \dot{I}_1 &= i_1 + ji_1' = I_1(\cos \theta_1 + j \sin \theta_1) \text{ amp,} \\ \dot{I}_2 &= i_2 + ji_2' = I_2(\cos \theta_2 + j \sin \theta_2) \text{ amp,} \\ &. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\ \dot{I}_n &= i_n + ji_n' = I_n(\cos \theta_n + j \sin \theta_n) \text{ amp,} \end{aligned} \quad (216)$$

where

$$i_n = I_n \cos \theta_n \text{ amp,} \quad (281)$$

and

$$i_n' = I_n \sin \theta_n \text{ amp,} \quad (282)$$

each current being referred to the common emf,

$$\dot{E} = E + j0 \text{ volts,} \quad (283)$$

used as the reference. The total current will be, by Kirchhoff's laws,

$$\dot{I}_0 = \dot{I}_1 + \dot{I}_2 + \ . \ . \ . \ + \dot{I}_n \text{ amp,} \quad (284)$$

or

$$\begin{aligned}\dot{I}_0 &= (i_1 + ji_1') + (i_2 + ji_2') + \ . \ . \ . + (i_n + ji_n') \text{ amp} \\ &= (i_1 + i_2 + \ . \ . \ . + i_n) + j(i_1' + i_2' + \ . \ . \ . + i_n') \\ &= i_0 + ji_0' \text{ amp}, \qquad (285)\end{aligned}$$

where

$$i_0 = (i_1 + i_2 + \ . \ . \ . + i_n) \text{ amp}, \qquad (279)$$

and

$$i_0' = (i_1' + i_2' + \ . \ . \ . + i_n') \text{ amp}. \qquad (280)$$

The currents represented by these equations and corresponding to the circuit of Fig. 65 are represented by a vector diagram in Fig. 66. The components of the branch-circuit currents as i_1, i_2, i_3, etc., and of the total current i_0, are called the "in-phase components" of the currents, and the components i_1', i_2', i_3', i_0', etc., of these currents are called the "quadrature components." It is seen that the *in-phase component of the total current* i_0 *is equal to the sum of the in-phase components of the branch currents and that the quadrature component of the total current* i_0' *is the algebraic sum of the quadrature components of the branch currents.* The branch currents are in phase with, behind, or ahead of the emf according to whether the individual branch is nonreactive, inductive, or capacitive. When the emf is

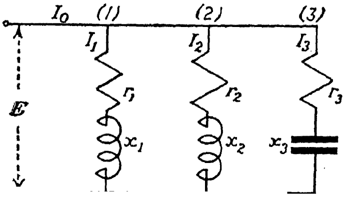

FIG. 65.—A simple parallel circuit.

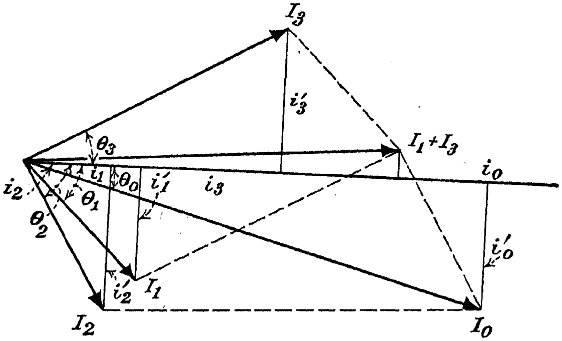

FIG. 66.—Vector diagram for a parallel circuit.

taken as the reference, the in-phase component of current is the projection of the current vector upon the emf line. This component is in phase with the emf and is known as the "active component" of the current. The quadrature component in this case is at 90 deg. with the emf and is known as the "reactive component" of the current. The sign of the active component of the current is positive for power input to the circuit, that of the reactive component is positive for capacitive loads and negative for inductive loads. The sum of the active components, as indicated in Eq. (262), is an algebraic sum, and the sign of the reactive component of the total current may be positive or negative depending upon whether the sum of the numerical values of i' with positive signs is greater or less than the sum of those with negative signs.

Active Component.—The active component of current in a circuit is the active power divided by the voltage. It is the projection of the current upon the emf.

Reactive Component.—The reactive component of current in a circuit is the square root of the difference between the square of the current and the square of the active component of the current.

The magnitude or ammeter reading of the total current is

$$I_0 = \sqrt{i_0^2 + i_0'^2} \text{ amp.} \tag{286}$$

The vector relations of these currents is shown in Fig. 66, corresponding to the circuit of Fig. 65. The total current is the vector sum of the currents in the several branches or loads and is less than the numerical sum of the several load currents unless these branch currents are in phase with each other. This fact has an important bearing on the total load current to be supplied by any source of energy, and consequently upon the investment required for the equipment necessary to furnish this energy as well as upon the losses in the equipment and line wires. As will be seen later, advantage is taken of this condition in the control of transmission-line voltages and losses.

64. Power Consumed and Power Factor.—The power consumed by each of the branches becomes

$$\begin{aligned} P_1 &= EI_1 \cos \theta_1 = Ei_1 \text{ watts}, \\ P_2 &= EI_2 \cos \theta_2 = Ei_2 \text{ watts}, \\ &\cdots\cdots\cdots\cdots \\ P_n &= EI_n \cos \theta_n = Ei_n \text{ watts}, \end{aligned} \tag{287}$$

and the total power consumed is

$$\begin{aligned} P_0 &= P_1 + P_2 + \ . \ . \ . \ + P_n \text{ watts} && (288) \\ &= Ei_1 + Ei_2 + \ . \ . \ . \ + Ei_n \\ &= E(i_1 + i_2 + \ . \ . \ . \ + i_n) \\ &= Ei_0 = EI_0 \cos \theta_0 \text{ watts.} && (289) \end{aligned}$$

From Eqs. (287) and (289), it is seen that the power factor of each individual circuit is

$$P.F._n = \frac{P_n}{EI_n} = \frac{Ei_n}{EI_n} = \frac{i_n}{I_n}, \tag{290}$$

and of the complete circuit it is

$$P.F._0 = \frac{P_0}{EI_0} = \frac{Ei_0}{EI_0} = \frac{i_0}{I_0}. \tag{291}$$

If the power consumed and the power factors of the loads are known, the active components of the currents may be found by definition as

$$i_n = \frac{P_n}{E} = \frac{EI \cos \theta_n}{E} = I_n \cos \theta_n \text{ amp} \tag{292}$$

and

$$i_0 = \frac{P_0}{E} = \frac{EI_0 \cos \theta_0}{E} = I_0 \cos \theta_0 \text{ amp.} \tag{293}$$

The reactive components of the currents are, by definition,

$$i_n' = \sqrt{I_n^2 - i_n^2} = I_n \sin \theta_n \text{ amp,} \tag{294}$$

$$i_0' = \sqrt{I_0^2 - i_0^2} = I_0 \sin \theta_0 \text{ amp.} \tag{295}$$

Example 21

Four circuits taking the currents,

I_1 = 100 amp, nonreactive,
I_2 = 100 amp at 0.8 P.F., lagging current,
I_3 = 100 amp at 0.6 P.F., leading current,
I_4 = 40 amp at 0.2 P.F., lagging current,

respectively, are supplied from a 1,000-volt a-c line. What is the total current and its power factor?

From Eqs. (292) and (294),

$$\begin{aligned} i_1 &= 100 \text{ amp,} & i_1' &= 0 \text{ amp,} \\ i_2 &= 100 \times 0.8 = 80 \text{ amp,} & i_2' &= -\sqrt{100^2 - 80^2} = -60 \text{ amp,} \\ i_3 &= 100 \times 0.6 = 60 \text{ amp,} & i_3' &= \sqrt{100^2 - 60^2} = 80 \text{ amp,} \\ i_4 &= 40 \times 0.2 = 8 \text{ amp,} & i_4' &= -\sqrt{40^2 - 8^2} = -39.2 \text{ amp.} \end{aligned}$$

The components of the total current are

$$i_0 = 100 + 80 + 60 + 8 = 248 \text{ amp},$$

and

$$i_0' = -60 + 80 - 39.2 = -19.2 \text{ amp},$$

and its magnitude is

$$I_0 = \sqrt{\overline{248}^2 + \overline{19.2}^2} = 248.7 \text{ amp}.$$

Its power factor is

$$P.F._0 = \frac{248}{248.7} = 0.997.$$

The total current lags slightly behind the emf E, as indicated by the negative sign before the reactive component. These currents and their components are illustrated in Fig. 67.

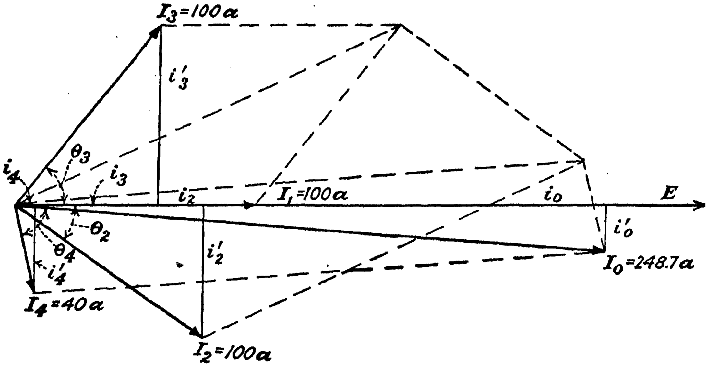

FIG. 67.—Vector diagram for the currents of Example 21.

65. Admittance Method of Considering Parallel Circuits.—If the impedances of the branches are given, it is necessary to determine each branch current, the total current, and the individual and total power factors. For convenience, use E, the common emf, as the reference, and take the values of the branch impedances as

$$\begin{aligned} \dot{z}_1 &= r_1 + jx_1 \text{ ohms}, \\ \dot{z}_2 &= r_2 + jx_2 \text{ ohms}, \\ &\dots\dots\dots\dots \\ \dot{z}_n &= r_n + jx_n \text{ ohms}, \end{aligned} \qquad (200)$$

where $\dot{z}_1, \dot{z}_2 \ldots \dot{z}_n$ represent any impedances with either inductive or capacitive reactance or any combinations of these.

The signs of the numerical values of the constants of such circuits will, of course, be determined by the particular kind of circuit considered, as was discussed in the chapter on series circuits. The individual branch currents will be found by the application of Eq. (133), using the emf E as the reference as

$$\dot{I}_1 = \frac{E}{\dot{z}_1} = \frac{E}{r_1 + jx_1} \text{ amp} \tag{296}$$

$$= \frac{E(r_1 - jx_1)}{r_1^2 + x_1^2} = E\left[\frac{r_1}{r_1^2 + x_1^2} + j\frac{-x_1}{r_1^2 + x_1^2}\right] \tag{297}$$

$$= E[g_1 + jb_1] = E\dot{y}_1 = i_1 + ji_1' \text{ amp}, \tag{298}$$

where

$$g_1 = \frac{r_1}{r_1^2 + x_1^2} = \frac{r_1}{z_1^2} \text{ mhos}, \tag{299}$$

and

$$b_1 = \frac{-x_1}{r_1^2 + x_1^2} = \frac{-x_1}{z_1^2} \text{ mhos}, \tag{300}$$

or

$$\dot{y}_1 = g_1 + jb_1 \text{ mhos}, \tag{301}$$

and

$$y_1 = \sqrt{g_1^2 + b_1^2} \text{ mhos}. \tag{302}$$

From Eq. (298), the components of the current are

$$i_1 = Eg_1 \text{ amp}, \tag{303}$$

and

$$i_1' = Eb_1 \text{ amp}, \tag{304}$$

the magnitude of the current being

$$I_1 = \sqrt{i_1^2 + i_1'^2} = Ey_1 \text{ amp}. \tag{305}$$

From Eqs. (297) to (305), the relations of the current components in the branch circuits, in respect to the constants of these circuits, are established, and from these relations the definitions of the new constants g, b, and y for parallel circuits are obtained as follows:

Conductance.—Conductance is the factor by which the emf across a circuit is multiplied to obtain the component of the current (the active component) in phase with the emf. Its value is equal to the resistance divided by the square of the impedance of the circuit, and its unit is the reciprocal ohm or mho.

Susceptance.—Susceptance is the factor by which the emf across a circuit is multiplied to obtain the component of current (the reactive component) at 90 *deg. with that emf. Its value is equal to the negative of the value of the reactance divided by the square of the impedance, and its unit is the reciprocal ohm or mho.*

Admittance.—Admittance is the factor by which the emf across a circuit is multiplied to obtain the current flowing in that circuit. Its value is equal to the reciprocal of the impedance, and its unit is the reciprocal ohm or mho.

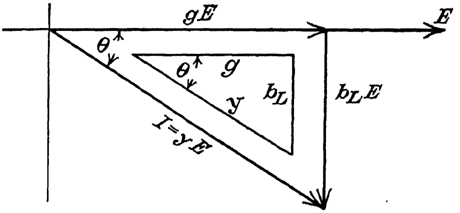

FIG. 68.—Vector and admittance diagram for an inductive circuit.

The quantities conductance, susceptance, and admittance are used, for convenience, for the calculation of parallel circuits to replace their equivalent expressions. They have the same physical significance as the factors resistance, reactance, and impedance used with series circuits, since they are defined in terms of these factors. In d-c circuits, the conductance is defined as the reciprocal of the resistance. From the new definition and Eq. (299), this is seen to hold only at the limiting frequency of zero where x becomes zero or for a circuit where x^2 may be neglected as small with respect to r^2. The susceptance is defined with the negative sign before the value of x in the numerator, in order to retain the positive sign before the j in Eq. (300). This is done also so that the numerical value of x, determined by the characteristic of the reactance, may be substituted with its proper sign in this definition to give the proper sign to the numerical value of b. The sign of the numerical value of b must also agree with the position of the reactive component of the current in reference to the emf impressed on the branch. The emf being used as the reference, the current vector diagram for an inductive circuit is shown in Fig. 68. If the vector quantities in Fig. 68 are divided by the voltage, the admittance triangle is obtained. The sign of the numerical value of b must be the same as that of the reactive component of current from which it is derived. From the preceding equations and definitions,

$$\dot{I}_n = \frac{E}{r_n + jx_n} \tag{296}$$

$$= E[g_n + jb_n] = Ey_n \text{ amp}, \tag{298}$$

where

$$g_n = \frac{r_n}{r_n^2 + x_n^2} \text{ mhos} \tag{299}$$

and

$$b_n = \frac{-x_n}{r_n^2 + x_n^2} \text{ mhos.} \tag{300}$$

The total current taken by the circuit is

$$\dot{I}_o = \dot{I}_1 + \dot{I}_2 + \cdots + \dot{I}_n \text{ amp} \tag{284}$$

$$= E[\dot{y}_1 + \dot{y}_2 + \cdots + \dot{y}_n] \text{ amp}; \tag{306}$$

or

$$\dot{I}_o = E[(g_1 + jb_1) + (g_2 + jb_2) + \cdots + (g_n + jb_n)] \text{ amp}$$

$$= E[(g_1 + g_2 + \cdots + g_n) + j(b_1 + b_2 + \cdots + b_n)] \text{ amp.} \tag{307}$$

This current may be expressed as

$$\dot{I}_0 = E(g_0 + jb_0) = E\dot{y}_0 \text{ amp}, \tag{298}$$

since

$$\dot{y}_0 = \dot{y}_1 + \dot{y}_2 + \cdots + \dot{y}_n \text{ mhos}, \tag{308}$$

where

$$g_0 = g_1 + g_2 + \cdots + g_n \text{ mhos}, \tag{309}$$

and

$$b_0 = b_1 + b_2 + \cdots + b_n \text{ mhos}, \tag{310}$$

from Eqs. (307) and (308). The ammeter reading or magnitude of the total current taken by the circuit becomes

$$I_0 = \sqrt{i_0^2 + i_0'^2} \text{ amp}, \tag{286}$$

where

$$i_0 = Eg_0 \text{ amp}, \tag{303}$$

and

$$i_0' = Eb_0 \text{ amp}. \tag{304}$$

Hence, in a parallel circuit the total conductance is equal to the sum of the individual conductances, and the total susceptance is equal to the sum of the individual susceptances of the branches. From these values of admittance and its components, the admittance diagram may be drawn. In this diagram, it should be noted that inductance causes the current to lag the impressed emf, the corresponding numerical value of b being negative, as shown in Fig. 68; whereas capacitance causes the current to lead the impressed emf, the corresponding numerical value of b being positive.

Finding the total admittance y_0 for a group of parallel admittances gives the admittance of one circuit that is equivalent to the combined group. The equivalent admittance may be substituted for the group under the given conditions in determining the current and potential relations associated with the group as a whole, *i.e.*, at the points of entry into the circuit.

The power consumed in a circuit when the constants of the circuit and the current flowing are known is usually obtained from the expression

$$P = RI^2 \text{ watts}. \tag{311}$$

This power may be computed directly when the conductance and the impressed emf are known, since from Eq. (311)

$$P = \frac{RE^2}{Z^2} = \frac{R}{Z^2}E^2 = gE^2 \text{ watts}. \tag{312}$$

Example 22

Three impedances represented by

$$\dot{z}_1 = 0.8 + j0.6 \text{ ohms},$$
$$\dot{z}_2 = 0.3 - j0.4 \text{ ohms},$$

and

$$\dot{z}_3 = 1.0 + j0 \text{ ohms},$$

are connected in parallel across a 100-volt 60-cycle a-c circuit; find the branch currents and the total current.

$$g_1 = \frac{0.8}{0.8^2 + 0.6^2} = 0.8 \text{ mho}, \qquad i_1 = 100 \times 0.8 = 80 \text{ amp},$$
$$b_1 = \frac{-0.6}{0.8^2 + 0.6^2} = -0.6 \text{ mho}, \qquad i_1' = 100 \times (-0.6) = -60 \text{ amp},$$
$$y_1 = \sqrt{0.8^2 + (-0.6)^2} = 1.0 \text{ mho}, \qquad I_1 = 100 \times 1.0 = 100 \text{ amp},$$
$$g_2 = \frac{0.3}{0.3^2 + 0.4^2} = 1.2 \text{ mhos}, \qquad i_2 = 100 \times 1.2 = 120 \text{ amp},$$
$$b_2 = \frac{0.4}{0.3^2 + 0.4^2} = 1.6 \text{ mhos}, \qquad i_2' = 100 \times 1.6 = 160 \text{ amp},$$
$$y_2 = \sqrt{\overline{1.2}^2 + \overline{1.6}^2} = 2.0 \text{ mhos}, \qquad I_2 = 100 \times 2.0 = 200 \text{ amp},$$
$$g_3 = \frac{1.0}{1.0} = 1.0 \text{ mho}, \qquad i_3 = 100 \times 1.0 = 100 \text{ amp},$$
$$b_3 = \frac{0}{1.0} = 0 \text{ mho}, \qquad i_3' = 100 \times 0 = 0 \text{ amp},$$
$$y_3 = 1.0 \text{ mho}, \qquad I_3 = 100 \times 1.0 = 100 \text{ amp}.$$

Hence, the values of conductance, susceptance, and admittance for the total circuit become

$$g_0 = 0.8 + 1.2 + 1.00 = 3.0 \text{ mhos},$$
$$b_0 = -0.6 + 1.6 + 0 = 1.0 \text{ mho},$$

and

$$y_0 = \sqrt{\overline{3.0}^2 + \overline{1.0}^2} = 3.17 \text{ mhos},$$

and the total current and its components,

$$i_0 = 100 \times 3.0 = 300 \text{ amp},$$
$$i_0' = 100 \times 1.0 = 100 \text{ amp},$$

and

$$I_0 = \sqrt{\overline{300}^2 + \overline{100}^2} = 317 \text{ amp} = 100 \times 3.17 \text{ amp}.$$

The power factor of the circuit is

$$P.F._0 = \frac{300}{317} = 0.945.$$

The vectors of current are illustrated in Fig. 69, the admittance diagram being shown in Fig. 70.

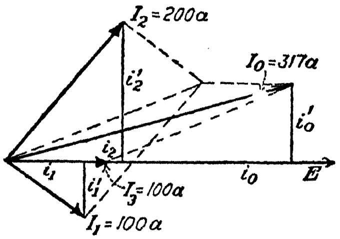

FIG. 69.—Vector diagram for Example 22.

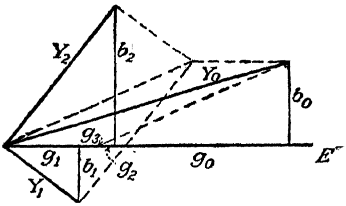

FIG. 70.—Admittance diagram for Example 22.

Example 23

Three impedances $\dot{Z}_1 = 3 + j4$, $\dot{Z}_2 = 6 + j8$, and $\dot{Z}_3 = 0 - j20$ ohms are connected in parallel to a 120-volt 60-cycle circuit. Find the admittance of each path, the admittance of a circuit equivalent to these three paths in parallel, and the current flowing in the circuit.

From the above impedances,

$$\dot{y}_1 = \frac{1}{3 + j4} = 0.12 - j0.16 \text{ mho},$$
$$\dot{y}_2 = \frac{1}{6 + j8} = 0.06 - j0.08 \text{ mho},$$
$$\dot{y}_3 = \frac{1}{0 - j20} = 0 + j0.05 \text{ mho},$$
$$\dot{y}_0 = \dot{y}_1 + \dot{y}_2 + \dot{y}_3 = 0.18 - j0.19 \text{ mho}.$$

Taking the impressed emf as the reference,

$$\dot{I}_0 = E\dot{y}_0 = 120(0.18 - j0.19) \text{ amp}$$
$$= 21.6 - j22.8 \text{ amp},$$

having the magnitude,

$$I_0 = \sqrt{21.6^2 + 22.8^2} = 31.4 \text{ amp}.$$

66. Resonance in Parallel Circuits.—In a parallel circuit as in a series circuit, a resonant condition is produced when the effect of the inductive reactance is equal to the effect of the capacitive reactance so that the sum of the currents (the total current) is in phase with the impressed emf. In Fig. 71, let the circuits 1 and 2 be in resonance. Representing the currents in these

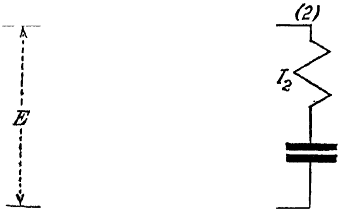

FIG 71.—A resonant parallel circuit.

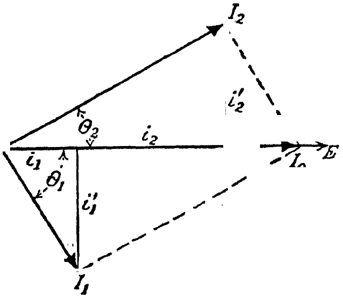

FIG. 72.—Vector diagram for a resonant parallel circuit.

circuits by

$$\dot{I}_1 = i_1 + ji_1' \text{ amp,}$$
$$\dot{I}_2 = i_2 + ji_2' \text{ amp,} \tag{216}$$

and the total current as the sum of these branch currents as

$$\dot{I}_0 = \dot{I}_1 + \dot{I}_2 \text{ amp} \tag{284}$$

or

$$= (i_1 + i_2) + j(i_1' + i_2') = i_0 + ji_0' \text{ amp,} \tag{285}$$

resonance can occur only when this total current is in phase with the impressed emf E, or when

$$i_0' = i_1' + i_2' = 0 \text{ amp,} \tag{313}$$

or as seen in Fig. 72. Hence, from Eq. (313)

$$i_1' = -i_2' \text{ amp} \tag{314}$$

is the condition of resonance in this circuit. Dividing both sides of Eq. (314) by E,

$$\frac{i_1'}{E} = \frac{-i_2'}{E} \text{ amp,}$$

or

$$b_1 = -b_2 \text{ mhos,} \tag{315}$$

corresponding to Fig. 72. Hence, substituting the values of b_1 and b_2 from Eq. (300),

$$\frac{-x_1}{r_1^2 + x_1^2} = \frac{x_2}{r_2^2 + x_2^2} \text{ mhos} \tag{316}$$

is the condition of resonance in terms of the resistances and reactances of the circuits. It is seen from Eq. (316) that x_1 and x_2 must be opposite in sign or that if x_1 is an inductive reactance, x_2 must be a capacitive reactance. Letting

$$x_1 = 2\pi fL = \omega L \text{ ohms,} \tag{71}$$

and

$$x_2 = \frac{-1}{2\pi fC} = \frac{-1}{\omega C} \text{ ohms,} \tag{118}$$

and, substituting these values into Eq. (316),

$$\frac{-\omega L}{r_1^2 + \omega^2 L^2} = \frac{\dfrac{-1}{\omega C}}{r_2^2 + \left(\dfrac{-1}{\omega C}\right)^2},$$

$$\frac{\omega L}{r_1^2 + \omega^2 L^2} - \frac{\omega C}{r_2^2\omega^2 C^2 + 1} = 0,$$

$$\omega\left[\frac{L}{r_1^2 + \omega^2 L^2} - \frac{C}{r_2^2\omega^2 C^2 + 1}\right] = 0,$$

$$\omega[\omega^2 C^2 r_2^2 L + L - C r_1^2 - \omega^2 L^2 C] = 0,$$

$$\omega[\omega^2 LC(Cr_2^2 - L) - (Cr_1^2 - L)] = 0. \qquad (317)$$

There are three conditions that satisfy Eqs. (317) and (316), *viz.,*

$$\omega = 0 \text{ or } f = 0,$$

$$\omega = \frac{1}{\sqrt{LC}}\sqrt{\frac{Cr_1^2 - L}{Cr_2^2 - L}} \text{ or } f = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{(Cr_1^2 - L)}{(Cr_2^2 - L)}}, \qquad (318)$$

and

$$r_1 = r_2 = \sqrt{\frac{L}{C}}. \qquad (319)$$

The condition shown in Eq. (319) is interesting, since when

$$r_1^2 = r_2^2 = r^2 = \frac{L}{C}, \qquad (319)$$

$$b_1 = \frac{-x_1}{r_1^2 + x_1^2} = \frac{-\omega L}{\dfrac{L}{C} + \omega^2 L^2}$$

$$= \frac{-\omega LC}{L(\omega^2 LC + 1)} = \frac{-\omega C}{\omega^2 LC + 1} \qquad (320)$$

and

$$b_2 = \frac{-x_2}{r_2^2 + x_2^2} = \frac{1/\omega C}{\dfrac{L}{C} + \dfrac{1}{\omega^2 C^2}}$$

$$= \frac{\omega C}{\omega^2 LC + 1}; \qquad (321)$$

hence

$$b_0 = b_1 + b_2 = 0,$$

for all values of ω, and *the circuit is in resonance for all values of frequency.* Also, from this condition,

$$g_1 = \frac{r_1}{r_1^2 + x_1^2} = \frac{r}{r^2 + \omega^2 L^2},$$

$$g_2 = \frac{r_2}{r_2^2 + x_2^2} = \frac{r}{r^2 + \dfrac{1}{\omega^2 C^2}},$$

and since

$$C = \frac{L}{r^2},$$

$$g_2 = \frac{r}{r^2 + \dfrac{r^4}{\omega^2 L^2}} = \frac{r\omega^2 L^2}{r^2\omega^2 L^2 + r^4} = \frac{\omega^2 L^2}{r(r^2 + \omega^2 L^2)}, \tag{322}$$

and

$$\begin{aligned} g_0 &= g_1 + g_2 \\ &= \frac{r}{r^2 + \omega^2 L^2} + \frac{\omega^2 L^2}{r(r^2 + \omega^2 L^2)} \\ &= \frac{r^2 + \omega^2 L^2}{r(r^2 + \omega^2 L^2)} = \frac{1}{r}. \end{aligned} \tag{323}$$

Therefore, at any frequency the total line current will be

$$\dot{I}_0 = \dot{E}\dot{Y}_0 = \dot{E}\left(\frac{1}{r} + j0\right) = \frac{E}{r}, \tag{324}$$

a constant quantity and in phase with the emf impressed.

The condition shown in Eq. (318) gives the value of frequency to produce resonance in a parallel circuit, except for the special case of Eq. (319). This equation may be written

$$f = \frac{1}{2\pi\sqrt{LC}}K,$$

where

$$K = \sqrt{\frac{Cr_1^2 - L}{Cr_2^2 - L}}, \tag{325}$$

and $1/2\pi\sqrt{LC}$ is the value of frequency to produce resonance in a series circuit having the same values of L and C. Where r_1 and r_2 are equal and have any other value than $\sqrt{L/C}$, Eq. (318) reduces to Eq. (134), or the resonant frequency is the same as for a series circuit.

Resonance in a parallel power circuit is a desirable condition since, at resonance, the total current will be the least and its power factor will be unity. Resonance in a series power circuit may be an undesirable condition since this condition is usually accompanied by dangerous voltages across some part of the circuit.

Problems

1-4. Three circuits taking the currents $I_1 = 100$ amp at 0.90 power factor, lagging current; $I_2 = 50$ amp at 0.85 power factor, lagging current; and $I_3 = 100$ amp at 0.50 power factor, leading current; respectively, are supplied from a 40,000-volt a-c line. What is the total current and its power factor? Draw the complete vector diagram, using the impressed emf as the reference.

2-4. A coil having a resistance of 10 ohms and a reactance of 10 ohms is connected in parallel with a resistor having a resistance of 50 ohms. If a current $\dot{I} = 100 - j100$ flows in the coil, what are the total current, the impressed emf, and the current through the resistance? Draw the vector diagram.

3-4. A branch having a resistance of 20 ohms and a capacitive reactance of 20 ohms is connected in parallel with a resistance of 50 ohms across a 480-volt 60-cycle circuit. What are (*a*) the total current, (*b*) the total active power consumed, and (*c*) the power factor?

4-4. A resistance of 12 ohms, an inductive reactance of 10 ohms, and a capacitive reactance of 10 ohms are connected in parallel across a 250-volt 60-cycle a-c circuit. (*a*) Calculate the values of the total and branch currents. (*b*) Find the active and reactive components of the total current, the power factor, and the total active power consumed. (*c*) Draw the vector diagram, using the impressed emf as the reference. (*d*) Draw the vector diagram with total current as the reference.

5-4. An inductor having a resistance of 10 ohms and an inductance of 0.015 henry is connected in parallel with a capacitor and resistance in series, the capacitance being 400μf and the resistance 12 ohms. An emf of 2,300 volts at 60 cycles is impressed on the circuit. (*a*) Determine the value of each of the currents and the total current. (*b*) Find the active and the reactive components of the total current, the active and reactive power inputs, and the power factor. (*c*) Draw the vector diagram showing E_o, I_o, I_1, and I_2.

6-4. Calculate the values of conductance, susceptance, and admittance for a coil having a self-inductance of $L = 0.036$ henry and a resistance of $R = 8.6$ ohms, when used on a 35-cycle and also on a 750-cycle circuit.

7-4. Calculate the values of conductance, susceptance, and admittance in a 60-cycle circuit having $C = 30$ μf and $R = 40$ ohms.

8-4. Two impedances $\dot{Z}_1 = 6 + j8$ and $\dot{Z}_2 = 0 - j8$ are connected in parallel across a 400-volt circuit. Find the values of the branch and total currents using the admittance method of calculation. Draw the vector and admittance diagrams.

9-4. A coil having a resistance of 0.5 ohm and an inductance of 0.0082 henry is connected in parallel with a capacitor having a capacitance of 6.3×10^{-3} farad on a 110-volt 25-cycle circuit. Find the current in each branch and the total current using the admittance method of calculation. Draw the admittance and vector diagrams.

10-4. Two impedances, $\dot{Z}_1 = 12\underline{/45°}$ and $\dot{Z}_2 = 5\underline{/-60°}$, are connected in parallel across a 110-volt 60-cycle circuit. Find the current in each branch and the total current, using the admittance method of calculation. Draw the admittance and vector diagrams.

11-4. A circuit having a resistance of 20 ohms, an inductance of 0.2 henry, and a capacitance of 200 μf is connected in parallel with a reactor having $R = 5$ ohms and $X_L = 5$ ohms across a 2,500-volt 60-cycle a-c circuit. Find the total current and the current in each branch. Draw the vector diagram, using the impressed emf as the reference, and the admittance diagram.

12-4. An emf $\dot{E}_o = 500 - j866$ is impressed on three impedances in parallel, the values of the impedances being $\dot{Z}_1 = 30 + j110$, $\dot{Z}_2 = 50 + j0$, and $\dot{Z}_3 = 10 - j10$. Find the total current and the current in each branch. Draw the admittance and vector diagrams.

13-4. Three impedances $\dot{Z}_1 = 16\underline{/60°}$, $\dot{Z}_2 = 5\underline{/45°}$, and $\dot{Z}_3 = 10\underline{/-30°}$ are connected in parallel across a 240-volt a-c circuit. Find I_o and each of the branch currents.

14-4. Two impedances $\dot{Z}_1 = 0 - j10$ and $\dot{Z}_2 = 10 + j10$ are connected in parallel across an a-c circuit. The total current taken by the combination is $\dot{I}_o = 100 + j100$. Find the value of the current in each branch and the impressed emf, and draw the complete vector and admittance diagrams.

15-4. A circuit with a voltage $\dot{E}_o = 200 + j0$ volts has a total current of $\dot{I}_o = 60 + j30$ amp. It is supplying two branches, the impedance of one branch being $\dot{Z}_1 = 3 + j4$ ohms. Determine the impedance of the other branch and the current in each of the branches. Solve the problem by the admittance method. Draw the vector and admittance diagrams.

16-4. An inductor having a resistance of 20 ohms and a reactance of 40 ohms is connected in parallel with a branch having a resistance of 5 ohms and a capacitive reactance of 10 ohms. A current of 100 amp flows through the second branch; determine the total current and the impressed emf, and draw the complete vector diagram, using the current in the second branch as the reference.

17-4. Three admittances $\dot{Y}_1 = 0.05\underline{/45°}$, $\dot{Y}_2 = 0.20\underline{/-40°}$, and $\dot{Y}_3 = 0.10\underline{/0°}$ are connected in parallel across an a-c circuit. The current in branch one is $\dot{I}_1 = 100\underline{/0°}$, when I_1 is used as the reference; find the other branch currents and the total current and the impressed emf. Draw the vector diagram, with I_1 as the reference.

18-4. A resistance of 7.9 ohms, an inductance of 0.012 henry, and a capacitance of 93 μf are placed in parallel across a variable-frequency 115-volt circuit. What frequency is required to produce resonance? Calculate the current in each branch and the total current at resonant frequency. Draw the vector diagram and the wave forms for this condition. Find the branch and total currents at 20 cycles above and below resonant frequency.

19-4. A capacitor with a capacitance $C_1 = 40$ μf and an inductor having a resistance of 10 ohms and an inductance of 0.2 henry are placed in parallel across a 2,300-volt circuit. Find the frequency at which parallel-phase resonance occurs. Find the values of the total and branch currents at this frequency. Draw the vector diagram.

20-4. A coil having a resistance of 20 ohms and an inductance of 0.05 henry is connected in parallel with a circuit having a resistance of 10 ohms in series with a capacitor having a capacitance of 100 μf, 200 volts at resonant frequency being impressed across the two branches. Find the frequency for resonance, the total current, and the current in each branch. Draw the complete admittance diagram.

21-4. A coil having a resistance of 10 ohms and an inductance of 0.2 henry is connected in parallel with a capacitor across a 230-volt circuit. Find the capacitance of the capacitor required to produce resonance at 60 cycles. Determine the vector values and magnitudes of the currents in both branches and the total current. Draw the vector diagram.

22-4. Two impedances $\dot{Z}_1 = 10 + j8$ and $\dot{Z}_2 = 12 - j20$ are connected in parallel, and a current $\dot{I}_1 = 300 + j400$ flows in the first impedance. Find the current in the second impedance, the impressed emf, and the total current.

23-4. Same as Prob. 4-4 but with 250 volts at 25 cycles impressed.

24-4. A coil having a resistance of 2 ohms and an inductance of 0.0016 henry is connected in parallel with a capacitor and resistor in series, the capacitance being 4×10^{-3} farad and the resistance 1.8 ohms. An emf of 87 volts at 625 cycles is impressed on the circuit. (*a*) Determine the value of each of the branch currents and the total current. (*b*) Find the active and reactive components of the total current, the active and reactive power input and the power factor. (*c*) Draw the vector diagram.

25-4. A voltage $\dot{E} = 230\underline{/-40^\circ}$ is impressed on three impedances in parallel where $\dot{Z}_1 = 8\underline{/70^\circ}$, $\dot{Z}_2 = 15\underline{/-20^\circ}$, and $\dot{Z}_3 = 5\underline{/30^\circ}$. What are the current taken by each branch, the total current, and total admittance of the circuit? Draw the admittance and vector diagrams.

26-4. Two admittances Y_1 and Y_2 are placed in parallel across an a-c circuit. The emf applied is $E = 110$ volts, and the total current is $I_0 = 21.8$ amp. The power factor is 0.60 with current lagging, and the admittance is $\dot{Y}_1 = 0.1 + j0.1$; find the admittance Y_2.

27-4. An emf $\dot{E}_0 = 500 - j866$ is impressed across three impedances in parallel, a current $\dot{I}_1 = 30 - j10$ flowing in the first impedance. If the values of the other impedances are $\dot{Z}_2 = 100 + j0$ and $\dot{Z}_3 = 10 - j10$, find the current in each of these impedances and the total current.

28-4. A coil having a resistance of 40 ohms and an inductance of 0.3 henry is connected in parallel with a circuit having a resistance of 40 ohms and a capacitance of 100 μf. Find the frequency to give resonance. Calculate the total admittance of the circuit for frequencies of 10, 30, 40, 60, and 100 cycles.

29-4. An inductor having a resistance of 12 ohms and a reactance of 8 ohms is placed in parallel on a 220-volt 60-cycle circuit with a capacitor having a capacitance of such a value as to produce resonance. What is the capacitance of the capacitor? Find the current in each branch and the total current. Draw the vector diagram and the wave forms.

CHAPTER V

ACTIVE POWER, REACTIVE POWER, VECTOR POWER, APPARENT POWER, AND POWER DIAGRAMS

This chapter presents a method, which will be found convenient in many cases, for the rapid solution of composite single-phase circuits with fundamental symmetrical alternating currents and voltages by the use of power diagrams. This method will also be found useful for polyphase circuits with balanced loads where only fundamentals of voltages and currents are to be considered.

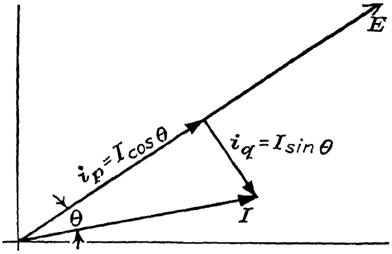

FIG. 73.—Emf and current in an inductive circuit.

67. The Power Diagram.—The emf impressed on an inductive circuit and the current flowing in the circuit are shown in Fig. 73, the phase displacement being indicated by the angle θ. Projecting the current vector upon the emf vector gives $i_p = I \cos \theta$, the active component of current. Projecting the current upon a line at right angles to the emf gives $i_q = I \sin \theta$, the reactive component of current, the two components with I forming a vector triangle of currents. The relations expressed by the diagram are independent of the reference axis, and so, for convenience in making calculations. the emf may be chosen as the reference, as in Fig. 74.

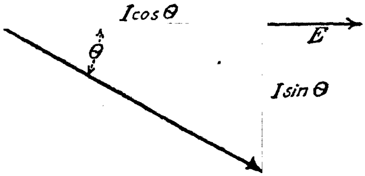

FIG. 74.—Vector diagram for an inductive circuit, emf as the reference.

If each side of the current diagram of Fig. 74 be multiplied by the emf impressed on the circuit, Fig. 75 results, having the same relative sides as Fig. 74, but to a different scale, that of power. The components of current for Fig. 74 may be expressed as

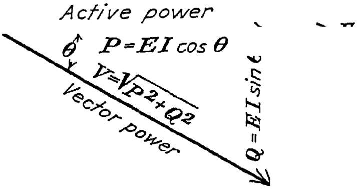

FIG. 75.—Power diagram for the circuit of Fig. 74.

$$\dot{I} = I \cos \theta + jI \sin \theta \text{ amp,} \qquad (326)$$

which, when multiplied by E, gives the equation

$$E\dot{I} = EI \cos \theta + jEI \sin \theta \text{ vector volt-amp.} \qquad (327)$$

Equation (327) may be expressed as

$$\dot{V} = P + jQ \text{ vector volt-amp,} \qquad (328)$$

where P is the active power in the circuit;
Q is the reactive power in the circuit; and
V is the vector power in the circuit.

The magnitude of V is given by

$$V = \sqrt{P^2 + Q^2} \text{ vector volt-amp..} \qquad (329)$$

Active Power.—Active power is the time average of the values of instantaneous power when the average is taken over a complete cycle of alternating current.

Active power was shown in Art. 22 to be equal to $EI \cos \theta$. Typical illustrations of the power wave were shown in Figs. 16, 17, and 18 for different values of the phase angle θ.

Reactive Power of a Two-wire Single-phase Circuit.—Reactive power of a two-wire single-phase circuit is equal to the product obtained by multiplying the effective value of the current by the effective value of the potential difference and by the sine of the angular phase difference by which the current leads the potential difference. Under sinusoidal conditions, the definition gives a positive sign to reactive power, if the current leads the potential difference as in the *case when the flow of energy is into a capacitive circuit. The unit of reactive power is the var,* which is derived from the expression "volt-ampere reactive."

In Art. 41, the energy relations are shown for an inductive circuit. It is found that the energy stored in the magnetic field during the first quarter cycle after the current passes through zero, *i.e.*, during the time the current is increasing in magnitude, is returned to the circuit during the next quarter cycle, during which interval the current is decreasing in magnitude. This process is repeated for each half cycle, so that no net energy is required and there is no active power in the circuit. There is, however, a continual flow of energy between the source and the circuit, creating the inductive-reactive power in the circuit. The same phenomenon occurs in the dielectric field of the capacitor, the energy being stored in the field each quarter cycle in which the voltage is building up and being returned to the source during the next quarter cycle when the emf is decreasing, as shown in Art. 42. This continual change in energy creates the capacitive-reactive power in the circuit. It was shown in Art. 42 that the flow of reactive power is opposite in direction in the two types of fields, so that when the two fields occur in one circuit, energy flows from one field to the other, only the excess energy required by one over that required by the other coming from the source of supply.

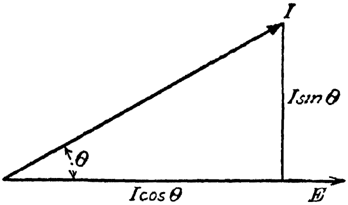

FIG. 76.—Vector diagram for a capacitive circuit, emf as the reference.

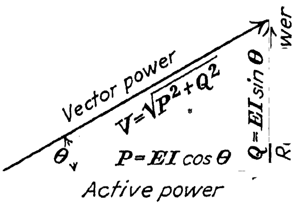

FIG. 77.—Power diagram for the circuit of Fig. 76.

Figure 76 shows the vector diagram for a capacitive load and Fig. 77 the corresponding power diagram. The numerical value of Q, the reactive power, is positive for the capacitive circuit, Fig. 77, and is negative for the inductive circuit, Fig. 75.

Vector Power in a Two-wire Single-phase Circuit.—Vector power is equal in magnitude to the square root of the sum of the squares of the active and reactive power. Vector power is conveniently treated as a vector of which the components are two mutually perpendicular vectors that have the magnitudes of active and reactive power. The unit of vector power is the vector volt-ampere.

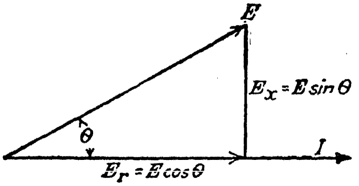

FIG. 78.—Vector diagram for an inductive circuit, current as the reference.

The power diagram may be derived, with current as the reference, as shown in Fig. 79, which is the power diagram corresponding to Fig. 78. The equation for the voltage referred to the current, from Fig. 78, is

$$\dot{E} = E\cos\theta + jE\sin\theta \text{ volts,} \tag{330}$$

which, when multiplied by the current flowing in the circuit, gives

$$\dot{E}I = EI\cos\theta + jEI\sin\theta \text{ volts,} \tag{331}$$

or expressed as

$$\dot{V} = P + jQ \text{ vector volt-amp.} \tag{332}$$

Equation (332) is the same in form as Eq. (328); however, the sign of the numerical value of reactive power in an inductive circuit, as used in Eq. (332), is positive whereas it is negative in Eq. (328).

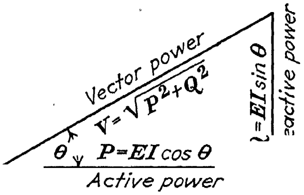

FIG. 79.—Power diagram for the circuit of Fig. 78.

Comparison should be made between Figs. 74 and 78 and also between Figs. 75 and 79. In the case of the diagrams with current as the reference, the reactive power is positive for an inductive circuit; whereas, in the case of the diagrams with emf as the reference, the reactive power is negative (*cf.* the signs of x and b for the same circuit). The sign of the numerical value of reactive power in an inductive circuit may be taken as either positive or negative as long as the initial assumptions are given. Throughout this text the sign of the inductive reactive power will be taken as negative in agreement with present Standardization Rules, A.I.E.E., and with the definition.

There is another type of power that is frequently used in the circuit, the apparent power.

Apparent Power.—Apparent power of a two-wire single-phase circuit is equal to the product of the effective current in one conductor multiplied by the effective potential difference across the circuit. The unit of apparent power is the volt-ampere. The apparent power is, from the above definition,

$$U = EI \text{ volt-amp.} \tag{333}$$

In the single-phase circuit with simple sinusoidal waves of emf and current of the same frequency, the magnitudes of the vector power and the apparent power are equal, *i.e.*,

$$V = \sqrt{P^2 + Q^2} = \sqrt{(EI \cos \theta)^2 + (EI \sin \theta)^2} \text{ vector volt-amp}$$
$$= EI = U \text{ volt-amp.} \tag{334}$$

The power factor of the circuit is defined in Art. 22 as the factor by which the apparent power must be multiplied to obtain the active power. It may also be expressed as the ratio of the active power to the apparent power, *i.e.*, the ratio of P to EI. The term "reactive factor" has been derived in a similar manner to express the relation between the reactive power and the apparent power.

Reactive Factor.—Reactive factor is the ratio of the reactive power to the apparent power. It is the factor by which the apparent power must be multiplied to obtain the reactive power in the circuit. For the single-phase circuit with sinusoidal current and emf it is equal to the sine of the phase angle between current and emf. Its sign may be either positive or negative according to the sign of the reactive power, since the apparent power is always positive.

68. Expressions for Active Power, Reactive Power, and Vector Power.—Expressions for active and reactive power in terms of the in-phase and quadrature components of emf and current may be developed as follows: The active power consumed in any circuit was found to be

$$P = EI \cos \theta \text{ watts}, \tag{51}$$

where θ is the angle between E and I. Let Fig. 80 represent the current and emf vectors of any circuit, both referred to the same axis, and with the phase displacement of θ deg. between them.

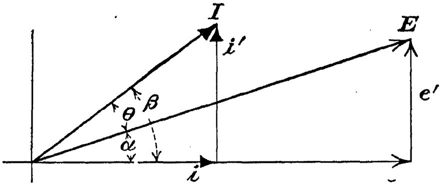

FIG. 80.—Vector diagram for a simple circuit, both E and I being in the first quadrant.

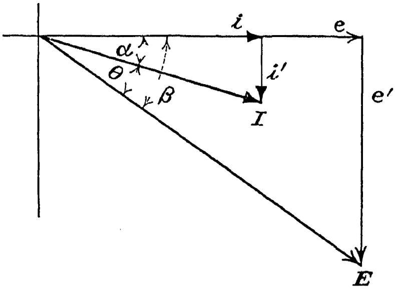

FIG. 81.—Vector diagram for a simple circuit, both E and I in fourth quadrant.

Referring to Fig. 80, the angle θ may be expressed as $\theta = \beta - \alpha$, where β is the angle between I and the reference axis and α is the angle between E and the same reference axis. Hence,

$$\begin{aligned} P &= EI \cos (\beta - \alpha) \text{ watts} && (335) \\ &= EI(\cos \beta \cos \alpha + \sin \beta \sin \alpha) \text{ watts} \\ &= E \cos \alpha I \cos \beta + E \sin \alpha I \sin \beta \text{ watts} \\ &= ei + e'i' \text{ watts}. && (336) \end{aligned}$$

Equation (336) is a general expression for the power in a circuit where neither E nor I is in phase with the axis of reference. In Fig. 80 both E and I are in the first quadrant; however, the derivation is general, holding for any values of the angle θ, whether positive or negative, where α and β are measured in a positive direction from the reference axis. From Eq. (336), *the active power may be expressed as the sum of the product of the in-phase components of current and potential difference and the product of the quadrature components of current and potential difference.* In applying this expression, the signs of the numerical values of the components must be included in the multiplication, and the algebraic sum of the products must be taken. For example, in Fig. 81 the values of emf, current, and phase angle are the same as in Fig. 80. The values of e' and i' are both negative, but the algebraic sum of the products is the same as for Fig. 80. When θ becomes more than 90 deg., the numerical value of $\cos\theta$ becomes negative and the numerical value of power becomes negative.

The expression for the reactive power may be derived in a manner similar to that for the active power. The expression for reactive power in the single-phase circuit is

$$\begin{aligned} Q &= EI \sin(\beta - \alpha) \text{ vars} && (337) \\ &= EI(\sin\beta \cos\alpha - \cos\beta \sin\alpha) \text{ vars} \\ &= E\cos\alpha I \sin\beta - E\sin\alpha I \cos\beta \text{ vars} \\ &= ei' - e'i \text{ vars.} && (338) \end{aligned}$$

This expression is independent of the position of the vectors with respect to the reference axis and gives the positive sign for capacitive-reactive power and the negative sign for inductive-reactive power.

The vector power, which is expressed as

$$\dot{V} = P + jQ \text{ vector volt-amp,} \tag{328}$$

may be written in terms of the components as

$$\dot{V} = ei + e'i' + j(ei' - e'i) \text{ vector volt-amp.} \tag{339}$$

If the vector expression for current is multiplied by the conjugate of the vector expression for the emf, the equation becomes

$$\begin{aligned} \dot{I}\hat{E} &= (i + ji')(e - je') \text{ vector volt-amp} && (340) \\ &= ei + e'i' + j(ei' - e'i) \text{ vector volt-amp.} && (339) \end{aligned}$$

The right-hand member of Eq. (339) is identical with that of the expression for vector power derived by trigonometry, so that Eq. (340) may be written

$$\dot{V} = \dot{I}\hat{E} \text{ vector volt-amp} \tag{341}$$

and the vector power may be found as the product of the vector expression of current times the conjugate of the expression for the potential difference.

Example 24

A current of $\dot{I} = 40 - j30$ is supplied from a station whose voltage, referred to the same axis, is $\dot{E} = 240 + j40$. What are the values of the active power, the reactive power, the apparent power, the power factor, and the reactive factor of the load referred to the station voltage?

From the problem,

$$e = 240 \text{ volts}, \qquad e' = 40 \text{ volts},$$
$$i = 40 \text{ amp}, \qquad i' = -30 \text{ amp}.$$

Then, since

$$P = ei + e'i' \text{ watts}, \tag{336}$$

the active power becomes

$$\begin{aligned} P &= 240 \times 40 + 40 \times (-30) \\ &= 9{,}600 - 1{,}200 = 8{,}400 \text{ watts}. \end{aligned}$$

The reactive power is

$$\begin{aligned} Q &= ei' - e'i \text{ volt-amp} \\ &= 240 \times (-30) - 40 \times 40 \\ &= -7{,}200 - 1{,}600 = -8{,}800 \text{ volt-amp}, \end{aligned} \tag{337}$$

and the apparent power is

$$U = EI,$$

but

and

$$I = \sqrt{\overline{40}^2 + \overline{30}^2} = 50 \text{ amp},$$

or

$$U = 243.3 \times 50 = 12{,}160 \text{ volt-amp}$$

and

$$\begin{aligned} V &= \sqrt{P^2 + Q^2} \\ &= \frac{\sqrt{\overline{8{,}400}^2 + \overline{8{,}800}^2}}{1{,}000} = 12.16 \text{ vector kva}. \end{aligned}$$

The power factor becomes

$$P.F. = \cos\theta = \frac{8{,}400}{12{,}160} = 0.69,$$

and the reactive factor,

$$R.F. = \sin\theta = \frac{-8{,}800}{12{,}160} = -0.725.$$

Example 25

An inductive circuit has an impressed emf of 110 volts and a current of 12 amp flowing in it, the phase displacement between them being 30 deg. Find the active and reactive power if the emf and current are represented by the vectors: (*a*) $\dot{E} = 110\underline{/60°}$ and $\dot{I} = 12\underline{/30°}$; (*b*) $\dot{E} = 110\underline{/150°}$ and $\dot{I} = 12\underline{/120°}$; and (*c*) $\dot{E} = 110\underline{/285°}$ and $\dot{I} = 12\underline{/255°}$.

a. $\dot{E} = 110\underline{/60°} = 55 + j95.3$ volts,

$\dot{I} = 12\underline{/30°} = 10.39 + j6$ amp,

$P = 55 \times 10.39 + 95.3 \times 6 = 1{,}143$ watts,

$Q = 55 \times 6 - 95.3 \times 10.39 = -660$ vars.

b. $\dot{E} = 110\underline{/150°} = -95.3 + j55$ volts,

$\dot{I} = 12\underline{/120°} = -6 + j10.39$ amp,

$P = (-95.3)(-6) + 55 \times 10.39 = 1{,}143$ watts,

$Q = (-95.3)(10.39) - (55)(-6) = -660$ vars.

c. $\dot{E} = 110\underline{/285°} = 28.5 - j106.3$ volts,

$\dot{I} = 12\underline{/255°} = -3.11 - j11.59$ amp,

$P = (28.5)(-3.11) + (-106.3)(-11.59) = 1{,}143$ watts,

$Q = (28.5)(-11.59) - (-106.3)(-3.11) = -660$ vars.

69. Addition of Values of Power.—In series circuits, it was found necessary to add the emfs in their proper phase relations referred to the common current as

$$\dot{E}_0 = \dot{E}_1 + \dot{E}_2 + \cdots + \dot{E}_n \text{ volts} \tag{207}$$

$$= (e_1 + e_2 + \cdots + e_n) + j(e_1' + e_2' + \cdots + e_n') \text{ volts.} \tag{210}$$

Multiplying both sides of Eq. (210) by the current I, there results

$$\dot{E}_0 I = (e_1 I + e_2 I + \cdots + e_n I) + j(e_1' I + e_2' I + \cdots + e_n' I) \text{ vector volt-amp,} \tag{342}$$

having the magnitude,

$$E_0 I = \sqrt{(e_1 I + e_2 I + \cdots + e_n I)^2 + (e_1' I + e_2' I + \cdots + e_n' I)^2} \text{ volt-amp.} \tag{343}$$

The value $E_0 I$ is the total apparent power taken by the circuit; the quantities $e_1 I$, $e_2 I$, $e_3 I$, etc., are the values of the active power consumed by the several parts of the circuit; and the quantities $e_1' I$, $e_2' I$, $e_3' I$, etc., are the values of the reactive power consumed in these same parts of the circuit. Since I is a positive number, these components of reactive power will take the same signs as the respective components of $\dot{E}_1$, $\dot{E}_2$, $\dot{E}_3$, etc.

From Eq. (343), it is seen that the individual values of apparent power cannot be added directly as numbers but that their component parts must first be added for the same reasons as those applied to the addition of emfs in series circuits.

Equation (342) may be written

$$\dot{E}_0 I = (P_1 + P_2 + \cdot \cdot \cdot + P_n) + j(Q_1 + Q_2 + \cdot \cdot \cdot + Q_n) \text{ vector volt-amp,} \qquad (344)$$

which is the expression for the vector power in the circuit. It can be seen from Eq. (344) that the total active power consumed in the circuit is equal to the sum of the active power consumed in the individual parts of the circuit and that the total reactive power consumed in the circuit is equal to the algebraic sum of the reactive power consumed in the individual parts of the circuit, and Eq. (344) may be expressed as

$$\dot{E}_0 I = \dot{V}_0 = P_0 + jQ_0 \text{ vector volt-amp,} \qquad (345)$$

where

$$P_0 = P_1 + P_2 + \cdot \cdot \cdot + P_n \text{ watts,} \qquad (346)$$

and

$$Q_0 = Q_1 + Q_2 + \cdot \cdot \cdot + Q_n \text{ vars.} \qquad (347)$$

The magnitude of the vector power is

$$V_0 = \sqrt{P_0^2 + Q_0^2} \text{ vector volt-amp,} \qquad (348)$$

which is the same as the value of apparent power expressed by Eq. (343).

In parallel circuits the currents are added, as

$$\dot{I}_0 = \dot{I}_1 + \dot{I}_2 + \cdot \cdot \cdot + \dot{I}_n \text{ amp} \qquad (284)$$
$$= (i_1 + i_2 + \cdot \cdot \cdot + i_n) + j(i_1' + i_2' + \cdot \cdot \cdot + i_n') \text{ amp.} \qquad (285)$$

Multiplying both sides of Eqs. (284) and (285) by the impressed emf E,

$$E\dot{I}_0 = E\dot{I}_1 + E\dot{I}_2 + \cdot \cdot \cdot + E\dot{I}_n \text{ volt-amp} \qquad (349)$$
$$= (Ei_1 + Ei_2 + \cdot \cdot \cdot + Ei_n) + j(Ei_1' + Ei_2' + \cdot \cdot \cdot + Ei_n') \text{ volt-amp,} \qquad (350)$$

having the magnitude,

$$EI_0 = \sqrt{(Ei_1 + Ei_2 + \ . \ . \ . \ + Ei_n)^2 + (Ei_1' + Ei_2' + \ . \ . \ . \ + Ei_n')^2} \text{ volt-amp.} \qquad (351)$$

Since Ei_1, Ei_2 . . . Ei_n are equal to the values of active power consumed in the circuit and Ei_1', Ei_2' . . . Ei_n' are equal to the values of reactive power consumed in the circuit, Eq. (351) may be expressed as

$$E\dot{I}_0 = (P_1 + P_2 + \ . \ . \ . \ + P_n) + j(Q_1 + Q_2 + \ . \ . \ . \ + Q_n) \text{ vector volt-amp,} \qquad (352)$$

which is the vector power in the circuit. Since the sum of the active powers in the parts is equal to the total active power in the circuit and the algebraic sum of the reactive powers in the parts is equal to the total reactive power in the circuit, Eq. (352) may be written

$$E\dot{I}_0 = \dot{V} = P_0 + jQ_0 \text{ vector volt-amp,} \qquad (353)$$

where

$$P_0 = P_1 + P_2 + \ . \ . \ . \ + P_n \text{ watts,} \qquad (354)$$

and

$$Q_0 = Q_1 + Q_2 + \ . \ . \ . \ + Q_n \text{ vars.} \qquad (355)$$

The magnitude of the vector power is given by Eq. (348), the same as for series circuits.

70. Derivation of E and I from Power Diagrams.—The preceding method of analysis is convenient in many cases where the emf or current in the circuit is not important in the calculation. The emfs and currents and their components may be derived from the foregoing values of power as follows: the active component of the emf is

$$e_r = \frac{\text{active power}}{I} = \frac{P}{I} \text{ volts,} \qquad (356)$$

the total impressed emf,

$$E_0 = \frac{\text{apparent power}}{I} = \frac{U}{I} = \frac{\text{vector power}}{I} = \frac{V}{I} \text{ volts,} \qquad (357)$$

the reactive component of emf,

$$e_x = \frac{\text{reactive power}}{I} = \frac{Q}{I} \text{ volts,} \tag{358}$$

the active component of the current,

$$i_p = \frac{\text{active power}}{E} = \frac{P}{E} \text{ amp,} \tag{359}$$

the reactive component of the current,

$$i_q = \frac{\text{reactive power}}{E} = \frac{Q}{E} \text{ amp,} \tag{360}$$

and the total current,

$$I_o = \frac{\text{apparent power}}{E} = \frac{U}{E} = \frac{\text{vector power}}{E} = \frac{V}{E} \text{ amp.} \tag{361}$$

71. Determination of the Constants of the Circuit from the Power Diagram.—Figure 82 shows graphically the relation between the impedance diagram and the power diagram for the series circuit; and Fig. 83 shows a similar relation for the parallel circuit between the admittance diagram and the power diagram. Referring to these two figures, the following relations are seen to hold:

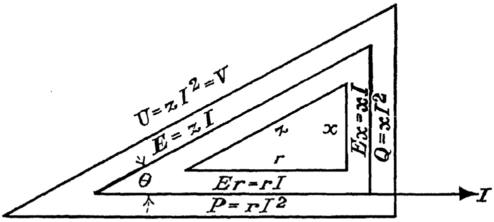

FIG. 82.—Impedance, vector, and power diagram for a series circuit.

the resistance,

$$r = \frac{P}{I^2} \text{ ohms,} \tag{362}$$

the reactance,

$$x = \frac{Q}{I^2} \text{ ohms,} \tag{363}$$

the impedance,

$$z = \frac{U}{I^2} = \frac{V}{I^2} \text{ ohms,} \tag{364}$$

the conductance,

$$g = \frac{P}{E^2} \text{ mhos,} \tag{365}$$

the susceptance,

$$b = \frac{Q}{E^2} \text{ mhos,} \tag{366}$$

and the admittance,

$$y = \frac{U}{E^2} = \frac{V}{E^2} \text{ mhos.} \tag{367}$$

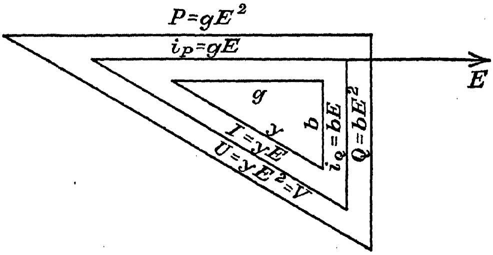

FIG. 83.—Admittance, vector, and power diagram for a parallel circuit.

The preceding relations and the uses of the equations may be shown by the following examples.

Example 26

An induction motor takes 10 kw from a line at 0.80 power factor. What is the apparent power, the reactive power, the reactive factor, and the current and its components if the impressed emf is 250 volts?

Apparent power,

$$U = \frac{10}{0.80} = 12.5 \text{ kva,}$$

reactive power,

$$Q = -\sqrt{12.5^2 - 10^2} = -7.5 \text{ kvar,}$$

the minus sign indicating inductive load, emf as reference. The *reactive factor* becomes

$$\text{reactive factor} = \frac{-7.5}{12.5} = -0.60.$$

Hence, the total current is

$$I = \frac{12.5 \times 1{,}000}{250} = 50 \text{ amp,}$$

the active component of current,

$$i = \frac{10 \times 1{,}000}{250} = 40 \text{ amp,}$$

and the reactive component,

$$i' = \frac{-7.5 \times 1{,}000}{250} = -30 \text{ amp.}$$

Example 27

A lighting load of 10 kw is added to the line in parallel with the motor in Example 26. Find the values required in Example 26 for the combined loads.

Active power for the motor = 10 kw.
Active power for the lights = 10 kw.

The total active power is then

$$P_0 = 10 + 10 = 20 \text{ kw,}$$

Reactive power for motor = −7.5 kvar
Reactive power for lights = 0.0 kvar.

The total reactive power is then

$$Q = -7.5 + 0 = -7.5 \text{ kvar.}$$

The vector power is

$$V_0 = \sqrt{20^2 + 7.5^2} = 21.4 \text{ vector kva,}$$

(not 22.5 kva, the arithmetical sum of the two apparent powers). The total power factor becomes

$$P.F._0 = \frac{20}{21.4} = 0.935 = 93.5 \text{ per cent}$$

and

$$R.F._0 = \frac{-7.5}{21.4} = -0.35 = -35 \text{ per cent.}$$

The total current taken by the two loads is

$$I_0 = \frac{21.4 \times 1{,}000}{250} = 85.6 \text{ amp,}$$

the active component,

$$i_0 = \frac{20 \times 1{,}000}{250} = 80 \text{ amp.}$$

and the reactive component,

$$i_0' = \frac{-7.5 \times 1{,}000}{250} = -30 \text{ amp.}$$

The total current for the lighting load is

$$I_1 = \frac{10 \times 1{,}000}{250} = 40 \text{ amp,}$$

and the current for the motor load

$$I_2 = \frac{12.5 \times 1{,}000}{250} = 50 \text{ amp.}$$

The lighting load has a power factor of 1.00 and a reactive factor of 0; the motor has a power factor of 0.80 and a reactive factor of −0.60; and the total load has a power factor of 0.935 and a reactive factor of −0.35. The results are shown in diagram in Fig. 84.

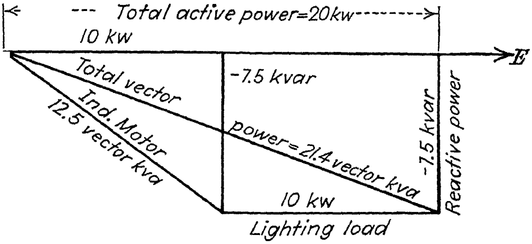

FIG. 84.—Power diagram for Example 27.

Example 28

A synchronous motor consuming 10 vector kva (leading) with 2-kw losses is connected in parallel with the induction motor of Example 26. What are the total results of the combination of loads?[1]

Active power for the induction motor = 10 kw
Active power for the synchronous motor = 2 kw
Total active power = 12 kw
Reactive power for induction motor = −7.5 kvar
Reactive power for synchronous motor = 9.8 kvar
Total reactive power = 2.3 kvar.

The reactive power for the synchronous motor is found as

$$Q_s = \sqrt{10^2 - 2^2} = 9.8 \text{ kvar.}$$

The total vector power for the two loads is

$$V_0 = \sqrt{12^2 + 2.3^2} = 12.2 \text{ vector kva.}$$

The total load power factor is

$$P.F._0 = \frac{12}{12.2} = 0.984,$$

and the reactive factor for the total load,

$$R.F._0 = \frac{2.3}{12.2} = 0.188,$$

---

[1] The synchronous motor current leads the impressed emf in this case, owing to the excitation of the machine, and produces the same effect as a capacitive reactance.

the current leading the impressed emf. The summaries of the problem are as follows:

Power factor for induction motor load = 0.8, reactive factor, −0.60
Power factor for synchronous motor = 0.20, reactive factor, 0.98
Power factor for total load = 0.984, reactive factor, 0.188.
Current taken by the induction motor load = 50 amp

Current taken by synchronous motor load $= \frac{10 \times 1{,}000}{250} = 40$ amp

Total current taken from the line $\frac{12.2 \times 1{,}000}{250} = 48.8$ amp.

Less current is required for the combination of the two loads than is required for the induction-motor load in Example 26. The addition of the synchronous motor load has lowered the total current and improved the power factor of the system. This use of apparatus for the control of load conditions is discussed in its physical and economic aspects when the machines are studied. The power diagram for this load is shown in Fig. 85.

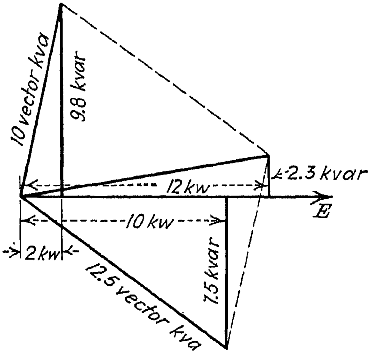

FIG. 85.—Power diagram for Example 28.

Example 29

What should be the vector kva input to the synchronous motor in Example 27 to bring the power factor of the circuit to unity, the active power input to this motor being assumed constant?

To bring the total power factor to unity the reactive power for the synchronous motor must exactly balance that for the induction motor, or

Reactive power for induction motor = −7.5 kvar
Reactive power for synchronous motor = 7.5 kvar
Total reactive power = 0.0 kvar.

For the synchronous motor then,

$$\text{Vector power} = \sqrt{2^2 + 7.5^2} = 7.77 \text{ vector kva}$$

Total active power = 12 kw
Total vector power = 12 vector kva
Current for induction motor = 50 amp

$$\text{Current for synchronous motor} = \frac{7.77 \times 1{,}000}{250} = 31.2 \text{ amp}$$

$$\text{Total current taken from line} = \frac{12 \times 1{,}000}{250} = 48 \text{ amp.}$$

This example is illustrated in Fig. 86.

Example 30

What should be the vector kva input to the synchronous motor in Example 27 to bring the power factor of the circuit to 0.90 with lagging current, with the same active power input to the synchronous motor as given in Example 27?

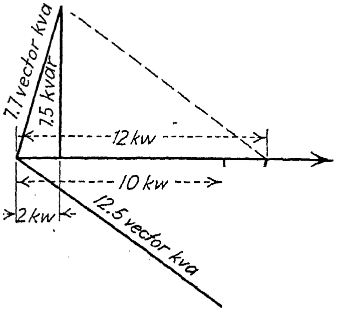

FIG. 86.—Power diagram for Example 29.

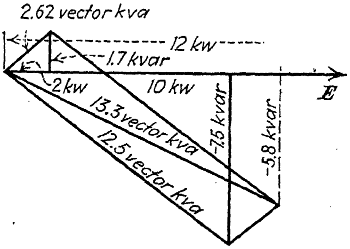

FIG. 87.—Power diagram for Example 30.

For a power factor of 0.90 lagging current the total reactive factor becomes

$$\text{R.F.} = -\sqrt{1 - \overline{0.90}^2} = -0.436.$$

The total active power is

$$10 + 2 = 12 \text{ kw},$$

the vector power,

$$\frac{12}{0.90} = 13.3 \text{ vector kva},$$

and the reactive power,

$$Q_0 = 13.3 \times (-0.436) = -5.8 \text{ kvar}.$$

The reactive power for the induction motor is -7.5 kvar, hence, the reactive power to be consumed by the synchronous motor becomes

$$Q_s = -5.8 - (-7.5) = 1.7 \text{ kvar}.$$

The vector power input to the synchronous motor is

$$V_s = \sqrt{\overline{2}^2 + \overline{1.7}^2} = 2.62 \text{ vector kva}.$$

These results are shown in diagram in Fig. 87.

Problems

1-5. An induction motor takes a load of 4.2 kw at 0.75 power factor from a 230-volt line. What are the reactive power, the vector power, and the reactive factor? Draw the power diagram.

2-5. An inductive load of 7.8 kva at 0.82 power factor is taken from a 240-volt source. What are the values of reactive power, active power, and reactive factor? Draw the power diagram.

3-5. An impedance of $\dot{Z} = 5.1 + j6.8$ is connected to a 220-volt source. Find the active component, the reactive component, the total current and the active, reactive, and vector power. Draw the current vector diagram, using emf as the reference, and the corresponding power diagram.

4-5. A load on a 160-volt circuit draws 42 amp at a phase angle of 35 deg. current lagging the emf. (*a*) Find the active and reactive components of current and from them the active and reactive power. Draw current vector diagram using emf as the reference. (*b*) Find the active and reactive components of emf and from them the active and reactive power. Draw the voltage vector diagram and the power diagram, using current as the reference.

5-5. A test on an electric motor showed the reading of the voltmeter was 219 volts, of the ammeter 39 amp, and of the wattmeter 5.7 kw. What are the apparent power and the reactive power demand of the load? What is the power factor of the load?

6-5. From calculations on the input to an electric circuit, it is found that the current is $\dot{I} = 9.53 - j5.5$ amp when the applied emf is $\dot{E} = 104 + j60$. From the components of emf and current, find the active and reactive power. What are the vector power and the power factor?

7-5. The input into a group of impedances was found to be $\dot{I} = -26 + j16.8$ amp when the impressed emf to the same reference was $\dot{E} = -164 - j67$. By means of these component expressions, find the active and reactive power. What are the vector power and the power factor?

8-5. A lighting load of 8 kw is connected in parallel with an induction motor taking 15 kw at 0.75 power factor to a 230-volt supply. Find the active power, the reactive power, the vector power, the power factor, and the reactive factor for the combined loads. Draw the power diagram for each component load and the total load. Find the active component, the reactive component, and the total current taken by the combined load. Draw the current vector diagram, using emf as the reference.

9-5. Two induction motors are supplied from the same circuit. One of the motors takes 8 kva at 0.72 power factor, and the other takes 10 kva at 0.50 power factor. Find the total active, reactive, and vector power supplied by the circuit to these two loads. Find the power factor of the supply circuit, and draw the power diagram.

10-5. A lighting load of 125 kw at unity power factor and an induction furnace drawing 200 kva at 0.40 power factor are connected in parallel to a feeder. Find the active, reactive, and vector power demand on the feeder, and draw the power diagram.

11-5. An inductive load of 30 kw at 0.90 power factor and a capacitive synchronous motor load of 18 kw at 0.65 power factor are taken from the same line. Find the active, reactive, and vector power taken by these two loads, and draw the power diagram.

12-5. A substation has three feeders supplied from a transformer. At the time of the peak load, the feeder loads are as follows: On feeder *A*, the reading of the wattmeter is 185 kw and the reactive power meter is 134 kvar inductive. On feeder *B*, the reading of the wattmeter is 217 kw and of the reactive power meter is 183 kvar capacitive. On feeder *C*, the reading of the wattmeter is 296 kw and of the reactive power meter is 384 kvar inductive. Find the total active, reactive, and vector power supplied by the transformer and the power factor at which it is operating. Draw the complete power diagram.

13-5. An inductive load of 12 kw operates at a power factor of 0.52. It is designed to bring the power factor of the load demanded from the feeder to unity by the addition of capacitors. What is the kilovolt-ampere rating of the capacitors? Draw the complete power diagram. What is the total kilovolt-ampere input to the combined loads?

14-5. A small shop has a lighting load of 7.5 kw operating in parallel with an induction motor load of 15 kva at 0.65 power factor. Capacitors are to be connected in parallel with these loads to bring the power factor to unity. What is the kilovolt-ampere rating of capacitors required? What are the active, reactive, and apparent power of the combined load? Draw the power diagram.

15-5. An industrial substation supplies a lighting load of 100 kw and an induction motor load of 150 kva at 0.50 power factor. What kilovolt-ampere rating in capacitors must be connected in parallel to bring the power factor of the combined loads to 0.85, current lagging? How much was the apparent power of the load reduced by the addition of these capacitors? Draw the power diagram.

16-5. A substation supplies a lighting load of 200 kw and a motor load of 400 kva at 0.70 power factor, current lagging. How many kilovolt-amperes in capacitors must be added in parallel to bring the power factor of the combined load to 0.95, current lagging? What are the active, reactive, and vector power supplied by the substation before and after the addition of the capacitors? Draw the complete power diagram.

17-5. A factory has an induction motor load of 500 kw at a power factor of 0.65. A synchronous motor is to be installed, having sufficient capacity to take 200 kw additional load and to raise the power factor of the combined load to 0.90, current lagging. What is the kilovolt-ampere rating of the required synchronous motor? At what power factor does it operate? What are the active, reactive, and vector power supplied to the factory? Draw the power diagram.

18-5. A 110-volt washing-machine motor is rated at 1/4 hp. It has an efficiency of 70 per cent and a power factor of 65 per cent at full load. What current does it draw from the line? What are the active and reactive components of current? What are the reactive and vector power? Draw the power and the current diagrams.

19-5. A feeder circuit has two branches with inductive loads. There are meters in the feeder circuit and in one of the branches. On a test, the following readings were taken: voltage 2,300 volts, feeder ammeter 90 amp, feeder wattmeter 152 kw, branch *A* ammeter 40 amp, branch *A* wattmeter 83 kw. What are the active power, reactive power, and vector power, and the power factor of the feeder and of each branch?

20-5. A group of impedances is tested as a unit, the impressed emf being 196 volts, the current 0.68 amp, and the active power 88 watts. Find the active and reactive components and the total current. Draw the current vector diagram. Find the equivalent resistance, impedance, conductance, susceptance, and admittance of the circuit.

21-5. The load on a distribution circuit was found to be 6.1 kw of active power and 4.3 kvar of inductive reactive power. The emf impressed was 117 volts. Find the active and reactive components and the total current. Draw the current vector diagram. Find the equivalent conductance, susceptance, resistance, and reactance of the load.

22-5. A 288-volt load of 8,000 watts, unity power factor, is made up of two branches. The impedance of one branch is $\dot{Z}_1 = 8.4 + j6.3$ ohms; what is the impedance of the other branch?

23-5. A load of 80 kw at 0.65 power factor, current lagging, on a 2,450-volt circuit is made up of two impedances in series. The first impedance is $\dot{Z}_1 = 25\underline{/75^\circ}$ ohms; find the value of the second impedance.

24-5. A transformer carries a load to three feeders. At the time of the peak load, the loads on the separate feeders are: feeder 1 taking 300 kva at 0.40 power factor inductive; feeder 2 taking 100 kva at unity power factor; and feeder 3 taking 200 kva at 0.70 power factor inductive. Find the active-power, the reactive-power, and the apparent-power output of the transformer.

25-5. A lighting load of 80 kw at unity power factor is connected in parallel with an induction motor taking 100 kva at 0.70 power factor. What kilovolt-amperes of capacitors must be connected in parallel with these loads to bring the combined power factor to unity? What is the total input to these three loads at this power factor?

26-5. An industrial plant has an induction motor load of 500 kva. at 0.50 power factor. A synchronous motor is to be installed to carry an additional load requiring 500 kva input and to bring the power factor of the combined load to 0.90 inductive. What is the rating of the synchronous motor?

27-5. A 3,000-volt load of 100 kw at unity power factor is composed of two branch loads. The impedance of one branch load is $\dot{Z}_1 = 50\underline{/45^\circ}$; what is the impedance of the second load?

28-5. Three loads $\dot{Y}_1 = 0.1\underline{/45^\circ}$, $\dot{Y}_2 = 0.2\underline{/-80^\circ}$, and $\dot{Y}_3 = 0.05\underline{/50^\circ}$ are connected in parallel across a 1,000-volt a-c circuit. Calculate the total active power, the total reactive power, and the total apparent power, the power factor, and the reactive factor of the combined loads. Draw the power diagram.

29-5. Three impedances $\dot{Z}_1 = 2.5 - j6.8$, $\dot{Z}_2 = 7\underline{/60^\circ}$, and $\dot{Z}_3 = 10\underline{/-80^\circ}$ are connected in parallel across a 500-volt a-c circuit. Find the total and branch currents and the total and branch active power, reactive power, and apparent power.

CHAPTER VI

COMPLEX ALTERNATING-CURRENT CIRCUITS

In the study of transmission lines, distribution circuits, and electrical machines, many of the circuits are composed of parallel branches in series with impedances. Such circuits are called "complex" because they are composed of combinations of simple series and simple parallel circuits. The solution of this type of circuit involves the successive reduction of groups of parallel and series impedances into equivalent single impedances according to the laws already developed for series and parallel circuits. Though the process of solution of the complex circuit will vary, depending upon the particular portions of the circuit that are known, the procedure in general is that of the application in successive steps of the principles of the series and parallel circuit. Two particular cases of the complex circuit that occur frequently will be outlined.

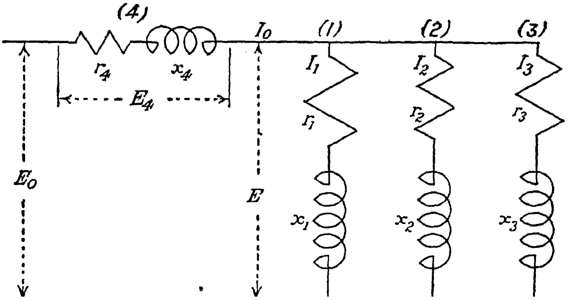

FIG. 88.—A complex circuit.

72. Case I. Electromotive Force across the Parallel Branches Given.—This case is one that frequently occurs in the solution of transmission, distribution, and equivalent machine circuits, the branch circuits constituting the loads with the voltage across them being known. The impedances of the supply circuit are known, and it is required to determine the supply voltage.

wattmeter 152 kw, branch A ammeter 40 amp, branch A wattmeter 83 kw. What are the active power, reactive power, and vector power, and the power factor of the feeder and of each branch?

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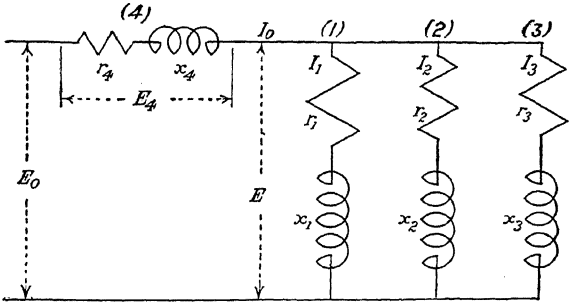


FIG. 88.—A complex circuit.

72. Case I. Electromotive Force across the Parallel Branches Given.—This case is one that frequently occurs in the solution of transmission, distribution, and equivalent machine circuits, the branch circuits constituting the loads with the voltage across them being known. The impedances of the supply circuit are known, and it is required to determine the supply voltage.

Let Fig. 88 represent the circuit diagram to be considered. Assume that E is given and remains constant and that either the current in branches 1, 2, and 3 or their impedances are known; it is required to find E_0, I_0, and the total power factor of the circuit. The given emf E will be used as the reference vector, or

$$\dot{E} = E + j0 \text{ volts.} \qquad (283)$$

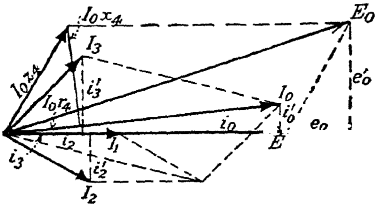


FIG. 89.—Vector diagram for a complex circuit.

The total current taken by the parallel branches is

$$\dot{I}_0 = \dot{I}_1 + \dot{I}_2 + \ . \ . \ . \ + \dot{I}_n \text{ amp,} \qquad (284)$$

or, if the impedances are given, the total current may be found as

$$\dot{I}_0 = \qquad + \dot{y}_2 + \ . \ . \ . \ + \dot{y}_n] \text{ amp} \qquad (306)$$
$$= E(g_o + jb_o) = Ey_0\underline{/\theta_0} \text{ amp} \qquad (307)$$
$$= i_o + ji_0' = I_0\underline{/\theta_0} \text{ amp,} \qquad (285)$$

where

$$\tan \theta_0 = \frac{b_0}{g_0}. \qquad (368)$$

This current, flowing through the impedance $\dot{z}_4$, causes an impedance drop that the impressed emf must supply in addition to the emf E across the parallel branches, or

$$= E + \dot{E}_4 \text{ volts}$$
$$= E + \dot{I}_0\dot{z}_4 \qquad (369)$$
$$= E + (i_0 + ji_0')(r_4 + jx_4) \qquad (370)$$
$$= E + i_0r_4 - i_0'x_4 + j(i_0x_4 + i_0'r_4) \qquad (371)$$
$$= e_0 + je_0' \text{ volts,}$$

where

$$e_0 = E + i_0r_4 - \qquad \text{volts,} \qquad (372)$$

and

$$e_0' = \qquad + i_0'r_4 \text{ volts.} \qquad (373)$$

The magnitude of the impressed emf, or voltmeter reading, is

$$E_0 = \sqrt{e_0^2 + e_0'^2} \text{ volts,}$$

and the magnitude of the current, or ammeter reading, is

$$I = \sqrt{i_0^2 + i_0'^2} \text{ amp.} \tag{286}$$

The vector diagram with E as the reference is shown in Fig. 89.

Example 31

A 220-volt motor takes 50 amp at 0.8 power factor inductive over a circuit having a resistance of 0.5 ohm and a reactance of 1 ohm. What is the station voltage, the power factor at the station, and the power supplied by the station?

$$\begin{aligned} I_o &= 50 \text{ amp}, \\ i_o &= 0.8 \times 50 = 40 \text{ amp}, \\ i_o' &= -\sqrt{\overline{50}^2 - \overline{40}^2} = -30 \text{ amp}. \end{aligned}$$

From Eq. (370),

$$\begin{aligned} \dot{E}_0 &= 220 + (40 - j30)(0.5 + j1.0) \text{ volts}, \\ \dot{E}_o &= 270 + j25 \text{ volts}, \end{aligned}$$

and

$$\begin{aligned} E_o &= \sqrt{\overline{270}^2 + \overline{25}^2} = 271 \text{ volts}. \\ P_o &= \qquad + \\ e_o i_o &= 270 \times 40 = 10{,}800 \text{ watts}, \\ e_o' i_o' &= 25 \times (-30) = -750 \text{ watts}, \end{aligned}$$

and

$$\begin{aligned} P_o &= 10{,}050 \text{ watts}, \\ P &= 220 \times 40 = 8{,}800 \text{ watts}. \end{aligned}$$

This example is illustrated in Fig. 90.

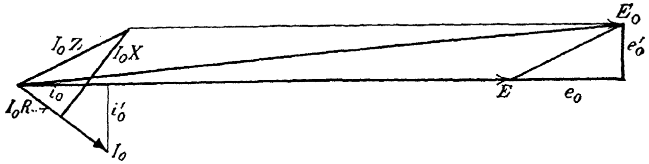


FIG. 90.—Vector diagram for the complex circuit of Example 31.

73. Case II. Impressed Emf and Constants Given.—Another common case is the one in which the emf applied to the circuit and the circuit constants are known, and it is desired to determine the unknown currents and emfs. Such a circuit is shown in Fig. 91, where E_0 and the impedances z_1, z_2, z_3, z_4, and z_5 are known. The steps are, first, to reduce the impedances in the parallel groups to equivalent admittances; then each parallel group is combined into a single admittance. These single admittances are in series and must be converted into equivalent impedances so that they may be combined into a single resultant equivalent impedance.

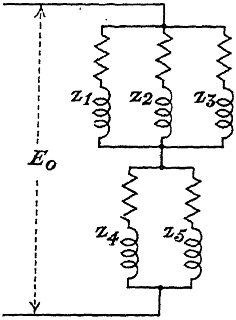


FIG. 91.—A complex circuit of five impedances.

In the first group of parallel impedances, the equivalent admittances are

$$\dot{y}_1 = \frac{1}{\dot{z}_1} = g_1 + jb_1 \text{ mhos,} \tag{374}$$

$$\dot{y}_2 = \frac{1}{\dot{z}_2} = g_2 + jb_2 \text{ mhos,} \tag{375}$$

$$\dot{y}_3 = \frac{1}{\dot{z}_3} = g_3 + jb_3 \text{ mhos,} \tag{376}$$

and the combined equivalent admittance is

$$\dot{y}_{13} = \dot{y}_1 + \dot{y}_2 + \dot{y}_3 \text{ mhos.}$$

This combined admittance must be converted into an equivalent impedance, the impedance being expressed as the reciprocal of the admittance, or

$$\dot{z}_{13} = \frac{1}{\dot{y}_{13}} \text{ ohms} \tag{377}$$

$$= \frac{1}{g_{13} + jb_{13}} \text{ ohms} \tag{378}$$

$$= \frac{g_{13}}{g_{13}^2 + b_{13}^2} + j\frac{-b_{13}}{g_{13}^2 + b_{13}^2} \text{ ohms} \tag{379}$$

$$= r_{13} + jx_{13} \text{ ohms,} \tag{380}$$

where

$$r_{13} = \frac{g_{13}}{g_{13}^2 + b_{13}^2} = \frac{g_{13}}{y_{13}^2} \text{ ohms} \tag{381}$$

and

$$x_{13} = \frac{-b_{13}}{g_{13}^2 + b_{13}^2} = \frac{-b_{13}}{y_{13}^2} \text{ ohms} \tag{382}$$

are the equivalent resistance and reactance of the parallel branches. In the definition of x, the numerator of the fraction is made negative so as to retain the positive sign in the general definition of x. When the proper sign is given to the numerical value of b, the numerical value of the resulting x will have its proper sign, capacitive susceptance being positive whereas capacitive reactance is negative; and inductive susceptance is negative whereas inductive reactance is positive, The equivalent impedance of the second group of parallel impedances is formed in a similar manner as

$$\dot{z}_{45} = r_{45} + jx_{45} \text{ ohms.} \tag{383}$$

The single impedance equivalent to these two combined impedances is

$$\dot{z}_0 = \dot{z}_{13} + \dot{z}_{45} \text{ ohms} \tag{384}$$

$$= (r_{13} + r_{45}) + j(x_{13} + x_{45}) \text{ ohms}$$

$$= r_0 + jx_0 \text{ ohms.} \tag{385}$$

The total current taken by the combined circuit is then

$$\dot{I}_0 = \frac{E_0}{\dot{z}_0} = E_0\dot{y}_0 \text{ amp,} \tag{386}$$

where E_0, the known value of emf, is taken as the reference.

If the current and potential distributions in the complex circuit or network are desired, the inverse process is followed. In this case, the impedance drops across the equivalent impedances may be obtained from the equations

$$\dot{E}_{13} = \dot{I}_0\dot{z}_{13} \text{ volts} \tag{387}$$

and

$$\dot{E}_{45} = \dot{I}_0\dot{z}_{45} = E_0 - \dot{E}_{13} \text{ volts.} \tag{388}$$

Then each branch current may be obtained from the emf across the branch and its admittance, *e.g.*, as in branch 1,

$$\dot{I}_1 = \dot{E}_{13}\dot{y}_1 = i_1 + ji_1' \text{ amp.} \tag{389}$$

In this manner, the emfs and currents may be broken up into components, until the drop across each impedance and the current in each impedance are known in reference to the emf E_0.

Example 32

In Fig. 88 consider the impedances as

$$\dot{z}_1 = 10 + j0 = 10\underline{/0^\circ} \text{ ohms,}$$

$$\dot{z}_2 = 3 + j4 = 5\underline{/53^\circ 8'} \text{ ohms,}$$

$$\dot{z}_3 = 0 - j4 = 4\underline{/-90^\circ} \text{ ohms, and}$$

$$\dot{z}_4 = 1 + j2 = 2.24\underline{/63^\circ 26'} \text{ ohms,}$$

with the impressed emf $E_0 = 100$ volts. Find the emf E, the total current, and the current taken by each branch.

For the parallel branches,

$$\dot{y}_1 = 0.10 + j0 = 0.10\underline{/0^\circ} \text{ mhos,}$$
$$\dot{y}_2 = 0.12 - j0.16 = 0.20\underline{/-53^\circ 8'} \text{ mhos,}$$
$$\dot{y}_3 = 0 + j0.25 = 0.25\underline{/90^\circ} \text{ mhos.}$$

Then

$$\dot{y} = \dot{y}_1 + \dot{y}_2 + \dot{y}_3 = 0.22 + j0.09 = 0.2375\underline{/22^\circ 13'} \text{ mhos, and}$$
$$\dot{z} = \frac{1}{\dot{y}} = 4.22\underline{/-22^\circ 13'} = 3.88 - j1.59 \text{ ohms.}$$

The total impedance of the circuit becomes

$$\dot{z}_0 = \dot{z} + \dot{z}_4 = 4.88 + j0.41 = 4.89\underline{/4^\circ 49'} \text{ ohms,}$$
$$\dot{y}_0 = \frac{1}{\dot{z}_0} = 0.2037\underline{/-4^\circ 49'} \text{ mhos.}$$

Using E_0 as the reference,

$$\dot{I}_0 = E_0\dot{y}_0 = 100 \times 0.2037\underline{/-4^\circ 49'} \text{ amp}$$
$$= 20.37\underline{/-4^\circ 49'} = 20.3 - j1.71 \text{ amp.}$$

The emf across the parallel branches is

$$\dot{E} = \dot{I}_0\dot{z} = 20.37\underline{/-4^\circ 49'} \times 4.22\underline{/-22^\circ 13'} \text{ volts}$$
$$= 85.5\underline{/-27^\circ 2'} = 76.08 - j39.05 \text{ volts.}$$

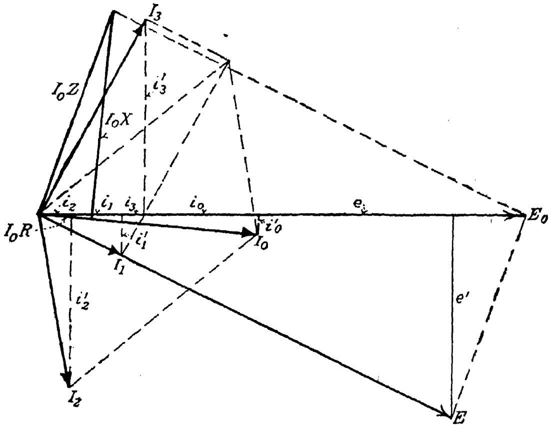


FIG. 92.—Vector diagram for the complex circuit of Example 32.

The currents in the branches are

$$
\begin{aligned}
\dot{I}_1 &= \dot{E}\dot{y}_1 = 85.5\underline{/-27^\circ 2'} \times 0.10\underline{/0^\circ} \text{ amp} \\
&= 8.55\underline{/-27^\circ 2'} = 7.608 - j3.905 \text{ amp}, \\
\dot{I}_2 &= \dot{E}\dot{y}_2 = 85.5\underline{/-27^\circ 2'} \times 0.20\underline{/-53^\circ 8'} \text{ amp} \\
&= 17.1\underline{/-80^\circ 10'} = 2.90 - j15.88 \text{ amp}, \\
\dot{I}_3 &= \dot{E}\dot{y}_3 = 85.5\underline{/-27^\circ 2'} \times 0.25\underline{/90^\circ} \text{ amp} \\
&= 21.38\underline{/62^\circ 58'} = 9.77 + j19.02 \text{ amp}.
\end{aligned}
$$

These values of current and voltage are shown in Fig. 92, E_0 being used as the reference.

74. Simplification of Networks.—Sometimes it is necessary to convert a mesh circuit to a star circuit or the reverse before combining the circuit with other impedances or admittances, or to reduce a network of impedances to a single equivalent impedance. There are three general cases given below for such transformations.

a. Mesh-star Transformation. Three impedances may be connected into a mesh circuit, Fig. 93. Assume that these three impedances z_{13}, z_{32}, and z_{21} are connected into a mesh; that it is required to replace these impedances by three other impedances z_{a-1}, z_{a-2}, and z_{a-3}, connected in star to the common point a; and that these new impedances will produce the same drop in a given a-c circuit as is produced by the original impedances. By transformations similar to those in the previous paragraphs,

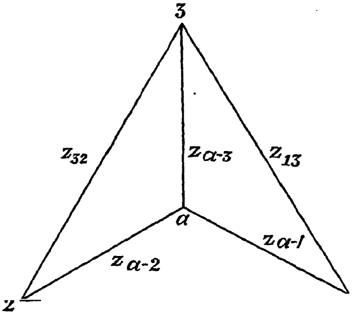


FIG. 93.—Star- and mesh-connected impedances.

$$\dot{z}_{a-1} = \frac{\dot{z}_{13}\dot{z}_{21}}{\dot{z}_{13} + \dot{z}_{21} + \dot{z}_{32}} \text{ ohms,} \tag{390}$$

$$\dot{z}_{a-2} = \frac{\dot{z}_{21}\dot{z}_{32}}{\dot{z}_{13} + \dot{z}_{21} + \dot{z}_{32}} \text{ ohms,} \tag{391}$$

$$\dot{z}_{a-3} = \frac{\dot{z}_{32}\dot{z}_{13}}{\dot{z}_{13} + \dot{z}_{21} + \dot{z}_{32}} \text{ ohms.} \tag{392}$$

These impedances may be changed to equivalent admittances, giving

$$\dot{y}_{a-1} = \frac{\dot{y}_{21}\dot{y}_{32} + \dot{y}_{32}\dot{y}_{13} + \dot{y}_{13}\dot{y}_{21}}{\dot{y}_{23}} \text{ mhos,} \tag{393}$$

$$\dot{y}_{a-2} = \frac{\dot{y}_{21}\dot{y}_{32} + \dot{y}_{32}\dot{y}_{13} + \dot{y}_{13}\dot{y}_{21}}{\dot{y}_{13}} \text{ mhos,} \tag{394}$$

$$\dot{y}_{a-3} = \frac{\dot{y}_{21}\dot{y}_{32} + \dot{y}_{32}\dot{y}_{13} + \dot{y}_{13}\dot{y}_{21}}{\dot{y}_{21}} \text{ mhos.} \tag{395}$$

b. Star-mesh Transformation.—When the given circuit, Fig. 93, is connected in star and the equivalent mesh is required, the transformation equations become

$$\dot{z}_{21} = \frac{\dot{z}_{a-1}\dot{z}_{a-2} + \dot{z}_{a-2}\dot{z}_{a-3} + \dot{z}_{a-3}\dot{z}_{a-1}}{\dot{z}_{a-3}} \text{ ohms,} \tag{396}$$

$$\dot{z}_{32} = \frac{\dot{z}_{a-1}\dot{z}_{a-2} + \dot{z}_{a-2}\dot{z}_{a-3} + \dot{z}_{a-3}\dot{z}_{a-1}}{\dot{z}_{a-1}} \text{ ohms,} \tag{397}$$

$$\dot{z}_{13} = \frac{\dot{z}_{a-1}\dot{z}_{a-2} + \dot{z}_{a-2}\dot{z}_{a-3} + \dot{z}_{a-3}\dot{z}_{a-1}}{\dot{z}_{a-2}} \text{ ohms,} \tag{398}$$

or

$$\dot{y}_{21} = \frac{y_{a-1}y_{a-2}}{\dot{y}_{a-1} + \dot{y}_{a-2} + \dot{y}_{a-3}} \text{ mhos,} \tag{399}$$

$$\dot{y}_{32} = \frac{\dot{y}_{a-2}\dot{y}_{a-3}}{\dot{y}_{a-1} + \dot{y}_{a-2} + \dot{y}_{a-3}} \text{ mhos,} \tag{400}$$

$$\dot{y}_{13} = \frac{\dot{y}_{a-1}\dot{y}_{a-3}}{\dot{y}_{a-1} + \dot{y}_{a-2} + \dot{y}_{a-3}} \text{ mhos.} \tag{401}$$

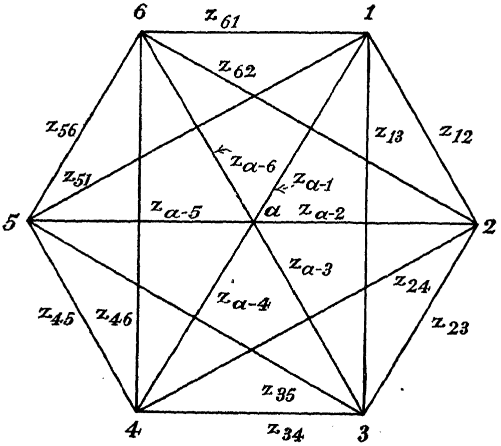


FIG. 94.—Star- and complete mesh-connected impedances.

c. Star-complete Mesh.—A star-connected circuit may be converted into a complete or partial mesh-connected circuit by the use of the proper transformation equations. The star-connected circuit, shown in Fig. 94, may be converted into equivalent mesh-connected triangular mesh connections and these in turn into the outside mesh by transformations given in Eqs. (396) to (401). Applying these equations to any star circuit,

$$\dot{z}_{12} = \dot{z}_{a-1}\dot{z}_{a-2}(\dot{y}_{a-1} + \dot{y}_{a-2} + \ . \ . \ . \ + \dot{y}_{a-n}) \text{ ohms,} \tag{402}$$

$$\dot{z}_{23} = \dot{z}_{a-2}\dot{z}_{a-3}(\dot{y}_{a-1} + \dot{y}_{a-2} + \ . \ . \ . \ + \dot{y}_{a-n}) \text{ ohms,} \tag{403}$$

$$. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .$$

$$\dot{z}_{mn} = \dot{z}_{a-m}\dot{z}_{a-n}(\dot{y}_{a-1} + \dot{y}_{a-2} + \ . \ . \ . \ + \dot{y}_{a-n}) \text{ ohms,} \tag{404}$$

or in terms of the admittances,

$$\dot{y}_{12} = \frac{\dot{y}_{a-1}\dot{y}_{a-2}}{\dot{y}_{a-1} + \dot{y}_{a-2} + \ . \ . \ . \ + \dot{y}_{a-n}} \text{ mhos,} \tag{405}$$

$$\dot{y}_{23} = \frac{\dot{y}_{a-2}\dot{y}_{a-3}}{\dot{y}_{a-1} + \dot{y}_{a-2} + \ . \ . \ . \ + \dot{y}_{a-n}} \text{ mhos,} \tag{406}$$

$$. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .$$

$$\dot{y}_{mn} = \frac{\dot{y}_{a-m}\dot{y}_{a-n}}{\dot{y}_{a-1} + \dot{y}_{a-2} + \ . \ . \ . \ + \dot{y}_{a-n}} \text{ mhos.} \tag{407}$$

These equations will hold for any star-mesh transformation, but a general case of a mesh-star transformation does not hold for all circuits.

75. Mutual Inductance.—In Chap. II, Art. 25, it was noted that a current flowing in a coil produces a flux which interlinks that coil. This flux in a circuit of constant permeability was found to be proportional to the current, to the number of turns in the coil, and to the dimensions of the coil. By the application of Faraday's law it was found that the emf induced in the coil is given by

$$e_i = -L\frac{di}{dt} \text{ volts,} \tag{57}$$

where L is the coefficient of self-inductance of the coil.

If the flux from this coil also interlinks a second coil, there is an emf induced in this second coil by the change of flux, or current, in the first coil. This second induced emf is proportional to the number of turns in the second coil and to the rate of change of the flux threading that coil, or

$$e_{2i} = -N_2\frac{d\phi_1}{dt}10^{-8} \text{ volts.} \tag{408}$$

If all the flux from the first coil threads the second coil, then by the application of Eq. (408),

$$\begin{aligned} e_{2i} &= -N_2\frac{d}{dt}\left(\frac{0.4\pi N_1 i_1 A_1}{l_1}\right)10^{-8} \text{ volts} \\ &= -\frac{0.4\pi N_1 N_2 A_1}{l_1}\frac{di_1}{dt} 10^{-8} \text{ volts,} \end{aligned} \tag{409}$$

an equation similar to Eq. (56). The coefficient of di_1/dt in Eq. (409) is known as the mutual inductance of coil 2 with respect to coil 1, and Eq. (409) may be written as

$$e_{2i} = -M_{12}\frac{di_1}{dt} \text{ volts,} \tag{410}$$

where

$$M_{12} = \frac{0.4\pi N_1 N_2 A_1}{l_1} 10^{-8} \text{ henrys.} \tag{411}$$

Mutual Inductance.—Mutual inductance is the common property of two associated electric circuits that determine, for a given rate of change of current in one of the circuits, the emf induced in the other.

In most circuits, not all the flux from the first coil threads the second, but Eq. (410) holds for such a condition, the value of M being less than when all the flux threads both coils, since for this condition ϕ_1 in Eq. (408) becomes $k_1\phi_1$, where k_1 is a factor less than unity, it being the proportional part of the flux enclosed by coil 2. From Eq. (409), it is seen that the emf induced in the second coil is proportional both to the value of flux interlinking the second coil and to the number of turns in the second coil. This principle of interlinked circuits has a wide application in electrical engineering problems, especially in its application in the transformer. In the transformer, the relative leakage of flux between the coils is kept to a low value, giving a high value of mutual flux, whereas in most other circuits there may be very little mutual flux threading a second circuit.

If the current is made to flow in coil 2, a similar flux to that already mentioned threads coil 1, and Eq. (410) may be written

$$e_1 = -M_{21}\frac{di_2}{dt} \text{ volts,} \tag{412}$$

where M_{21} is the mutual inductance of coil 1 with respect to coil 2. The mutual inductance of each coil with respect to the other is the same, so that

$$= M \text{ henrys.} \tag{413}$$

If the coils are wound on a common core with turns side by side, nearly all the flux produced in coil 1 threads coil 2 and the condition of perfect mutual inductance is approached, where there is no leakage of flux between the coils. For such a condition, Eq. (411) may be written

$$M_{12} = \frac{N_2}{N_1}L_1 \text{ henrys,} \tag{414}$$

and similarly,

$$= \frac{N_1}{N_2}L_2 \text{ henrys.} \tag{415}$$

Multiplying Eqs. (414) and (415) there results

$$M = \sqrt{L_1L_2} \text{ henrys.} \tag{416}$$

For all practicable coupled coils, the actual flux in the second coil is less than that in the first, so that

$$M = k\sqrt{L_1L_2} \text{ henrys,} \tag{417}$$

where k is the coefficient of coupling of the coils, a quantity less than 1.

Coefficient of Coupling.—The coefficient of coupling is the ratio of the mutual inductance to the square root of the product of the coefficients of self-induction.

For transformers with steel cores, the value of k approaches unity, whereas, for air-cored coils and most mutual circuits, k is much less than 1. In Eqs. (416) and (417), the value of M may be either positive or negative, whereas the value of L is always positive.

When a current flows in both coils, the differential equations for the two sides of the circuit may be written

$$e_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \text{ volts,} \tag{418}$$

and

$$e_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \text{ volts,} \tag{419}$$

where e_1 is the instantaneous emf impressed on coil 1;
e_2 is the instantaneous emf impressed on coil 2;
L_1 is the coefficient of self-inductance of coil 1, henrys;
L_2 is the coefficient of self-inductance of coil 2, henrys;
M is the mutual inductance of the two coils, henrys;
i_1 is the instantaneous current flowing in coil 1, amp; and
i_2 is the instantaneous current flowing in coil 2, amp.

For steady-state conditions in the circuit, Eqs. (418) and (419) may be solved by methods similar to those used in the previous chapters into

$$\dot{E}_1 = R_1 \dot{I}_1 + j\omega L_1 \dot{I}_1 + j\omega M \dot{I}_2 \text{ volts,} \tag{420}$$

and

$$\dot{E}_2 = R_2 \dot{I}_2 + j\omega L_2 \dot{I}_2 + j\omega M \dot{I}_1 \text{ volts,} \tag{421}$$

where E_1 is the emf in volts impressed on coil 1;
E_2 is the emf in volts impressed on coil 2;
I_1 is the current, amp, flowing in coil 1;
I_2 is the current, amp, flowing in coil 2;
R_1 is the resistance in ohms of coil 1;
R_2 is the resistance in ohms of coil 2;

and L_1, L_2, and M have the same definitions as in Eqs. (418) and (419), it being understood that the emfs and currents are all referred to the same reference. Equations (420) and (421) may be solved simultaneously for I_1 and I_2.

Example 33

Two coils are placed close together so as to have their magnetic circuits interlinked. The self-inductance of coil A is 0.174 henry and of coil B is 0.113 henry. The mutual inductance between them is 0.105 henry. Find the coefficient of coupling of the coils.

$$k = \frac{M}{\sqrt{L_a L_b}} = \frac{0.105}{\sqrt{0.174 \times 0.113}}$$
$$= 0.748.$$

Problems

1-6. Three impedances represented by $\dot{Z}_1 = 10 + j0$, $\dot{Z}_2 = 7 - j8$, and $\dot{Z}_3 = 10 + j10$ are connected in parallel, and an emf drop of 1,000 volts results across the combination. This combination is fed through an impedance $\dot{Z}_4 = 0.3 + j0.4$, as shown in Fig. 1-6. Find the vector value and the magnitude of the impressed emf and the total and branch currents.

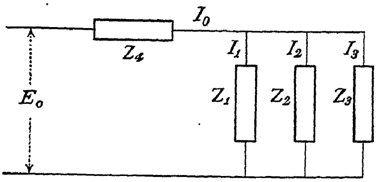


FIG. 1-6.—Figure for Prob. 1-6.

2-6. A group of three impedances in parallel, represented by $\dot{Z}_1 = 10\underline{/60^\circ}$, $\dot{Z}_2 = 5\underline{/-30^\circ}$, and $\dot{Z}_3 = 4\underline{/-90^\circ}$, is connected in series with a third impedance $\dot{Z}_4 = 2\underline{/-45^\circ}$, Fig. 1-6. If a voltage of 500 volts is impressed across the parallel branches, calculate the values of the branch currents, the total current, and the impressed emf.

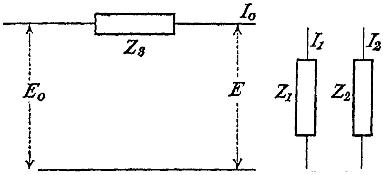


FIG. 3-6.—Figure for Prob. 3-6.

3-6. Two loads having impedances $\dot{Z}_1 = 4 + j0$ and $\dot{Z}_2 = 5 + j8$ are connected in parallel and fed from a 440-volt circuit. This combination is supplied from a station over lines having an impedance Z_3 composed of a resistance of 0.05 ohm and an inductive reactance of 0.06 ohm as shown in Fig. 3-6. Find the station voltage and the total current. Find the total active, reactive, and vector power taken by the loads, and also delivered from the substation.

4-6. Two loads having admittances $\dot{Y}_1 = 0.08\underline{/-60^\circ}$ and $\dot{Y}_2 = 0.10\underline{/30^\circ}$ are connected in parallel and an emf $\dot{E} = 1{,}000\underline{/45^\circ}$ is applied to these two branches. This combination is supplied over lines having the impedance $\dot{Z}_3 = 0.2\underline{/20^\circ}$, the circuit being shown in Fig. 3-6. Calculate the station voltage E_o, the current I_o, and the power and power factor of the total circuit.

5-6. In the circuit shown in Fig. 3-6, $\dot{Z}_3 = 0.1\underline{/45^\circ}$, $\dot{Y}_1 = 0.6\underline{/-30^\circ}$, $\dot{Y}_2 = 0.5\underline{/90^\circ}$, and $\dot{I}_2 = 100\underline{/45^\circ}$. Find the vector values of E, I, and E_o, and draw the vector diagram.

6-6. Two loads having impedances $\dot{Y}_1 = 0.25 + j0$ and $\dot{Y}_2 = 0 + j0.125$ are connected in parallel across a distribution circuit in which the voltage E is kept constant. The current in branch 1 is 30 amp. The parallel branches are fed through a circuit having a resistance of 0.3 ohm and an inductive reactance of 0.6 ohm. Compute the other currents and emfs for this circuit, and draw a vector diagram using the known current as the reference.

7-6. In the circuit shown in Fig. 3-6, $\dot{Z}_1 = 5 + j5$, $\dot{Z}_2 = 0 - j5$, $\dot{Z}_3 = 4 + j4$, and $\dot{E}_o = 400$ volts. Using E_o as the reference, find I_o and E, and draw the vector diagram.

8-6. In Fig. 3-6, $\dot{Z}_1 = 3 + j6$, $\dot{Z}_2 = 3\underline{/-60^\circ}$, $\dot{Z}_3 = 2\underline{/30^\circ}$, and $\dot{E}_o = 1{,}000\underline{/20^\circ}$. Calculate the values of the total and branch currents and of the emf drop across the parallel branches. Draw the vector diagram.

9-6. A group of three impedances $\dot{Z}_1 = 0.3 + j0.6$, $\dot{Z}_2 = 0.8 - j0.3$, and $\dot{Z}_3 = 0.5 + j0$ are connected in parallel. A second group of two impedances $\dot{Z}_4 = 0.3 + j0.4$ and $\dot{Z}_5 = 0.5 - j1$ are also connected in parallel. The two groups are connected in series, and an emf of 84 volts is applied. Find the total current and the current in each branch.

10-6. Two impedances, $\dot{Z}_1 = 10\underline{/40^\circ}$ and $\dot{Z}_2 = 10\underline{/-90^\circ}$, are connected in parallel. This combination is connected in series with an impedance $\dot{Z}_3 = 5\underline{/45^\circ}$, Fig. 3-6, and an emf of 400 volts is applied to the total circuit. Calculate the values of the total current, the branch currents, and the emf across the branches. Draw the vector diagram with the impressed emf as the reference.

11-6. The current I_o taken by the circuit of Prob. 8-6 is 20 amp at 0.60 power factor lagging, and the emf is $E_o = 100$ volts. Find the values of E and I, and draw the vector diagram using E_o as the reference, assuming that $\dot{Y}_1 = 0.05\underline{/-60^\circ}$, $\dot{Z}_3 = 2\underline{/30^\circ}$ and that Y_2 is to be determined.

12-6. An emf $E = 86$ volts is applied to the network of Fig. 12-6. The impedances are $\dot{Z}_2 = 0.08 - j0.6$, $\dot{Z}_1 = 0.2 + j0.3$, $\dot{Z}_3 = 0.05 + j0.7$, $\dot{Z}_4 = 1.0 + j0.6$, and $\dot{Z}_5 = 0.09 - j0.2$. Find the current delivered to the network.

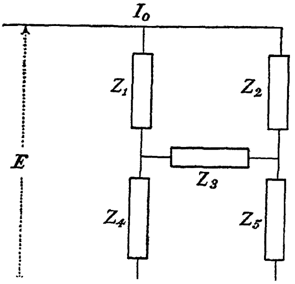


FIG. 12-6.—Figure for Prob. 12-6.

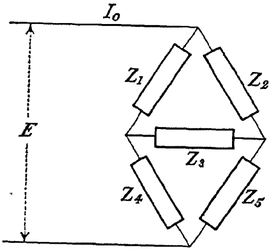


FIG. 13-6.—Figure for Prob. 13-6.

13-6. An emf of 293 volts is applied to the network of Fig. 13-6. The impedances are $\dot{Z}_1 = 8\underline{/34^\circ}$, $\dot{Z}_2 = 4.6\underline{/0^\circ}$, $\dot{Z}_3 = 7.3\underline{/-16^\circ}$, $\dot{Z}_4 = 1.8\underline{/78^\circ}$, and $\dot{Z}_5 = 2.7\underline{/-60^\circ}$. Find the current supplied to the network.

14-6. In the network of Fig. 14-6, the impedances are $\dot{Z}_1 = 106\underline{/30^\circ}$, $\dot{Z}_2 = 83\underline{/-20^\circ}$, $\dot{Z}_3 = 25\underline{/25^\circ}$, $\dot{Z}_4 = 45\underline{/0^\circ}$, $\dot{Z}_5 = 30\underline{/90^\circ}$, $\dot{Z}_6 = 126\underline{/0^\circ}$, $\dot{Z}_7 = 40\underline{/90^\circ}$, and $\dot{Z}_8 = 70\underline{/-45^\circ}$. An emf of 1,164 volts is applied to the network. Find the total current.

15-6. Two coils having self-inductances of $L_1 = 0.0645$ henry and $L_2 = 0.0218$ henry are so placed as to have a mutual inductance of 0.0263 henry. What is the coefficient of coupling?

16-6. Two coils have self-inductances of $L_1 = 0.738$ henry and $L_2 = 0.536$ henry. What must be the mutual inductance in order to have a coefficient of coupling of $K = 0.7$?

17-6. Two coils having self-inductances of $L_1 = 0.016$ henry and $L_2 = 0.021$ henry and a mutual inductance of $M = 0.009$ henry are connected in series. What is the combined inductance of the coils?

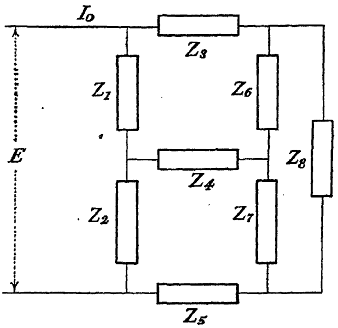


FIG. 14-6.—Figure for Prob. 14-6.

18-6. Two admittances, $\dot{Y}_1$ $0.25\underline{/45^\circ}$ and $\dot{Y}_2 = 0.125\underline{/-90^\circ}$, are connected in parallel across a distribution circuit that supplies an emf of E volts. The circuit supplying the two branches has an impedance $= 0.5\underline{/60^\circ}$, Fig. 3-6, and the current flowing through impedance 1 is $= 250\underline{/30^\circ}$. Calculate the value of the station voltage E_o, the emf drop across the branches, the current in branch 2, and the total current.

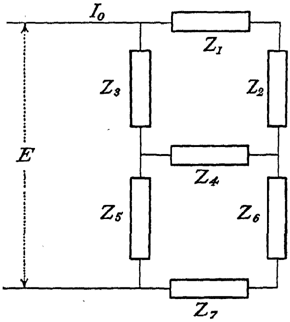


FIG. 21-6.—Figure for Prob. 21-6.

19-6. Given the same circuit and constants as in Prob. 1-6, but with the impressed emf given as 300 volts. Find the total current, the branch currents, and the emf drop across the branches.

20-6. Two loads, of which one is $\dot{Y}_1 = 0.02 - j0.08$, are fed from a station supplying 400 volts over a line having the impedance $\dot{Z}_3 = 4 - j4$ ohms. The current taken from the station is 20 amp at 0.90 power factor, current lagging; find the value of the other admittance Y_2.

21-6. In the network of Fig. 21-6, the impedances are $\dot{Z}_1 = 24\underline{/-75^\circ}$, $\dot{Z}_2 = 10\underline{/0^\circ}$, $\dot{Z}_3 = 16.1\underline{/30^\circ}$, $\dot{Z}_4 = 20\underline{/90^\circ}$, $= 5\underline{/10^\circ}$, $\dot{Z}_6 = 12\underline{/-90^\circ}$ and $\dot{Z}_7 = 8\underline{/0^\circ}$. A current of 26.4 amp flows into the network; find the emf impressed and the distribution of current in the network.

22-6. In the network of Fig. 21-6, the impedances are $\dot{Z}_1 = 0.05 + j0.1$, $\dot{Z}_2 = 0.1 + j0.05$, $\dot{Z}_3 = 0.005 + j0.06$, $\dot{Z}_4 = 0.1 + j0$, $\dot{Z}_5 = 0.05 - j0.06$, $\dot{Z}_6 = 0.1 + j0.8$, and $\dot{Z}_7 = 0.01 + j0.07$. Find the current if an emf of 45 volts is applied. Find the distribution of current in the network.

23-6. Two coils with self-inductances of $L_A = 3.25$ henrys and $L_B = 2.63$ henrys are so placed as to have a coefficient of coupling of 0.4. (*a*) Find the combined inductance if the two coils are connected in series with the fluxes in the same direction. (*b*) Find the inductance of the two if connected in series with the fluxes opposing.

CHAPTER VII

THREE-WIRE ALTERNATING-CURRENT CIRCUITS

In order to avoid confusion in making an analysis of a circuit involving a combination of emfs or currents, certain basic principles or rules must be established and followed in the solution of the problem. First, a definite reference time must be established, and all quantities must be stated with reference to this time in order that they may be combined. Second, the direction in which the quantities being considered are acting in the circuit must be established before they can be used in the combination. A direct emf may be connected into the circuit in either of two ways: either its emf is in the direction through which the circuit is being traversed when it is considered positive; or else it is opposite to the direction taken through the circuit, under which condition it is considered negative.

Double subscripts are used in equations and vector diagrams to indicate the direction in the circuit in which the quantities are being considered; *e.g.*, E_{ab} indicates that the emf between points a and b of a circuit is being considered from a to b, and the numerical value of E_{ab} is positive if the emf E_{ab} creates a rise in potential. If the circuit is being considered from b to a, the emf is denoted as E_{ba}. If the emf between a and b is in such a direction that E_{ab} is positive, then the same emf, denoted by E_{ba}, is negative, *i.e.*,

$$E_{ba} = -E_{ab} \text{ volts.} \qquad (422)$$

The same relations hold true in respect to current flowing, and

$$I_{ba} = -I_{ab} \text{ amp.} \qquad (423)$$

In the following paragraphs, a short discussion of the three-wire d-c circuit will first be given as a review and foundation for the analysis of the a-c three-wire circuit.

76. Three-wire D-c Circuit Equations.—Consider first the three-wire d-c circuit in Fig. 95, where the positive brush b

of machine bc is connected to the negative brush b of machine ba, and the neutral wire is connected to the common junction b. The generator ba maintains the point a 110 volts above the point b. Since a source of emf sends a current from a point of low potential to a point of high potential within the source, $E_{ba} = +110$ volts, and sends the current I_{ba} from b to a within the generator. In the external load, the current I_{AB} flowing through the resistance R_1 produces a voltage drop

$$E_{AB} = I_{AB}R_1 = 110 \text{ volts,} \tag{424}$$

so that B is 110 volts below A in potential. The current

$$I_{ba} = I_{AB}$$

may be considered as being produced by the potential rise in the generator E_{ba} or as producing the voltage drop E_{AB} across the load.

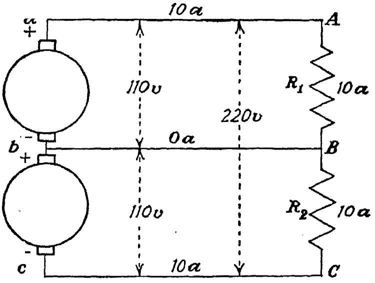


FIG. 95.—Three-wire d-c circuit with balanced loads.

Applying Kirchhoff's law that the sum of the potential rises in a closed circuit, such as $baAB$, must be equal to the sum of the voltage drops, it is seen that

$$E_{ba} = E_{AB}, \tag{425}$$

$$E_{ba} - E_{AB} = 0,$$

$$E_{ba} - I_{AB}R_1 = 0, \tag{426}$$

$$110 - 110 = 0 \text{ volts.}$$

Similar relations exist for the closed circuit $cbBC$.

The voltage relations in the entire three-wire circuit may be considered from the point of view of cC, bB, or aA. When the generator voltages are considered from the point of view c,

b is above c + 110 volts, or the potential rise $E_{cb} = 110$ volts,
a is above b + 110 volts, or the potential rise $E_{ba} = 110$ volts;

hence

a is above c + 220 volts, or the potential rise $E_{ca} = 220$ volts.

This may be expressed in the form of an equation,

$$E_{ca} = E_{cb} + E_{ba} = 110 + 110 = 220 \text{ volts.} \tag{427}$$

Considering the voltage across the load from the point of view C,
C is below B 110 volts, or the voltage drop

$$E_{BC} = I_{BC}R_2 = 110 \text{ volts}, \tag{428}$$

B is below A 110 volts, or the voltage drop

$$E_{AB} = I_{AB}R_1 = 110 \text{ volts}; \tag{424}$$

hence, C is below A 220 volts, or the voltage drop $E_{AC} = 220$ volts, and the equation of the voltage drops is

$$\begin{aligned} E_{AC} &= E_{AB} + E_{BC} \\ &= I_{AB}R_1 + I_{BC}R_2 \\ &= 110 + 110 = 220 \text{ volts}. \end{aligned} \tag{429}$$

Applying Kirchhoff's law to each of the closed circuits,

$$E_{cb} - E_{BC} = 0, \tag{430}$$

$$E_{ba} - E_{AB} = 0, \tag{431}$$

and adding Eqs. (430) and (431)

$$E_{cb} + E_{ba} - E_{BC} - E_{AB} = 0, \tag{432}$$

$$E_{cb} + E_{ba} - I_{BC}R_2 - I_{AB}R_1 = 0, \tag{433}$$

or

$$E_{ca} - E_{AC} = 0.$$

From the point of view b,

c is above b $-$ 110 volts, or the potential rise $E_{bc} = -110$ volts,
a is above b $+$ 110 volts, or the potential rise $E_{ba} = +110$ volts;

hence, a is above c $+$ 220 volts, or the potential rise $E_{ca} = 220$ volts, or

$$E_{ca} = E_{cb} + E_{ba} \tag{434}$$

$$\begin{aligned} &= -E_{bc} + E_{ba} \\ &= -(-110) + 110 = 220 \text{ volts}. \end{aligned} \tag{435}$$

From the point of view B,

B is below A $+$ 110 volts, or the voltage drop

$$E_{AB} = I_{AB}R_1 = 110 \text{ volts}, \tag{424}$$

B is below C $-$ 110 volts, or the voltage drop

$$E_{CB} = -I_{BC}R_2 = -110 \text{ volts}; \tag{428}$$

hence the voltage drop

$$\begin{aligned} E_{AC} &= E_{AB} + E_{BC} \\ &= E_{AB} - E_{CB} \\ &= I_{AB}R_1 - (-I_{BC}R_2) \\ &= I_{AB}R_1 + I_{BC}R_2 = 220 \text{ volts.} \end{aligned} \tag{429}$$

From the point of view a,

Potential rise $E_{ab} = -110$ volts,
Potential rise $E_{bc} = -110$ volts,
Potential rise $E_{ac} = -220$ volts,

$$E_{ac} = E_{ab} + E_{bc}. \tag{436}$$

From the point of view A,

Voltage drop $E_{AB} = I_{AB}R_1 = 110$ volts,
Voltage drop $E_{BC} = I_{BC}R_2 = 110$ volts,
Voltage drop $E_{AC} = 220$ volts,

$$E_{AC} = E_{AB} + E_{BC}. \tag{429}$$

The preceding consideration of a familiar problem is given to demonstrate the necessity of differentiating between a potential rise produced by a source of emf and a voltage drop produced by a current. It also shows the necessity for establishing a systematic form of notation for the consideration of several related emfs. This will be still more apparent in the later consideration of emfs out of phase as in a polyphase circuit. The point of view must always be known in the statement of the equations or in drawing the vector diagram of a circuit in order to determine the direction of the quantities under consideration. From a comparison of Eqs. (427) to (436) it is seen that *reversing the order of the letters in the subscript reverses the sign of the quantity, reversing the sign and the order of the letters of the subscript does not change the sign of the numerical value of the quantity, i.e.,* if $E_{ba} = +110$ volts then $E_{ab} = -110$ volts and $-E_{ab} = +110$ volts.

77. Current Equations in Three-wire D-c Circuits.—The currents in a three-wire d-c circuit may be considered in a similar manner, as shown below, where Kirchhoff's law of the sum of the currents flowing into, or out of, a point must apply at every point of consideration of the circuit. For the circuit of Fig. 95

the following conditions hold. At the point B of the load,

$$I_{bB} + I_{AB} + I_{CB} = 0 \text{ amp}, \tag{437}$$

or

$$I_{bB} = I_{BC} + I_{BA} \text{ amp}, \tag{438}$$
$$I_{Bb} = I_{CB} + I_{AB} \text{ amp}. \tag{439}$$

At the point b,

$$I_{bB} = I_{cb} + I_{ab} \text{ amp}. \tag{440}$$

Using the line currents,

$$I_{bB} = I_{Cc} + I_{Aa} \text{ amp} \tag{441}$$

or

$$I_{bB} + I_{cC} + I_{aA} = 0 \text{ amp} \tag{442}$$

expresses the conditions at the load. *From Eq.* (442) *it is seen that the sum of the currents in a circuit, considered in a given direction passing through a plane of intersection with the three wires, must be equal to zero.*

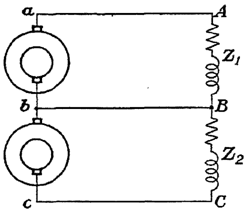


FIG. 96.—Three-wire a-c circuit furnished from two single-phase generators.

78. Electromotive Force in Three-wire A-c Circuits.—Consider a three-wire circuit supplied from two a-c generators mechanically coupled in such a manner as to have their emf waves always of the same frequency and in synchronism, as in Fig. 96. The emf and current relations, expressed by Kirchhoff's laws, must be satisfied at every instant of time in any closed circuit. When Kirchhoff's law of generated emfs and voltage drops is applied to the circuit $cbBC$, Fig. 96, and when instantaneous values are used, the equation will be

$$e_{cb} = e_{BC} \text{ volts}. \tag{443}$$

For the circuit $baAB$ and $caAC$, the equations are, respectively,

$$e_{ba} = e_{AB} \text{ volts}, \tag{444}$$

and

$$e_{cb} + e_{ba} = e_{AB} + e_{BC} \text{ volts}. \tag{445}$$

Equations (443) to (445) hold for the values of generated emf and impedance drop at all instants of time; and the trigonometric expression of the wave forms or the vectors may be substituted for the instantaneous values, provided that all expressions for the wave forms in the original equations are given with reference to the same starting or zero time.

Since Eqs. (443), (444), and (445) hold for the values of generated voltage and impedance drops at all instants of time, they may be written as vectors as follows:

$$\dot{E}_{cb} = \dot{E}_{BC} = \dot{I} \qquad {}_2 \text{ volts,} \tag{446}$$

$$\dot{E}_{ba} = \dot{E}_{AB} = \dot{I} \qquad {}_1 \text{ volts,} \tag{447}$$

$$\dot{E}_{cb} + \dot{E}_{ba} = \dot{E}_{AB} + \dot{I} \qquad = \dot{I}_{AB}\dot{Z}_1 + \dot{I}_{BC}Z_2 \text{ volts.} \tag{448}$$

In these vector equations, the time-phase relations of the various vector quantities are indicated by the magnitudes and the signs of the components in the complex form and by the angles in the polar form, while the double subscripts indicate their space-phase relations. The use of the double subscripts enables one to designate clearly the voltages and currents in a-c circuits.

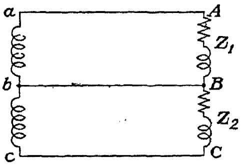


FIG. 97.—Three-wire a-c circuit furnished from an a-c source of energy, emfs in

Equations (446), (447), and (448) are expressions of the statement of Kirchhoff's law as applied to a-c circuits, *i.e.*, *the vector sum of the generated emfs taken around a circuit in a given direction is equal to the vector sum of the voltage drops taken around the circuit in the same direction, all vectors having the same time reference.*

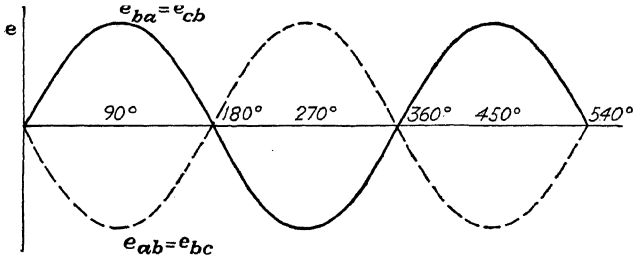


FIG. 98.—Wave forms for the a-c source of Fig. 97.

Consider the two generators of Fig. 96 to be so coupled that their emf waves are in time phase. Figure 97 illustrates a method of designating two a-c machines connected in series with their generated emfs in phase, their wave forms being shown in Fig. 98.

This method is commonly used and will be adopted for future representation of coils in different phase relations where they are associated in the same machine or by connection. In Fig. 97, the voltage drop E_{AB} is in time phase with the generated emf E_{ba}, and the current flowing in the circuit $baAB$ may be considered as I_{ba}, produced by the generated emf E_{ba}, or as the current I_{AB}, produced by the voltage drop $I_{AB}Z_1$. Since this text is limited to the consideration of outside receiving circuits and does not concern itself with the reactions within the source, further discussions will be based upon the voltage drop across the load rather than the generated emf within the source.

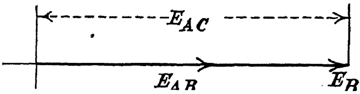


FIG. 99.—Vector diagram for Fig. 97, E_{AB} as the reference.

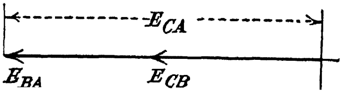


FIG. 100.—Vector diagram for Fig. 97, E_{AB} as the reference.

Applying Kirchhoff's law to the circuit of Fig. 97, using vector notation and point of view A, the equation is

$$\dot{E}_{AC} = \dot{E}_{AB} + \dot{E}_{BC} \text{ volts.} \tag{449}$$

With a double-subscript notation, the point of view is included in the subscript, the first letter indicating this point of view. This is true not only for the original vectors, as E_{AB} and E_{BC}, but also for the composite vector consisting of the sum of two or more vectors, such as E_{AC} in Eq. (449); the subscripts indicating that the voltage vector E_{AC} is composed of the sum of all voltage vectors in the circuit between A and C, *i.e.*, from the point of view A. The corresponding vector diagram is shown in Fig. 99, where E_{AB} is chosen as the reference vector.

From the point of view C, the equation would be written

$$\dot{E}_{CA} = \dot{E}_{CB} + \dot{E}_{BA} \text{ volts,} \tag{450}$$

which is illustrated in Fig. 100, the same time reference being used for both vector diagrams.

Since reversing the direction in which a circuit is being considered shifts the phase position of the vectors by 180 deg., *i.e.*, changes the sign of the voltage vectors, the solution of Fig. 97 may be expressed as

$$\dot{E}_{AC} = \dot{E}_{BC} - \dot{E}_{BA} \text{ volts.} \tag{451}$$

79. Current Relations in Three-wire A-c Circuits.—In considering the current relations in three-wire a-c circuits, the direction in which the currents are taken must be considered, and all quantities must have the same time reference, usually one of the voltage vectors. Kirchhoff's law expressing the current relations in the circuit may be stated thus: *The vector sum of the currents flowing toward (or away from) any point in a circuit is equal to zero.* Two methods of procedure are open for use in this case. First, a suitable reference, usually the voltage across the load, may be chosen for each single-phase or two-wire load, the current being expressed in respect to this reference. Then the emf and resulting currents of all the loads must be rotated until all are expressed in respect to the same common reference. When this has been done, the equations derived by applying Kirchhoff's laws may be used, or else graphical solutions made by means of vector diagrams.

In the second method, all emfs are first expressed in respect to a common reference, and the various currents are found by applying these new voltages to the impedances of the loads or branches. Both methods lead to the same result, the final expressions being identical if the same reference is chosen. Both methods will be illustrated in the examples.

Applying Kirchhoff's law to point B of Fig. 97. the equation for the currents is

$$\dot{I}_{bB} + \dot{I}_{AB} + \dot{I}_{CB} = 0 \text{ amp.} \tag{452}$$

Solving for the current in the common wire,

$$\dot{I}_{bB} = -\dot{I}_{AB} - \dot{I}_{CB} \text{ amp,} \tag{453}$$

or

$$\dot{I}_{bB} = \dot{I}_{BA} + \dot{I}_{BC} \text{ amp.} \tag{454}$$

Equation (454) may be written in terms of the in-phase and quadrature components of the currents as

$$\dot{I}_{bB} = i_{BA} + ji_{BA}' + i_{BC} + ji_{BC}' \text{ amp} \tag{455}$$

$$= (i_{BA} + i_{BC}) + j(i_{BA}' + i_{BC}') \text{ amp.} \tag{456}$$

It can be seen from Eq. (456) that there will be no current in the common return wire if I_{AB} is equal to I_{BC}, *i.e.*, if I_{BA} is equal and opposite to I_{BC}. However, if these currents differ in amount or have a different phase relation in respect to the same reference, there will be a resultant current in the common return wire. If the emfs across the loads are in phase with each other, as in Fig. 97, and the reference time is taken as coinciding with these two emfs, the in-phase components of the currents become the active components, and the quadrature components become the reactive components of their respective loads. This is the most useful method of analyzing this type of circuit, and the common emf vectors should normally be chosen as the reference.

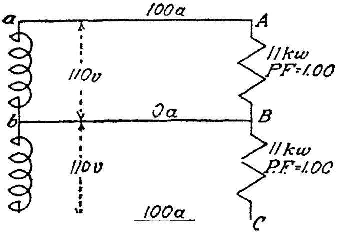


FIG. 101.—Three-wire single-phase circuit, with balanced nonreactive loads.

The vectors forming the currents and voltages of an electric circuit, according to the laws of vectors, may be moved to any position on the vector diagram so long as the magnitudes and directions of the vectors are kept the same. This leads to two types of diagrams in one of which the tails of all vectors are placed at the origin, the one commonly used heretofore, in this chapter, and the other in which the vectors are placed in consecutive order, the end of one vector being the starting point of the next, thus conforming, in general, to the relative space location of the loads. Such a diagram is shown in Fig. 102, where E_{CB}, E_{BA}, and E_{CA} have the same relative positions as the space positions of the loads in Fig. 101, the current vectors I_{CB} and I_{BA} starting at the same point as their respective voltage vectors. This type of diagram is often very useful in considering circuits containing two or more voltages that are not in phase. Combinations of these two types of diagrams are permissible, different combinations being applicable to different types of problems.

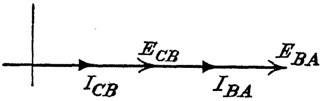


FIG. 102.—Vector diagram for the loads of Fig. 101, E_{CB} as the reference.

Example 34

Two nonreactive loads of 11 kw each are connected to a 110- to 220-volt three-wire a-c circuit (Fig. 101). Find the neutral current and the current in each outside wire.

$$i_{CB} = \frac{11 \times 1{,}000}{110} = 100 \text{ amp} = i_{BA} \text{ amp},$$

$$i_{CB}' = i_{BA}' = 0 \text{ amp.}$$

The neutral current becomes, from Eq. (454),

$$\dot{I}_{bB} = 100 + j0 - 100 - j0 = 0 \text{ amp},$$

illustrated in Fig. 102.

Example 35

With the same circuit as in Example 34 but with load BC reduced to 5.5 kw nonreactive (Fig. 103), find the current in each wire.

i_{BC} = -50 amp, i_{AB} -100 amp,

and from Eq. (454)

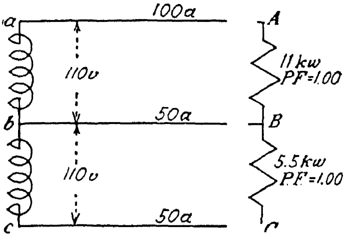


FIG. 103.—Three-wire single-phase circuit, with unbalanced nonreactive loads.

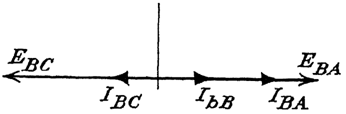


FIG. 104.—Vector diagram for the loads of Fig. 103, E_{BA} as the reference.

$$\dot{I}_{bB} = i_{BC} - i_{AB} = -50 + 100 = +50 \text{ amp},$$

flowing from b toward B, or negative in regard to the emf vector E_{AB}, or

$$\dot{I}_{Bb} = i_{AB} + i_{CB} = -100 + 50 = -50 \text{ amp},$$

as shown in Fig. 104.

Example 36

Two inductive loads (Fig. 105) are connected across the line in place of the loads of Example 34. If these loads are as follows:

$$\dot{I}_{CB} = 40 - j30 \text{ amp}$$

of magnitude,

$$I_{CB} = 50 \text{ amp},$$

and

$$\dot{I}_{BA} = 40 - j30 \text{ amp}$$

of magnitude,

$$I_{BA} = 50 \text{ amp},$$

find the current in each of the wires. From Eq. (457),

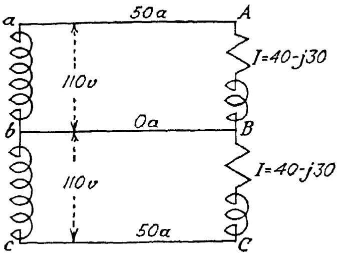


FIG. 105.—Three-wire single-phase circuit, with balanced inductive loads.

$$\begin{aligned}\dot{I}_{Bb} &= \dot{I}_{CB} - \dot{I}_{BA} \text{ amp}\\ &= (40 - 40) + j(30 - 30) = 0 \text{ amp},\end{aligned}$$

as shown in Fig. 106, or

$$\begin{aligned}\dot{I}_{bB} &= \dot{I}_{BA} + \dot{I}_{BC} \text{ amp}\\ &= (40 - 40) + j(-30 + 30) = 0 \text{ amp},\end{aligned}$$

as shown in Fig. 107.

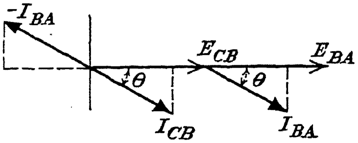


FIG. 106.—Vector diagram for the loads of Fig. 105, E_{CB} as the reference.

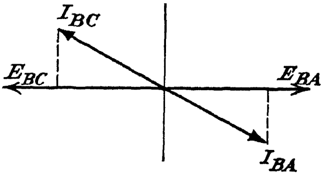


FIG. 107.—Vector diagram for the loads of Fig. 105, E_{BA} as the reference.

Example 37

A nonreactive load BC, and an inductive load AB, each of 50 amp, are connected as shown in Fig. 108. Find the currents in each of the three wires.

From the statement of the problem,

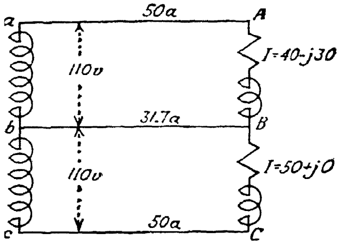


FIG. 108.—Three-wire single-phase circuit with loads balanced in magnitude, but unbalanced in phase.

$$\dot{I}_{BC} = 50 + j0 \text{ amp}$$

of magnitude,

$$I_{BC} = 50 \text{ amp},$$

and

$$\dot{I}_{AB} = 40 - j30 \text{ amp}$$

of magnitude,

$$I_{AB} = 50 \text{ amp}.$$

From Eq. (454),

$$\begin{aligned}\dot{I}_{bB} &= (50 - 40) + j(0 + 30) \text{ amp}\\ &= 10 + j30 \text{ amp}\end{aligned}$$

of magnitude,

$$I_{bB} = \sqrt{\overline{10}^2 + \overline{30}^2} = 31.7 \text{ amp},$$

with vector diagram as shown in Fig. 109. The following solution is also useful:

$$\begin{aligned}&= \dot{I}_{CB} + \dot{I}_{AB} \text{ amp}\\ &= -50 + 40 - j30 = -10 - j30 \text{ amp},\end{aligned}$$

of magnitude,

$$I_{Bb} = \sqrt{\overline{10}^2 + \overline{30}^2} = 31.7 \text{ amp}$$

and as shown in Fig. 110. The neutral wire carries all the reactive component and the unbalanced portion of the active component of current.

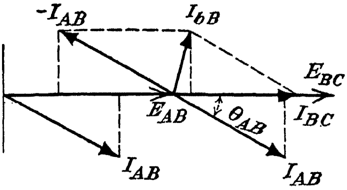


FIG. 109.—Vector diagram for loads of Fig 108, E_{AB} as the reference.

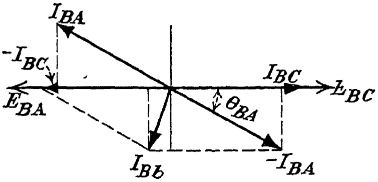


FIG. 110.—Vector diagram for loads of FIG. 108, E_{BC} as the reference.

Example 38

In Fig. 111, the emfs E_{cb} and E_{ba} are in time phase. The two load impedances are $\dot{Z}_{AB} = 5 + j8.66$ and $\dot{Z}_{BC} = 5 - j8.66$. What are the three line currents?

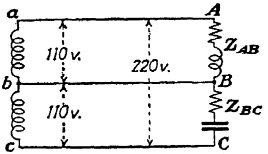


FIG. 111.—Circuit for Example 38.

$$\begin{aligned}
\dot{E}_{AB} &= \dot{E}_{ba} = 110\underline{/0^\circ} \text{ volts} \\
&= 110 + j0 \text{ volts}, \\
\dot{E}_{BC} &= \dot{E}_{cb} = 110\underline{/0^\circ} \text{ volts} \\
&= 110 + j0 \text{ volts}, \\
\dot{E}_{AC} &= \dot{E}_{ca} = 220\underline{/0^\circ} \text{ volts} \\
&= 220 + j0 \text{ volts}. \\
\dot{I}_{AB} &= \dot{I}_{ba} = \frac{\dot{E}_{AB}}{\dot{Z}_{AB}} \text{ amp} \\
&= \frac{110\underline{/0^\circ}}{10\underline{/60^\circ}} \text{ amp} \\
&= 11\underline{/-60^\circ} \text{ amp} \\
&= 5.5 - j9.53 \text{ amp}. \\
\dot{I}_{BC} &= \dot{I}_{cb} = \frac{\dot{E}_{BC}}{\dot{Z}_{BC}} \text{ amp} \\
&= \frac{110\underline{/0^\circ}}{10\underline{/-60^\circ}} \text{ amp} \\
&= 11\underline{/60^\circ} \text{ amp} \\
&= 5.5 + j9.53 \text{ amp}. \\
\dot{I}_{Bb} &= \dot{I}_{CB} + \dot{I}_{AB} \\
&= (-5.5 - j9.53) + (5.5 - j9.53) \text{ amp} \\
&= 0 - j19.06 \text{ amp}.
\end{aligned}$$

These relations are shown in a vector diagram in Fig. 112.

Example 39

An inductance $\dot{Z}_{Bb} = 0.433 + j0.25$ is placed in the line Bb of Example 38, as shown in Fig. 113. Find the line currents and the line voltages E_{AB} and E_{BC}.

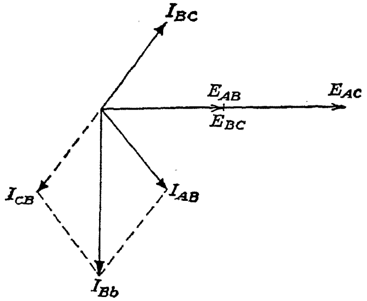


FIG. 112.—Vector diagram for the circuit of Fig. 111, Example 38.

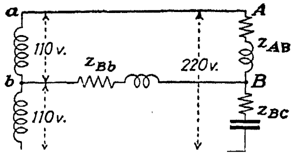


FIG. 113.—Three-wire a-c circuit with impedance in neutral wire for Example 39.

Applying Kirchhoff's laws to the closed circuits *baAB* and *caAC*,

$$\dot{E}_{ba} - \dot{I}_{AB}\dot{Z}_{AB} - \dot{I}_{Bb}\dot{Z}_{Bb} = 0 \text{ volts,} \tag{457}$$

$$\dot{E}_{ca} - \dot{I}_{AB}\dot{Z}_{AB} - \dot{I}_{BC}\dot{Z}_{BC} = 0 \text{ volts,} \tag{458}$$

$$\dot{I}_{Bb} = \dot{I}_{AB} - \dot{I}_{BC} \text{ amp.} \tag{459}$$

Substituting Eq. (459) in Eq. (457),

$$\dot{E}_{ba} - \dot{I}_{AB}(\dot{Z}_{AB} + \dot{Z}_{Bb}) + \dot{I}_{BC}\dot{Z}_{Bb} = 0 \text{ volts.} \tag{460}$$

Combining Eqs. (460) and (458),

$$\dot{I}_{AB} = \frac{\dot{E}_{ba}\dot{Z}_{BC} + \dot{E}_{ca}\dot{Z}_{Bb}}{\dot{Z}_{AB}\dot{Z}_{BC} + \dot{Z}_{Bb}(\dot{Z}_{AB} + \dot{Z}_{BC})} \text{ amp.}$$

Substituting the known values of the problem,

$$\begin{aligned}
\dot{I}_{AB} &= \frac{110\underline{/0^\circ}\ 10\underline{/-60^\circ} + 220\underline{/0^\circ}\ 0.5\underline{/30^\circ}}{10\underline{/60^\circ}\ 10\underline{/-60^\circ} + 0.5\underline{/30^\circ}\ (10\underline{/60^\circ} + 10\underline{/-60^\circ})} \text{ amp} \\
&= \frac{1100\underline{/-60^\circ} + 110\underline{/30^\circ}}{100\underline{/0^\circ} + 0.5(1\underline{/90^\circ} + 1\underline{/-30^\circ})} \text{ amp} \\
&= \frac{645.3 - j897.6}{104.33 + j2.5} \text{ amp} \\
&= \frac{1105.5\underline{/-54.3^\circ}}{104.35\underline{/1.35^\circ}} = 10.59\underline{/-55.65^\circ} \text{ amp} \\
&= 5.97 - j8.74 \text{ amp.} \\
\dot{E}_{AB} &= \dot{I}_{AB}\dot{Z}_{AB} = 10.59\underline{/-55.65^\circ}\ 10\underline{/60^\circ} \text{ volts} \\
&= 105.9\underline{/4.35^\circ} = 105.6 + j8.02 \text{ volts.}
\end{aligned}$$

Substituting in Eq. (454),

$$\dot{I}_{Bb} = \frac{\dot{E}_{ba} - \dot{I}_{AB}\dot{Z}_{AB}}{\dot{Z}_{Bb}} \text{ amp,}$$

$$\dot{I}_{Bb} = \frac{(110 + j0) - (105.6 + j8.02)}{0.5\underline{/30^\circ}} = \frac{4.4 - j8.02}{0.5\underline{/30^\circ}} \text{ amp}$$

$$
\begin{aligned}
&= \frac{9.14\underline{/-61.25^\circ}}{0.5\underline{/30^\circ}} = 18.28\underline{/-91.25^\circ} \text{ amp} \\
&= -0.398 - j18.28 \text{ amp.} \\
&= 18.28\underline{/-91.25^\circ}\ 0.5\underline{/30^\circ} \text{ amp} \\
&\qquad\qquad = 9.14\underline{/-61.25^\circ} \text{ amp} \\
&\qquad\qquad = 4.39 - j8.02 \text{ amp.} \\
\dot{E}_{BC} &= \dot{E}_{AC} - \dot{E}_{AB} \\
&= (220 + j0) - (105.6 + j8.02) \text{ volts} \\
&= 114.4 - j8.02 \text{ volts} \\
&= 114.7\underline{/-4^\circ} \text{ volts.} \\
\dot{I}_{BC} &= \frac{\dot{E}_{BC}}{\dot{Z}_{BC}} = \frac{114.7\underline{/-4^\circ}}{10\underline{/-60^\circ}} \\
&= 11.47\underline{/56^\circ} \text{ amp} \\
&= 6.4 + j9.5 \text{ amp} \\
\dot{I}_{AB} - \dot{I}_{Bb} - \dot{I}_{BC} &= 0 \text{ amp.}
\end{aligned}
$$

The vector diagram illustrating this example is shown in Fig. 114.

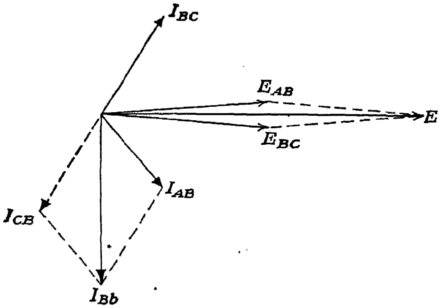


FIG. 114.—Vector diagram for circuit of Fig. 113, Example 39.

Example 40

Two loads represented by Fig. 115 and as

$$\dot{I}_{CB} = 40 + j30 \text{ amp}$$

and

$$\dot{I}_{BA} = 40 - j30 \text{ amp}$$

are connected to a three-wire a-c circuit. Find the currents in the three wires.

From Eq. (454),

$$\dot{I}_{Bb} = (40 - 40) + j(30 + 30) = j60 \text{ amp}$$

of magnitude,

$$I_{bB} = 60 \text{ amp.}$$

The neutral wire carries the entire reactive component of the currents for the two loads, but the energy components of the load currents balance, as illustrated in Fig. 116.

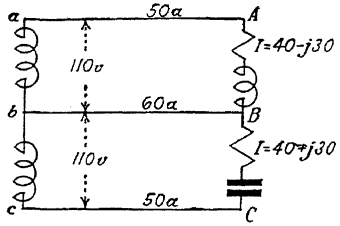


FIG. 115.—Three-wire single-phase circuit for Example 40.

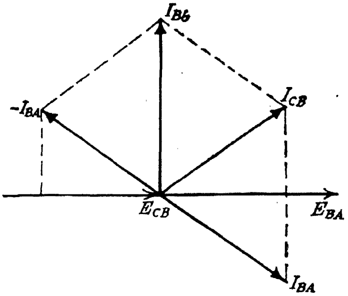


FIG. 116.—Vector diagram for the loads of Fig 115, E_{BA} as the reference.

Example 41

Using the same loads between wires as in Example 37 and a load of

$$\dot{I} = 40 - j30 \text{ amp}$$

connected between wires A and C, find the current in each wire.

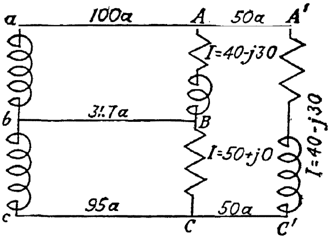


FIG. 117.—Three-wire single-phase circuit with unbalanced loads and a load between outside wires.

At point A, Fig. 117,

$$\dot{I}_{aA} + \dot{I}_{BA} + \dot{I}_{C'A'} = 0 \text{ amp}, \quad (461)$$

or

$$\dot{I}_{aA} = \dot{I}_{AB} + \dot{I}_{A'C'} \text{ amp}.$$

From the statement of the problem,

$$\dot{I}_{AB} = 40 - j30 \text{ amp},$$
$$\dot{I}_{A'C'} = 40 - j30 \text{ amp},$$

and

$$\dot{I}_{BC} = 50 + j0 \text{ amp};$$

hence from Eq. (461),

$$\begin{aligned}\dot{I}_{aA} &= 40 - j30 + 40 - j30 \text{ amp}\\ &= 80 - j60 \text{ amp},\end{aligned}$$

having the magnitude

$$I_{aA} = \sqrt{80^2 + 60^2} = 100 \text{ amp},$$

At the point B,

$$\dot{I}_{Bb} + \dot{I}_{BA} + \dot{I}_{BC} = 0 \text{ amp},$$
$$\begin{aligned}\dot{I}_{Bb} &= \dot{I}_{AB} - \dot{I}_{BC} \text{ amp}\\ &= 40 - j30 - 50 + j0 \text{ amp}\\ &= -10 - j30 \text{ amp},\end{aligned}$$

having the magnitude,

$$I_{Bb} = \sqrt{\overline{10}^2 + \overline{30}^2} = 31.7 \text{ amp},$$

as in Example 37.

At the point C,

$$\dot{I}_{Cc} + \dot{I}_{CB} + \dot{I}_{C'A'} = 0 \text{ amp},$$

or

$$\begin{aligned}\dot{I}_{Cc} &= \dot{I}_{BC} + \dot{I}_{A'C'} \text{ amp}\\ &= 50 + 40 - j30 \text{ amp}\\ &= 90 - j30 \text{ amp}\end{aligned}$$

of magnitude,

$$I_{Cc} = \sqrt{\overline{90}^2 + \overline{30}^2} = 95 \text{ amp},$$

as illustrated in Fig. 118. The current flowing in the neutral wire could also be found from the two line currents $\dot{I}_{Aa}$ and $\dot{I}_{Cc}$, by the use of Eq. (441).

For conditions where the emfs are of different values or where there are more than two emfs and three wires, the same methods apply as in the consideration of the three-wire a-c circuit in the preceding examples.

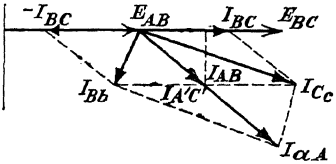


FIG. 118.—Vector diagram for loads of Fig. 117, E_{AB} as the reference.

80. Electromotive Forces 90 Deg. Out of Phase in a Three-wire A-c Circuit.—The emfs supplying the three-wire circuit may be out of time phase with each other, owing to the position of the coils on the armature of the generator or to other causes. The same methods and the same equations apply, however, as have been developed in the previous paragraphs. In view of the fact that the double subscripts indicate the space relations of the voltages across the terminals of the generator, the equations for the voltages across the generator of Fig. 119 may be written as follows:

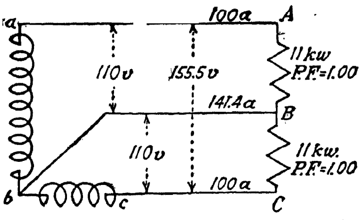


FIG. 119.—Three-wire a-c circuit with emfs at 90 deg., and balanced nonreactive loads.

$$\begin{aligned}\dot{E}_{ac} &= \dot{E}_{ab} + \dot{E}_{bc} \text{ volts}\\ &= -\dot{E}_{ba} + \dot{E}_{bc} \text{ volts}\\ &= \dot{E}_{ab} - \dot{E}_{cb} \text{ volts.}\end{aligned} \tag{462}$$

These equations hold regardless of the time-phase relations of the various voltages. The coils may be so arranged on the armature that, with a given direction of rotation,

$$e_{cb} = E_{m_{cb}} \sin \omega t \text{ volts} \tag{463}$$

and

$$e_{ab} = E_{m_{ab}} \sin (\omega t + \beta) \text{ volts.} \tag{464}$$

This means that the point b reaches its maximum potential above a, β deg. in time phase before b reaches its maximum potential above c, or that E_{ab} is β deg. ahead of E_{cb} in time phase. A reversed connection of the coil ab will make a reach its maximum potential above b, β deg. in time phase before b reaches its maximum potential above c, and the equations become

$$e_{cb} = E_{m_{cb}} \sin \omega t \text{ volts,} \tag{463}$$

$$e_{ba} = E_{m_{ba}} \sin (\omega t + \beta) \text{ volts,} \tag{465}$$

or E_{ba} is now β deg. ahead of E_{cb}. The time-phase angle β may have any value, positive or negative, between 0 and 180 deg. In any particular case, it is, in general, unknown whether β is positive or negative, since this depends on the direction of rotation of the generator field or armature. It is always necessary to know, however, the magnitude of the time-phase angle β and whether this is the angle between E_{cb} and E_{ab} or between E_{cb} and E_{ba}. Since the sign of β is, in general, immaterial for the solution of the problem, it is customary to assume this as the time-phase relation between the voltages E_{cb} and E_{ab} or between E_{bc} and E_{ba}. It is also convenient, but not at all necessary, to draw the figure representing the circuit with the space angle between the coils the same as the time-phase angle β.

Similar considerations apply, of course, to the load voltages. Thus, in Fig. 119, the emfs are given as being 90 deg. out of phase with each other. In such a circuit, it is common practice to consider the common point or junction of the coils as the "point of view," the emf vectors being drawn outward from that point as shown by E_{BA} and E_{BC} in Fig. 120. In the consideration of this diagram, it is assumed that the maximum voltage acting from B to A occurs one-quarter cycle before the maximum voltage acting from B to C, or that E_{BA} is 90 deg. ahead of E_{BC}. Applying the equations as previously derived, the emf from C to A will be

$$\dot{E}_{CA} = \qquad + \dot{E}_{BA} \text{ volts.} \tag{449}$$

It is to be noted that Eq. (449) is a basic equation and is obtained from the circuit connections of Fig. 119, independent of the phase positions of the emfs in the circuit. In the vector diagram of Fig. 120, E_{BC} is shown as the reference. For use in Eq. (449), E_{CB} must be obtained. This is very easily done by reversing E_{BC}, as shown in Fig. 120, since E_{CB} is the negative of E_{BC}. Then the vectors E_{CB} and E_{BA} are combined to give E_{CA}. E_{CA} is seen to be of the same length as, and parallel with, the vector joining the ends of the vectors E_{BC} and E_{BA}, and either of these constructions may be used according to convenience. It is easier to judge the phase relations of these vectors if the "tails" of their arrows are joined at O as the origin. It is seen that E_{CA} is 45 deg. ahead of E_{BA}, 45 deg. behind E_{CB}, and 135 deg. ahead of E_{BC}. The emf E_{AC} is found by reversing E_{BA} and adding this to E_{BC} vectorially, or

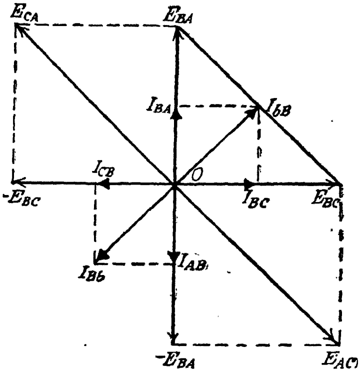


FIG. 120.—Vector diagram for loads of Fig. 119, E_{BC} as the reference.

$$\dot{E}_{AC} = \dot{E}_{BC} - \dot{E}_{BA} = -\dot{E}_{CA} \text{ volts.} \tag{466}$$

This emf is numerically equal to the emf E_{CA}, but it is 180 deg. out of phase with it.

81. Currents in a Three-wire Circuit with Emfs 90 Deg. Out of Phase.—The vector addition of currents is accomplished in a manner similar to that illustrated for the addition of currents when the emfs are in phase. The case of nonreactive loads is illustrated in Fig. 120. The current in the wire bB is found by

$$\dot{I}_{bB} + \dot{I}_{CB} + \dot{I}_{AB} = 0 \text{ amp,} \tag{453}$$

$$\dot{I}_{bB} = -\dot{I}_{CB} - \dot{I}_{AB} \text{ amp,}$$

or

$$= \dot{I}_{CB} + \dot{I}_{AB} \text{ amp.} \tag{467}$$

In applying the above equations, the currents must be referred to the same axis before the actual vector addition can be made mathematically. To perform this operation, assume that $\dot{E}_{BC}$ is in phase with the reference axis. To indicate the axis of reference, the letters of the emf in phase with it may be used in proper order as an additional subscript, as $\dot{E}_{BC_{BC}}$ means that $\dot{E}_{BC}$ is referred to the reference axis BC. In a similar manner, $\dot{E}_{BA_{BA}}$ means that $\dot{E}_{BA}$ is referred to the axis BA. To refer $\dot{E}_{BA}$ to the axis BC, it is necessary to multiply it by the operator $\underline{/\theta}$, the angle by which the axis BA differs in phase from the axis BC, as

$$\begin{aligned} \dot{E}_{BA_{BC}} &= \dot{E}_{BA} \, \underline{/90^\circ} \text{ volts} \\ &= j\dot{E}_{BA} \text{ volts.} \end{aligned} \tag{468}$$

Hence,

$$\begin{aligned} \dot{E}_{CA_{BC}} &= -\dot{E}_{BC} + j\dot{E}_{BA} \text{ volts} \\ &= -E_{BC} + jE_{BA} \text{ volts,} \end{aligned} \tag{469}$$

since

$$\dot{E}_{BC} = E_{BC} + j0 \text{ volts,}$$

and

$$\dot{E}_{BA} = E_{BA} + j0 \text{ volts.}$$

Hence, the magnitude of E_{CA} is

$$E_{CA} = \sqrt{E_{BC}^2 + E_{BA}^2} \text{ volts,} \tag{470}$$

and since $E_{BC} = E_{BA}$ by assumption,

$$E_{CA} = \sqrt{2}E_{BC} \text{ volts.} \tag{471}$$

From Eqs. (469) and (471), it is seen that the emf between outside wires is 135 deg. ahead of E_{BC} and is $\sqrt{2}$ times E_{BC} in value, corresponding with E_{CA} in the vector diagram (Fig. 120).

Consider the loads,

$$\dot{z}_{BA} = r_{BA} + jx_{BA} \text{ ohms}$$

and

$$\dot{z}_{BC} = r_{BC} + jx_{BC} \text{ ohms,}$$

connected between B and A and B and C, respectively, of the circuit of Fig. 119. The currents flowing will be

$$\dot{I}_{BA} = \frac{E_{BA}}{\dot{z}_{BA}} \text{ amp}$$

and

$$\dot{I}_{BC} = \frac{E_{BC}}{\dot{z}_{BC}} \text{ amp},$$

each referred to its own emf, as in single-phase circuits. These will be the currents in the wires A and C in reference to the emfs E_{BA} and E_{BC}. The current in B will be found as

$$\dot{I}_{bB} + \dot{I}_{AB} + \dot{I}_{CB} = 0 \text{ amp}, \tag{453}$$

or

$$\dot{I}_{bB} = \dot{I}_{BA} + \dot{I}_{BC} \text{ amp}, \tag{454}$$

where all the currents must be referred to the same axis before making the actual vector addition numerically. Choosing E_{BC}, in phase with the reference axis,

$$\dot{E}_{BA_{BC}} = E_{BA}\underline{/90^\circ} \text{ volts}, \tag{468}$$

and

$$\begin{aligned} \dot{I}_{BA_{BC}} &= \frac{\dot{E}_{BA_{BC}}}{\dot{z}_{BA}} \text{ amp} & (472)\\ &= \frac{E_{BA}\underline{/90^\circ}}{\dot{z}_{BA}} \text{ amp} \\ &= \dot{I}_{BA}\underline{/90^\circ} \text{ amp.} & (473) \end{aligned}$$

Hence, the current may be rotated into reference with another axis by the same method as is used with the emf. Another method that may be employed is to divide each emf by its impedance and then to rotate these currents to the common axis, as was done with the emfs.

Example 42

Consider the same emfs and loads as in Example 34, but with the emfs 90 deg. out of phase, as in Fig. 119. Find the currents in each wire and the emf between outside wires.

From Eq. (471),

$$E_{CA} = \sqrt{2} \times 110 = 155.5 \text{ volts.}$$

From the statement of the problem,

$$I_{BC} = \frac{11 \times 1{,}000}{110} \doteq 100 \text{ amp},$$
$$\dot{I}_{BC} = 100 + j0 \text{ amp},$$

and

$$\dot{I}_{BA} = 100 + j0 \text{ amp};$$

hence from Eq. (473),

$$\dot{I}_{BA_{BC}} = 100\underline{/90^\circ} = j100 \text{ amp}.$$

From Eq. (454),

$$\begin{aligned}\dot{I}_{bB_{BC}} &= \dot{I}_{BC} + \dot{I}_{BA_{BC}} \text{ amp} \\ &= 100 + j100 \text{ amp},\end{aligned} \tag{474}$$

having the magnitude,

$$I_{bB} = \sqrt{\overline{100}^2 + \overline{100}^2} = 141.4 \text{ amp}.$$

It is seen that the current I_{BA} and the emf E_{BA} are referred to the emf E_{BC} by multiplying each vector by $\underline{/90}$ deg., the angle of phase difference between the two reference axes. I_{BA} could first be found in reference to E_{BA} and then be rotated by $\underline{/90}$ deg. and added to I_{BC}, as shown in Fig. 120.

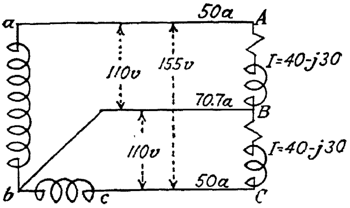


FIG. 121.—Three-wire a-c circuit with emfs at 90 deg. and balanced inductive loads.

Example 43

Consider the same loads as given in Example 36, each current being in reference to its own emf. Find the currents in each of the three wires and the emf between wires. From Example 36,

$$\dot{I}_{BC} = 40 - j30 \text{ amp}, \; I_{BC} = 50 \text{ amp},$$

and

$$\dot{I}_{BA} = 40 - j30 \text{ amp}, \; I_{BA} = 50 \text{ amp},$$

hence from Eq. (473),

$$\begin{aligned}\dot{I}_{BA_{BC}} &= (40 - j30)\underline{/90^\circ} \text{ amp} \\ & \quad 30 \qquad \text{amp}.\end{aligned}$$

From Eq. (454),

$$\begin{aligned}\dot{I}_{bB} &= (40 - j30) + (30 + j40) \text{ amp} \\ &= 70 + j10 \text{ amp},\end{aligned}$$

having the magnitude,

$$I_{bB} = \sqrt{\overline{70}^2 + \overline{10}^2} = 70.7 \text{ amp.}$$

This is illustrated in Figs. 121 and 122.

Example 44

Consider the same loads as given in Example 40, circuit of Fig. 123. Find the currents in each of the three wires.

From Example 40,

$$\dot{I}_{BC} = 40 + j30 \text{ amp}, \; I_{BC} = 50 \text{ amp},$$
$$\dot{I}_{BA} = 40 - j30 \text{ amp}, \; I_{BA} = 50 \text{ amp}.$$

From Eq. (473),

$$\dot{I}_{BA_{BC}} = 30 + j40 \text{ amp},$$

and from Eq. (454),

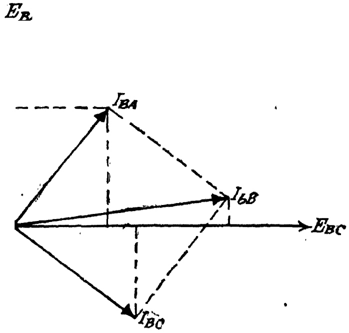


FIG. 122.—Vector diagram for loads of Fig. 121, E_{BC} as the reference.

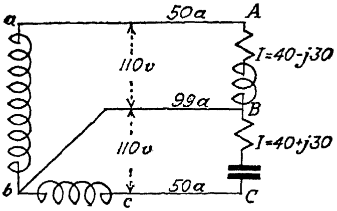


FIG. 123.—Three-wire a-c circuit with emfs at 90 deg. and loads balanced in numerical value but unbalanced in phase.

$$\dot{I}_{bB_{BC}} = (40 + j30) + (30 + j40) \text{ amp}$$
$$= 70 + j70 \text{ amp},$$

having the magnitude,

$$I_{bB} = \sqrt{\overline{70}^2 + \overline{70}^2} = 99 \text{ amp.}$$

and as illustrated in Fig. 124.

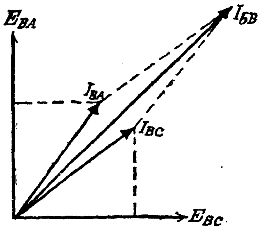


FIG. 124.—Vector diagram for loads of Fig. 123.

Example 45

Consider the loads of Example 44 interchanged, as shown in Fig. 125. Find the currents as in Example 44.

From Example 44,

$$\dot{I}_{BC} = 40 - j30 \text{ amp}, \; I_{BC} = 50 \text{ amp},$$

and

$$\dot{I}_{BA} = 40 + j30 \text{ amp}, \; I_{BA} = 50 \text{ amp}.$$

From Eq. (473),

$$\dot{I}_{BA_{BC}} = -30 + j40 \text{ amp.}$$

From Eq. (454),

$$\begin{aligned}\dot{I}_{bB_{BC}} &= 40 - j30 - 30 + j40 \text{ amp} \\ &= 10 + j10 \text{ amp}\end{aligned}$$

of magnitude,

$$I_{bB} = \sqrt{\overline{10}^2 + \overline{10}^2} = 14.1 \text{ amp,}$$

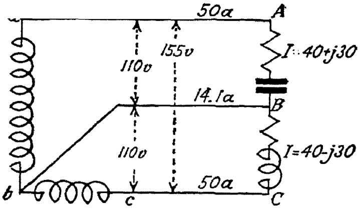


FIG. 125.—Three-wire circuit with emfs at 90 deg. and same loads as Fig. 123, but reversed in order.

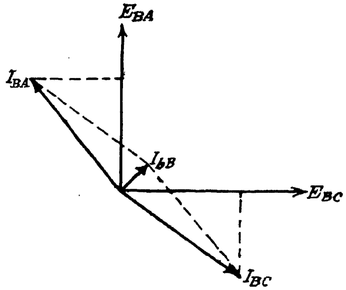


FIG. 126.—Vector diagram for loads of Fig. 125, E_{BC} as the reference.

as illustrated in Fig. 126. It is seen that a different association of the same loads produces a different value of current in the common return wire. This phenomenon will be discussed later in the chapter on polyphase emfs and loads.

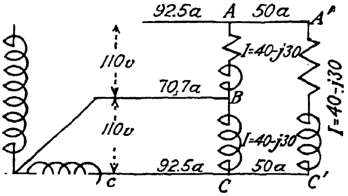


FIG. 127.—Three-wire circuit with emfs at 90 deg. and the same loads as in Fig. 121, but with a load added between outside wires.

Example 46

A load of $\dot{I}_{A'C'} = 40 - j30$ amp is added between the lines A and C of Example 43 as shown in Fig. 127. Find the currents in each of the wires.

The current in the middle wire will remain unchanged from its value as found in Example 43. The current in the new load is referred to E_{AC}. At the point A,

$$\dot{I}_{aA} + \dot{I}_{BA} + \dot{I}_{C'A'} = 0 \text{ amp,}$$

or

$$\dot{I}_{aA} = \dot{I}_{AB} + \dot{I}_{A'C'} \text{ amp.}$$

From Example 41,

$$\dot{I}_{AB} = -\dot{I}_{BA} = -40 + j30 \text{ amp,}$$

and

$$\dot{I}_{AB_{BC}} = -\dot{I}_{BA}\,\underline{/90^\circ}\text{ amp}$$
$$= -30 - j40\text{ amp.}$$

Since $E_{A'C'}$ is 45 deg. behind E_{BC},

$$\dot{I}_{A'C'_{BC}} = \dot{I}_{A'C'}\,\underline{/-45^\circ}\text{ amp}$$
$$= 7.1 - j49.5\text{ amp.}$$

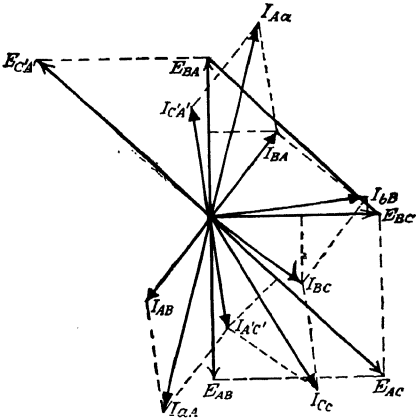


FIG. 128.—Vector diagram for loads of Fig. 127.

From Eq. (453),

$$\dot{I}_{aA_{BC}} = \dot{I}_{AB_{BC}} + \dot{I}_{A'C'_{BC}}\text{ amp}$$
$$= -30 - j40 + 7.1 - j49.5\text{ amp}$$
$$= -22.9 - j89.5\text{ amp}$$

having a magnitude

$$I_{aA} = \sqrt{22.9^2 + 89.5^2} = 92.5\text{ amp.}$$

At the point C,

$$\dot{I}_{Cc} + \dot{I}_{CB} + \dot{I}_{C'A'} = 0\text{ amp,}$$

or

$$\dot{I}_{Cc} = \dot{I}_{BC} + \dot{I}_{A'}$$

and

$$\dot{I}_{Cc_{BC}} = 40 - j30 + 7.1 - j49.5\text{ amp}$$
$$= 47.1 - j79.5\text{ amp,}$$

having the magnitude,

$$I_{Cc} = \sqrt{47.1^2 + 79.5^2} = 92.5\text{ amp.}$$

In Fig. 128, I_{aA} and I_{Cc} are shown as indicated in the foregoing equations. I_{bB} is shown as the sum of I_{BA} and I_{BC} and also as the sum of I_{Aa} and I_{Cc}.

82. Electromotive Forces 120 Deg. Out of Phase in a Three-wire A-c Circuit.—In the circuit of Fig. 129, with the emfs 120 deg. out of phase, consider, as in the case of emfs 90 deg. out of phase, that the common point, or junction, of the vectors is the origin and that it is the point of view of the vector diagram, as shown in Fig. 130, the emf vectors being drawn outward from that point. The emf E_{CA} must be the vector sum of the vectors E_{CB} and E_{BA}; hence

$$\dot{E}_{CA} = \dot{E}_{CB} + \dot{E}_{BA} \text{ volts,} \tag{475}$$

all vectors being referred to the same axis. In Fig. 130, the vector for E_{CA} is shown, in construction, to be the same in length and phase as that joining the ends of the vectors E_{BA} and E_{BC}. It is seen that E_{CA} is 30 deg. ahead of E_{BA} and 150 deg. ahead of E_{BC}. It is also seen that E_{CA} is 30 deg. behind the vector E_{CB}.

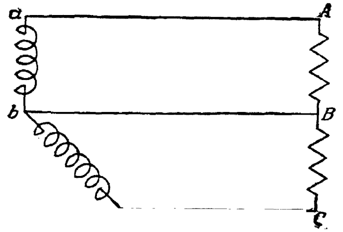


FIG. 129.—Three-wire circuit with emfs at 120 deg. and balanced, non-reactive loads.

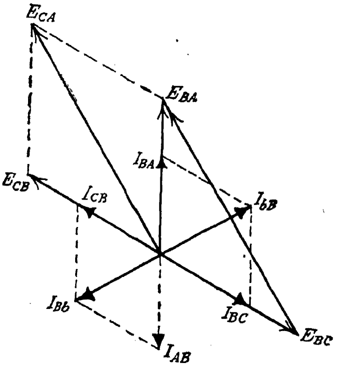


FIG. 130.—Vector diagram for loads of Fig. 129.

83. Currents in Three-wire A-c Circuits, Emfs 120 Deg. Out of Phase.—The sum of the currents at the point B (Fig. 129), must be equal to zero, as before, or

$$\dot{I}_{bB} + \dot{I}_{CB} + \dot{I}_{AB} = 0 \text{ amp,} \tag{453}$$

or

$$\dot{I}_{bB} = \dot{I}_{BC} + \dot{I}_{BA} \text{ amp,} \tag{454}$$

and

$$\dot{I}_{Bb} = \dot{I}_{CB} + \dot{I}_{AB} \text{ amp,} \tag{467}$$

all currents being referred to the same axis before adding the values. The addition of emf and currents may be performed by choosing some emf, as E_{BC}, in phase with the reference axis. Then, in reference to this axis,

$$\begin{aligned}\dot{E}_{BC} &= E_{BC} + j0 \text{ volts.} \\ \dot{E}_{BA_{BC}} &= E_{BA}/120^\circ \text{ volts} && (476) \\ &\qquad\qquad + j0.866E_{BA} \text{ volts.} && (477)\end{aligned}$$

Hence,

$$\begin{aligned}\dot{E}_{C} &= -\dot{E}_{BC} + \dot{E}_{B} \qquad \text{volts} \\ &= -E_{BC} - 0.5 \quad {}_{BA} + j0.866E_{BA} \text{ volts.}\end{aligned}$$

When

$$\begin{aligned}E_{BA} &= E_{BC} \text{ volts,} \\ \dot{E}_{CA_{BC}} &= -1.5E_{BA} + j0. \qquad \text{volts} && (478) \\ &= \sqrt{3}E_{BA}\left(-\frac{\sqrt{3}}{2} + j0.5\right) \text{volts} && (479) \\ &= \sqrt{3}E_{BA}(\cos 150^\circ + j \sin 150^\circ) \text{ volts,}\end{aligned}$$

or E_{CA} is $\sqrt{3}$ times the length of E_{BA} and 150 deg. ahead of E_{BC}, as shown in Fig. 130. The currents may be referred to the same axis as follows:

$$\dot{I}_{bB} = \dot{I}_{BC} + \dot{I}_{BA_{BC}} \text{ amp,} \tag{454}$$

but

$$\begin{aligned}\dot{I}_{BA_{BC}} &= \dot{I}_{BA}\underline{/120^\circ} = \frac{\dot{E}_{BA}\underline{/120^\circ}}{\dot{z}_{BA}} \text{ amp} \\ &= -0.5\,\dot{I}_{BA} + j0.866\dot{I}_{BA} \text{ amp.} && (480)\end{aligned}$$

For nonreactive load,

$$\dot{I}_{bB_{BC}} = I_{BC} - 0.5I_{BA} + j0.866I_{BA} \text{ amp} \tag{481}$$

and when,

$$\begin{aligned}I_{BC} &= I_{BA} \text{ amp} \\ I_{bB} &= I_{BA} - 0.5I_{BA} + j0.866I_{BA} \text{ amp} \\ &= 0.5I_{BA} + j0.866I_{BA} \text{ amp} \\ &= I_{BA}\underline{/60^\circ} \text{ amp,}\end{aligned}$$

which has the magnitude,

$$I_{bB} = \left[\sqrt{0.5^2 + 0.866^2}\right]I_{BA} = I_{BA} \text{ amp} \tag{482}$$

or the currents in all three wires are equal and the current in the wire *Bb* is 60 deg. ahead of E_{BC}.

Example 47

Consider the same values of emf and loads as in Example 42, but with the emfs 120 deg. out of phase. Find the current in each wire and the emf across the outside wires.

From Example 42 and Eq. (479),

$$E_{CA} = 110\sqrt{3} = 190.5 \text{ volts,}$$

and

$$\dot{I}_{BC} = \frac{11 \times 1{,}000}{110} = 100 + j0 \text{ amp, } I_{BC} = 100 \text{ amp.}$$

From Eq. (480)

$$\dot{I}_{BA_{BC}} = \frac{11 \times 1{,}000(-0.5 + j0.866)}{110} = -50 + j86.6 \text{ amp}$$

having the magnitude,

$$I_{BA} = \sqrt{(-50)^2 + \overline{86.6}^2} = 100 \text{ amp.}$$

Also from Eq. (438),

$$\begin{aligned} \dot{I}_{bB} &= 100 - 50 + j86.6 \text{ amp} \\ &= 50 + j86.6 \text{ amp,} \end{aligned}$$

having the magnitude,

$$I_{bB} = \sqrt{\overline{50}^2 + \overline{86.6}^2} = 100 \text{ amp,}$$

as illustrated in Fig. 130.

Example 48

Consider the same loads and values of emf as in Example 43 but with the emfs 120 deg. apart.

From Example 43,

$$\dot{I}_{BC} = 40 - j30 \text{ amp, } I_{BC} = 50 \text{ amp,}$$

and

$$\dot{I}_{BA} = 40 - j30 \text{ amp, } I_{BA} = 50 \text{ amp.}$$

From Eq. (480),

$$\begin{aligned} \dot{I}_{BA_{BC}} &= (40 - j30)\,\underline{/120^\circ} \text{ amp} \\ &= 6 + j49.6 \text{ amp,} \end{aligned}$$

and from Eq. (438),

$$\begin{aligned} \dot{I}_{bB_{BC}} &= 40 - j30 + 6 + j49.6 \text{ amp} \\ &= 46 + j19.6 \text{ amp,} \end{aligned}$$

having the magnitude,

$$I_{bB} = \sqrt{\overline{46}^2 + \overline{19.6}^2} = 50 \text{ amp},$$

as illustrated in Fig. 131.

Example 49

Consider the same loads and emfs as in Example 44, but with emfs 120 deg. out of phase. Find the emf between outside wires and the current in each of the three wires.

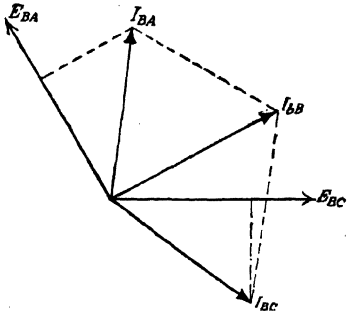


FIG. 131.—Vector diagram for a three-wire circuit with emfs at 120 deg. and balanced inductive loads between common and outside wires.

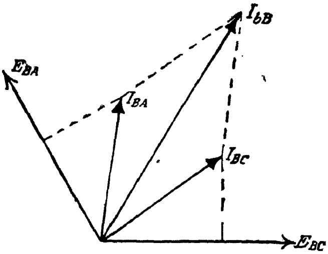


FIG. 132.—Vector diagram for a three-wire circuit with emfs at 120 deg., and with loads balanced in magnitude and power factor but unbalanced in phase.

From Example 44,

$$\dot{I}_{BC} = 40 + j30 \text{ amp}, \; I_{BC} = 50 \text{ amp},$$
$$\dot{I}_{BA} = 40 - j30 \text{ amp}, \; I_{BA} = 50 \text{ amp}.$$

From Eq. (480),

$$\begin{aligned}\dot{I}_{BA_{BC}} &= (40 - j30)(-0.5 + j0.866) \text{ amp} \\ &= 6 + j49.6 \text{ amp},\end{aligned}$$

and from Eq. (438),

$$\begin{aligned}\dot{I}_{bB_{BC}} &= 40 + j30 + 6 + j49.6 \text{ amp} \\ &= 46 + j79.6 \text{ amp},\end{aligned}$$

having the magnitude,

$$I_{bB} = \sqrt{\overline{46}^2 + \overline{79.6}^2} = 91.8 \text{ amp},$$

as illustrated in Fig. 132.

Problems

1-7. A building is wired for three-wire d-c supply of 120 volts between neutral and each outside wire. A lighting load of 50 kw is connected between A and B and one of 35 kw between B and C. Find the current in each wire.

2-7. A motor load of 100 kw is also connected between the outside wires in Prob. 1-7. Find the current in each wire.

3-7. A 125/250-volt three-wire d-c circuit is located 250 ft. from a load. The outside wires are each No. 1 B. & S. copper and the neutral wire has one-half the cross section. A current of 100 amp passes over one outside wire, and one of 75 amp returns in the other outside wire. Find the emf across each load at the distant end.

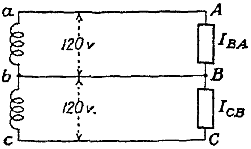


FIG. 4-7.—Figure for Prob. 4-7.

4-7. In Fig. 4-7, $\dot{I}_{BA} = 100 + j0$ amp, and $\dot{I}_{CB} = 50 + j0$ amp. Find the current in each of the wires, and draw the vector diagram, using E_{CB} as the reference, showing E_{AB}, E_{BC}, I_{AB}, I_{BC}, and I_{bB}, the vectors E_{ab} and E_{bc} being in time phase.

5-7. In the circuit of Fig. 4-7, with $\dot{I}_{CB} = 60 + j30$ and $\dot{I}_{BA} = 60 - j30$, find the current in each wire and draw the complete vector diagram, using E_{CB} as the reference.

6-7. An induction-motor load drawing 100 kva at 0.70 power factor is connected between one outside wire and the neutral, and a synchronous motor taking 500 amp at 0.30 power factor leading current is connected between the neutral and the other wire in Fig. 4-7. Find the current flowing in each wire and the total active power taken by the combination, and draw the vector diagram using E_{BA} as the reference.

7-7. In Fig. 4-7, $\dot{Z}_{AB} = 2 - j6$ and $\dot{Z}_{BC} = 4.36 + j9$; find the current in each wire. Draw the vector diagram.

8-7. The neutral wire in Prob. 7-7 is now opened; find the currents and the voltages E_{BA} and E_{CB}.

9-7. In Fig. 9-7, $\dot{I}_{CB} = 40 + j50$, $\dot{I}_{BA} = 25 + j30$, and $\dot{I}_{C'A'} = 30 + j30$. If the voltages E_{ab} and E_{bc} are in synchronism, find the current in each wire and draw the vector diagram, using E_{CA} as the reference.

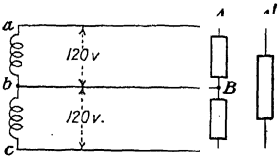


FIG. 9-7.—Figure for Prob. 9-7.

10-7. Three nonreactive loads of 150 kw each are connected in place of the three impedances of Fig. 9-7. (*a*) What current is taken by each load, and what current flows in each wire? (*b*) The load $A'C'$ is now disconnected, and load BC is changed to a 150 kva load of 0.707 power factor inductive; what currents flow? (*c*) Reconnect load $A'C'$, and answer the same question.

11-7. In the three-wire circuit of Fig. 11-7 with two emfs in synchronism, $\dot{Z}_{BA} = 5 - j5$, $\dot{Z}_{BC} = 4 + j4$, and $\dot{Z}_{A'C'} = 0 + j10$; find the current in each wire and the emf across the loads. A wattmeter is placed in the circuit with its current coil in aA and the potential coil across AB; find the reading of this meter.

12-7. In a three-wire system with the line wires a and c and the common return wire b, $\dot{E}_{ba} = 220\underline{/90^\circ}$ and $\dot{E}_{bc} = 220\underline{/0^\circ}$, $\dot{Z}_{ab} = 5 - j8$ and $\dot{Z}_{bc} = 4.36 + j9$. Find the currents in each wire. What is the total active power consumed? Draw the vector diagram.

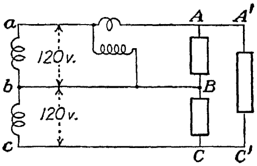


FIG. 11-7.—Figure for Prob. 11-7.

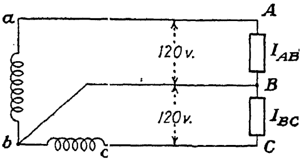


FIG. 13-7.—Figure for Prob. 13-7.

13-7. In Fig. 13-7, E_{ba} leads E_{bc} by 90 deg., $\dot{I}_{BC} = 70 - j80$, $\dot{I}_{BA} = 70 - j80$, each current being expressed in relation to its emf. Find the currents in the three wires and the emf between lines A and C. Draw the vector diagram, using E_{BC} as the reference.

14-7. Use the same circuit as shown in Fig. 13-7 with $\dot{Z}_{BA} = 5 - j7$ and $\dot{Z}_{BC} = 5 + j7$. Find the currents in each wire and the emf between the outside wires. Draw the vector diagram, using E_{BC} as the reference.

15-7. The neutral wire in Prob. 14-7 is now opened; find the new currents and the voltages E_{BA} and E_{BC}.

16-7. In Fig. 16-7, $\dot{I}_{BC} = 100 - j400$, $\dot{I}_{BA} = 100 - j400$, and $\dot{I}_{A'C'} = 100 - j400$, each current being expressed in reference to its own emf.

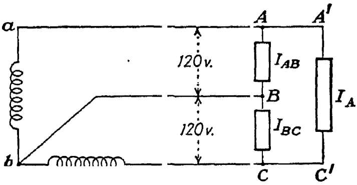


FIG. 16-7.—Figure for Prob. 16-7.

Find the currents in the three wires and the emf between wires if E_{ba} leads E_{bc} by 90 deg.

17-7. In Fig. 16-7, $\dot{Z}_{BA} = 2 + j6$, $\dot{Z}_{BC} = 0 - j10$, $\dot{Z}_{A'C'} = 3 + j4$. Find the currents in the three wires and the emf between wires. Draw the vector diagram, using E_{BC} as the reference.

18-7. An impedance $\dot{Z}_{Bb} = 0.3 + j0.6$ is inserted in the neutral wire of Prob. 14-7; find the currents in the three wires and the voltages across the impedances.

19-7. In the three-wire circuit of Fig. 19-7 with two emfs 90 deg. apart, load $ML = 25$ kw at unity power factor, load LN is an inductive load of 15 kva at 0.707 power factor, and load $M'N'$ is an inductive load of 30 kva at 0.60 power factor. Find the current in each wire and the potential difference between wires. A wattmeter is connected with the current coil in the line *bL* and the potential coil connected from *M* to *N*. What will be the reading of the wattmeter? Draw the vector diagram, using E_{LN} as the reference.

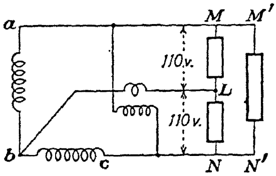


Fig. 19-7.—Figure for Prob. 19-7.

20-7. Two 220-volt single-phase generators *ab* and *bc* with their emfs held 90 deg. apart are connected in series, with *b* as their common terminal. (*a*) Two resistances of 15 ohms each are connected in series across the terminals *a* and *c* of these generators. What current flows in each resistance? What active power does each of the generators furnish? (*b*) The common connection point of the resistances is now connected to the point *b* of the generators. What current now flows in each wire, and what active power does each generator furnish?

21-7. Given the same generator combination as that used in Prob. 20-7, but with the resistances replaced by two impedances $\dot{Z}_{ab} = 8 + j6$ and $\dot{Z}_{bc} = 8 - j6$, answer the same questions as in Prob. 20-7.

22-7. Given the same generators and loads as in Prob. 21-7 but with the emfs offset by 120 deg., answer the same questions as in Prob. 20-7.

23-7. If E_{ba} in Fig. 13-7 is 120 deg. ahead of E_{bc}, the loads being as in Prob. 13-7, find the currents and emfs.

24-7. If E_{ba} in Fig. 13-7 is 120 deg. ahead of E_{bc}, the load impedances being as in Prob. 14-7, find the currents and emfs.

25-7. The neutral wire in Prob. 24-7 is now opened; find the new currents and the voltages E_{BA} and E_{BC}.

26-7. Figure 26-7 shows a three-wire circuit with two emfs at 120 deg. The load *AB* is 40 kva at 0.60 power factor inductive, and load *BC* is 40 kva at 0.60 power factor capacitive. Find the current in each of the wires. A wattmeter is connected in the circuit with its potential coil between *B* and *A* and its current coil in wire *bB*; find the reading of the wattmeter. Draw the vector diagram with E_{BC} as the reference.

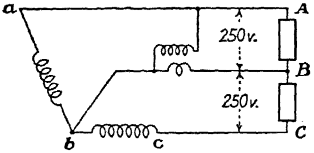


Fig. 26-7.—Figure for Prob. 26-7.

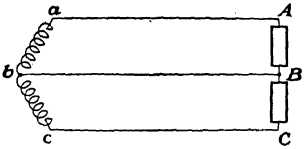


Fig. 28-7.—Figure for Prob. 28-7.

27-7. The same impedance is inserted in the neutral wire of Prob. 24-7 as was used in Prob. 18-7; find the new currents and voltages.

28-7. In Fig. 28-7, $E_{BA} = 100$ volts and $E_{BC} = 120$ volts, E_{BA} leading E_{BC} by 60 deg. If $\dot{Z}_{BA} = 10 + j10$ and $\dot{Z}_{BC} = 10 - j10$, find the currents in the three wires referred to E_{BA}, and draw the complete vector diagram with E_{BA} as the reference.

29-7. In Fig. 29-7, $E_{AB} = 100$ volts, $E_{AC} = 150$ volts, and $E_{AD} = 120$ volts. Write the equation for these emfs referred to E_{AB} and also to E_{AC}.

30-7. In Fig. 30-7, $E_{BA} = E_{BC} = E_{BD} = 150$ volts. Write the equation for these emfs referred to E_{BA} and also to E_{BC}.

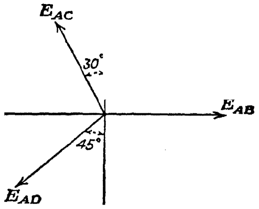


FIG. 29-7.—Figure for Prob. 29-7.

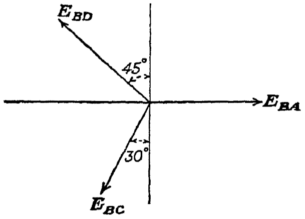


FIG. 30-7.—Figure for Prob. 30-7.

31-7. In Fig. 31-7, E_{BA} leads E_{BC} by 45 deg. The currents in the loads connected between A and B, B and C, and A' and C' are, respectively, $\dot{I} = 30 + j30$, $\dot{I} = 40 + j0$, and $\dot{I} = 0 - j40$, each referred to its own emf.

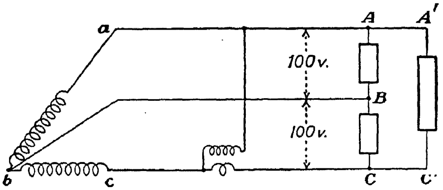


FIG. 31-7.—Figure for Prob. 31-7.

With E_{BC} as the reference, find each of the currents in the wires and the emf between wires. What is the reading of the wattmeter with its current coil connected in wire cC and its potential coil across CA? Draw the vector diagram.

CHAPTER VIII

POLYPHASE CIRCUITS

In Chap. I, the generation of an emf in an armature coil and the relation of two emfs induced in two coils on the same armature, but with coil sides not in the same slot, were studied. It is the purpose of this chapter to present the relations between sine waves of emf in several coils on the same armature, separated from each other by definite angles, as well as the conditions of emf and current in the outside circuit resulting from combinations of these coils to form polyphase circuits.

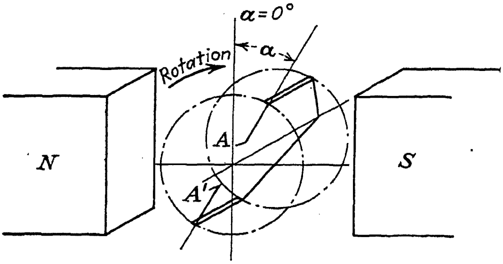


FIG. 133.—A coil rotating in a uniform magnetic field.

84. Generation of Polyphase Emfs.—In Fig. 133, the coil sides of coil $A'A$ are each assumed concentrated in one slot, the coil revolving at a uniform velocity in a uniform magnetic field. Counting time from the instant when the coil is perpendicular to the direction of the magnetic lines, the emf induced in this coil at each instant is

$$e_{A'A} = E_m \sin \alpha \text{ volts,} \tag{20}$$

where

$$E_m = 2\pi f N \Phi_m \times 10^{-8} \text{ volts.} \tag{18}$$

This emf has been represented in the previous chapters by means of a wave as in Fig. 134 and a vector as in Fig. 135. If two coils are considered on the same armature, each having the same number of turns and located θ deg. apart, where $B'B$ is θ degrees behind $A'A$ in position, as represented in Fig. 136 or Fig. 137, the emf waves generated in these two coils will have the same frequency, the same wave shape, and the same maxima. This may be represented by waves as in Fig. 138 and by the following equations, using $A'A$ as the reference coil:

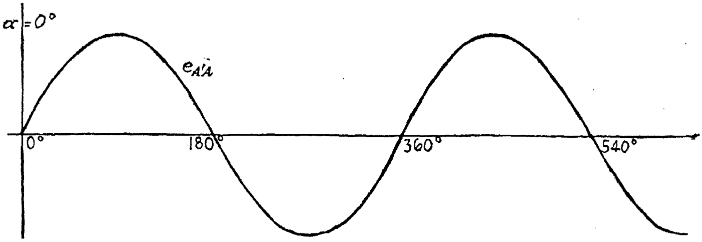


FIG. 134.—Sine wave of emf induced in the coil of Fig. 133.

$$e_{A'A} = E_m \sin \alpha \text{ volts,} \tag{20}$$
$$e_{B'B} = E_m \sin (\alpha - \theta) \text{ volts.} \tag{483}$$

The vectors representing these emfs will be θ deg. apart as shown by $E_{A'A}$ and $E_{B'B}$ in Fig. 139.

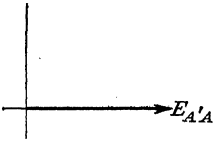


FIG. 135.

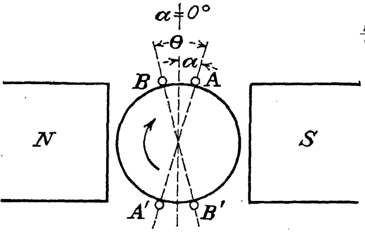


FIG. 136.

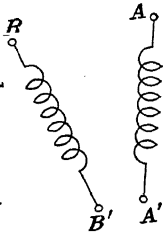


FIG. 137.

FIG. 135.—Vector representation of the emf induced in the coil of Fig. 133.
FIG. 136.—Two coils at a space angle rotating in a magnetic field.
FIG. 137.—Illustration for the two coils of Fig. 136.

85. Association of Polyphase Emfs.—Suppose that the terminals A' and B' of the coils in Fig. 136 are connected and that the value and position of the emf E_{BA} of the combination are to be determined. By Kirchhoff's law the emf $\dot{E}_{BA}$ is the vector sum of $\dot{E}_{BB'}$ and $\dot{E}_{A'A}$

$$\dot{E}_{BA} = \dot{E}_{BB'} + \dot{E}_{A'A} \text{ volts} \tag{484}$$
$$= \dot{E}_{A'A} - \dot{E}_{B'B} \text{ volts.} \tag{485}$$

Equation (485) shows that in going from B to A the direction is counter to the emf for $B'B$ and in the direction of the emf for $A'A$. In Fig. 140, the ends of the emf vectors OA, representing the emf $E_{A'A}$, and OB, representing the emf $E_{B'B}$, are joined at O, representing the metallic connection of the coils at A' and B'. By reversing the vector OB to OB', for the emf $E_{BB'}$, and drawing the parallelogram with OA and OB', the vector OC, representing the emf E_{BA}, is obtained.

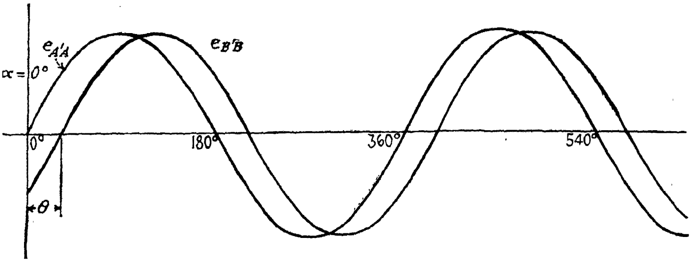


FIG. 138.—Sine waves of emf generated by the two coils of Figs. 136 and 137.

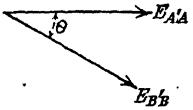


FIG. 139.—Vectors representing the emfs of the coils in Figs. 136 and 137.

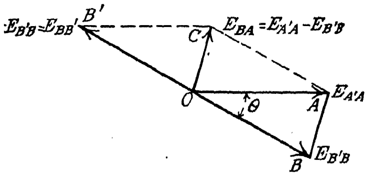


FIG. 140.—Vector sum of the emfs of the coils of Figs. 136 and 137, when A' and B' are connected.

The equations representing this new resultant wave are derived as follows, using instantaneous values:

$$e_{BA} = e_{A'A} - e_{B'B} \text{ volts,} \tag{486}$$

but, since

$$e_{A'A} = E_m \sin \alpha \text{ volts} \tag{20}$$

and

$$e_{B'B} = E_m \sin (\alpha - \theta) \text{ volts,} \tag{483}$$

then

$$e_{BA} = E_m \sin \alpha - E_m \sin (\alpha - \theta) \text{ volts.} \tag{487}$$

Expanding Eq. (487),

$$\begin{aligned} e_{BA} &= E_m[\sin \alpha - \sin (\alpha - \theta)] \text{ volts} \\ &= E_m[\sin \alpha - \sin \alpha \cos \theta + \cos \alpha \sin \theta] \text{ volts} \\ &= E_m[(1 - \cos \theta) \sin \alpha + \sin \theta \cos \alpha] \text{ volts,} \end{aligned}$$

but

$$(1 - \cos \theta) = 2 \sin^2 \frac{\theta}{2}$$

and

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2};$$

hence,

$$\begin{aligned} e_{BA} &= E_m\left[2 \sin^2 \frac{\theta}{2} \sin \alpha + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \alpha\right] \text{ volts} \\ &\sin \frac{\theta}{2}\left[\sin \frac{\theta}{2} \sin \alpha + \cos \frac{\theta}{2} \cos \alpha\right] \text{ volts} \\ &= 2 \sin \frac{\theta}{2} E_m \sin \left(\alpha + 90° - \frac{\theta}{2}\right) \text{ volts,} \end{aligned} \tag{488}$$

where

$$\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta}. \tag{489}$$

Special values will be assigned to θ in the later considerations of special problems. The waves representing these same operations are shown in Fig. 141, where $e_{BB'} = -e_{B'B}$, or a wave at 180 deg. from $e_{B'B}$ is first constructed, and this wave is then added to $e_{A'A}$ to obtain wave e_{BA}.

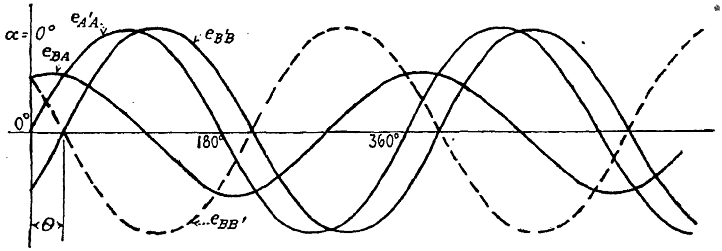


FIG. 141.—Sine waves representing the emf E_{BA} and its components when A' and B' are connected in the machine of Figs. 136 and 137.

Single-phase Circuit.—A single-phase circuit is a circuit energized by a single alternating emf.

Polyphase Circuit.—A polyphase circuit is a circuit of more than a single phase. This term is ordinarily applied to symmetrical systems.

Quarter-phase or Two-phase Circuit.—A quarter-phase or two-phase circuit is a combination of circuits energized by alternating emfs that differ in phase by a quarter of a cycle, i.e., 90 *deg.*

Three-phase Circuit.—A three-phase circuit is a combination of circuits energized by alternating emfs that differ in phase by one-third of a cycle, i.e., 120 *deg.*

Six-phase Circuit.—A six-phase circuit is a combination of circuits energized by alternating emfs that differ in phase by one-sixth of a cycle, i.e., by 60 *deg.*

Polyphase Set of Potential Differences.—A polyphase set of potential differences is a group of interrelated alternating potential differences all of which have the same fundamental frequency, but each of which differs from every other in phase and may differ in effective value and wave form.

Symmetrical Polyphase Set of Potential Differences.—A Symmetrical set of n polyphase potential differences is a set in which each potential difference has the same effective value and the same wave form as each other potential difference and in which the members of the set can be arranged in a sequence such that the phase difference between each member and the one following it, and between the first and last members, is $2\pi/n$ radians.

Characteristic Angular Phase Difference.—The characteristic angular phase difference of a symmetrical polyphase set of potential differences or currents is the angular phase difference between adjacent members of the sequence. In a symmetrical polyphase set of potential differences, the characteristic angular phase difference is $2\pi/n$ radians or $360/n$ deg.

Balanced Polyphase System.—A balanced polyphase system is a polyphase system in which both the currents and voltages are symmetrical.

86. Quarter-phase or Two-phase Systems.—In order to produce the voltages for a quarter-phase or two-phase system from an alternator, the coils must be alike in number of turns and in distribution on the armature surface, and they must be displaced by 90 electrical degrees, so as to produce equal emfs 90 deg. apart. Three general cases will be considered, the four-wire two-phase system, the three-wire two-phase system, and the five-wire two-phase system.

87. Four-wire Two-phase System.—In the four-wire two-phase system, each phase of the armature is electrically distinct from the other and is connected to a separate outside circuit. In Fig. 142, coil *ab* feeds the load *A* and coil *cd* the load *B*. These loads may be independent of each other, as the lights in different buildings, or they may be combined magnetically, as in the windings of two-phase motors. The vector diagram will be as shown in Fig. 143, where the emf vectors are not connected, since there is no electrical connection between the coils or circuits. These emfs, however, will be associated at 90 deg. with each other, since they are both furnished by the same machine.

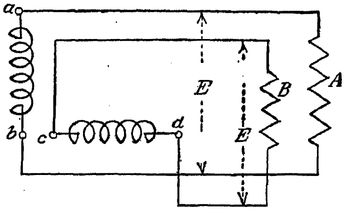


Fig. 142.—Two nonreactive loads connected to a four-wire two-phase system.

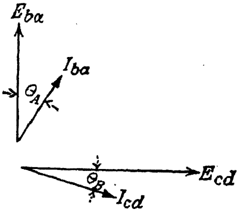


Fig. 143.—Vector representation of two loads on a four-wire two-phase system.

88. Three-wire Two-phase Circuits.—In order to reduce the number of wires required in the distribution system as well as the total weight of copper and the number of supporting insulators, sometimes two of the terminals, one for each phase, are joined at the generator, a third wire, a common return, being brought out from this point, which, together with the two wires from the free terminals, gives a three-wire two-phase system, as illustrated in Fig. 144.

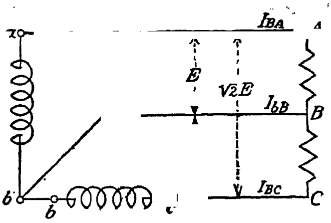


Fig. 144.—Two nonreactive loads on a three-wire two-phase system.

Common Return.—A common return is a return conductor common to several circuits.

Voltage Relations.—The vector expressions for the emfs corresponding to Fig. 144, with E_{BC} as the reference, are

$$E_{BC} = E_{BC}\underline{/0^\circ} \text{ volts,} \tag{490}$$

$$\dot{E}_{BA} = E_{BA}\underline{/90^\circ} \text{ volts,} \tag{491}$$

$$\dot{E}_{CA} = \dot{E}_{CB} + \dot{E}_{BA} \text{ volts} \tag{492}$$

$$= E_{BC}\underline{/180^\circ} + E_{BA}\underline{/90^\circ} \text{ volts}$$

$$= \sqrt{2}E_{BC}\underline{/135^\circ} \text{ volts,} \tag{493}$$

having a magnitude,

$$E_{CA} = \sqrt{2}E_{BC} = \sqrt{2}E_{BA} \text{ volts.} \tag{494}$$

This emf is 135 deg. ahead of E_{BC} and 45 deg. ahead of E_{BA}. The single-phase loads are usually connected between line B and each of the other lines so as to balance the load currents in the two phases of the machine; polyphase motors are connected to all three lines. The relation of emf vectors is shown in Fig. 145.

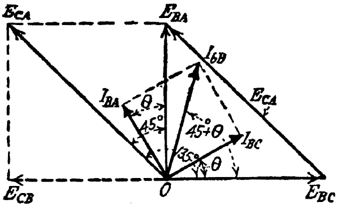


FIG. 145.—Vector diagram of a three-wire two-phase system with balanced capacitive loads.

Current Relations.—The current relations are found in a similar manner to that given for three-wire a-c circuits. Thus, for a balanced load with the current leading the voltage by θ deg., as shown in Fig. 145, the currents in the lines A and C will be equal in magnitude and displaced from each other by 90 deg., each current being θ deg. ahead of its voltage. The vector expressions for these currents with E_{BC} as the reference, are

$$\dot{I}_{BC} = \frac{\dot{E}_{BC}}{\dot{Z}_{BC}} = \frac{E_{BC}\underline{/0^\circ}}{Z_{BC}\underline{/\theta_{BC}}} = I_{BC}\underline{/-\theta_{BC}} \text{ amp,} \tag{495}$$

where

$$\dot{Z}_{BC} = R_{BC} + jX_{BC} \text{ ohms}$$

$$= Z_{BC}\underline{/\theta_{BC}} \text{ ohms,}$$

θ being positive for inductive reactance and negative for capacitive reactance.

$$\dot{I}_{BA} = \frac{\dot{E}_{BA}}{\dot{Z}_{BA}} = \frac{E_{BA}\underline{/90^\circ}}{Z_{BA}\underline{/\theta_{BA}}} = I_{BA}\underline{/90^\circ - \theta_{BA}} \text{ amp.} \tag{496}$$

The current in the common return wire is

$$\dot{I}_{bB} = \dot{I}_{BC} + \dot{I}_{BA} \text{ amp} \tag{497}$$

$$= I_{BC}\underline{/-\theta} + I_{AB}\underline{/90^\circ - \theta} \text{ amp} \tag{498}$$

$$= \sqrt{2}I_{BC}\underline{/45^\circ - \theta} \text{ amp}, \tag{499}$$

having a magnitude,

$$I_{bB} = \sqrt{2}I_{BC} = \sqrt{2}I_{BA} \text{ amp.} \tag{500}$$

Thus it is seen that the common-return current for balanced loads is $\sqrt{2}$ times as great as the load currents and that it is displaced 45 deg. from these currents. When the loads are unbalanced, the same equations hold as for balanced loads.

89. Power in Two-phase Circuits.—The active power output of three- and four-wire two-phase generators is the sum of the amounts of power required for the different loads on the two phases, or, referring to Fig. 142,

$$P_{ba} = E_{ba}I_{ba} \cos \theta_A \text{ watts,} \tag{504}$$

$$P_{cd} = E_{cd}I_{cd} \cos \theta_B \text{ watts,} \tag{505}$$

and

$$P_T = P_{ba} + P_{cd} \text{ watts.} \tag{506}$$

In a similar manner, the total reactive power is the sum of the reactive-power components of the different loads on each phase, as

$$Q_{ba} = E_{ba}I_{ba} \sin \theta_A \text{ vars,} \tag{507}$$

$$Q_{cd} = E_{cd}I_{cd} \sin \theta_B \text{ vars,} \tag{508}$$

and

$$Q_T = Q_{ba} + Q_{cd} \text{ vars.} \tag{509}$$

In considering the apparent power and the power factor, each phase must be considered separately except for balanced systems. For balanced systems,

$$P_T = 2EI \cos \theta \text{ watts,} \tag{510}$$

$$Q_T = 2EI \sin \theta \text{ vars,} \tag{511}$$

and both the apparent and the vector power become

$$U_T = V_T = 2EI \text{ vector volt-amp}$$

$$= 2EI \text{ volt-amp,} \tag{512}$$

since the vector power is identical with the apparent power in a balanced circuit. From these relations, the current taken by any balanced load on a two-phase system may be computed, since, from Eq. (512),

$$I = \frac{\text{vector power}}{2E} = \frac{\text{apparent power}}{2E} \text{ amp,} \tag{513}$$

or, from Eq. (510),

$$I = \frac{\text{active power}}{2E \cos \theta} \text{ amp.} \tag{514}$$

Example 50

What is the current for a 220-volt two-phase induction motor taking an input of 50 kva at 0.80 power factor from a four-wire two-phase circuit?

From Eq. (513),

$$I = \frac{50 \times 1{,}000}{2 \times 220} = 113.5 \text{ amp.}$$

Example 51

A lighting load of 20 kw is also connected across phase *A* of Example 50. What are the current and the power factor of each phase?

Phase A

Motor load:

$$\begin{aligned} \text{Apparent power} &= \frac{50}{2} = 25 \text{ kva,} \\ \text{Active power} &= 25 \times 0.80 = 20 \text{ kw,} \\ \text{Reactive power} &= 25 \times (-0.60) = -15 \text{ kvar.} \end{aligned}$$

Lighting load:

$$\begin{aligned} \text{Active power} &= 20 \text{ kw,} \\ \text{Reactive power} &= 0 \text{ kvar.} \end{aligned}$$

Total load:

$$\begin{aligned} \text{Active power} &= 20 + 20 = 40 \text{ kw,} \\ \text{Reactive power} &= -15 + 0 = -15 \text{ kvar,} \\ \text{Apparent power} &= \sqrt{40^2 + 15^2} = 42.7 \text{ kva,} \\ \text{Power factor} &= \frac{40}{42.7} = 93.5 \text{ per cent.} \end{aligned}$$

Phase B

Motor load:

$$\begin{aligned} \text{Apparent power} &= 25 \text{ kva,} \\ \text{Active power} &= 20 \text{ kw,} \\ \text{Reactive power} &= -15 \text{ kvar,} \\ \text{Power factor} &= 0.80 = 80 \text{ per cent,} \end{aligned}$$

The two phases have the same reactive components of current but different active components and different power factors.

Example 52

Consider a three-wire two-phase circuit with the same loads and voltages as in Example 50. Find the currents in the three wires.

The currents in the lines A and C will be

$$I_{BA} = I_{BC} = \frac{50 \times 1{,}000}{2 \times 220} = 113.5 \text{ amp.}$$

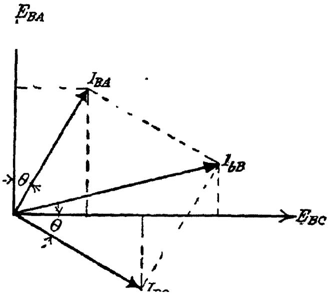


FIG. 146.—Vector diagram for a three-wire two-phase system with balanced inductive loads.

Using E_{BA} as a reference,

$$\begin{aligned} \dot{I}_{BA} &= 113.5\underline{/-36^\circ 52'} \text{ amp} \\ &= 90.9 - j68.1 \text{ amp.} \end{aligned}$$

In a similar manner, using E_{BC} as the reference,

$$\begin{aligned} \dot{I}_{BC} &= 113.5\underline{/-36^\circ 52'} \text{ amp} \\ &= 90.9 - j68.1 \text{ amp.} \end{aligned}$$

Referring I_{BA} to E_{BC} as the reference,

$$\begin{aligned} \dot{I}_{BA_{BC}} &= 113.5\underline{/-36^\circ 52'}\underline{/90^\circ} \text{ amp} \\ &= 113.5\underline{/53^\circ 8'} \text{ amp} \\ &= 68.1 + j90.9 \text{ amp.} \end{aligned}$$

The common-return current is

$$\begin{aligned} \dot{I}_{bB_{BC}} &= 68.1 + j90.9 + 90.9 - j68.1 \text{ amp} \\ &= 159 + j22.8 \text{ amp} \\ &= 160.5\underline{/8^\circ 8'} \text{ amp.} \end{aligned}$$

These relations are shown in Fig. 146.

Example 53

If a lighting load of 20 kw is also connected across phase A of Example 52, find the current in each wire.

For the lighting load, using E_{BA} as the reference,

$$I_{BA_L} = \frac{20 \times 1{,}000}{220} = 90.9 \text{ amp}$$

and

$$\dot{I}_{BA_L} = 90.9\underline{/0^\circ} = 90.9 + j0 \text{ amp.}$$

The total current in line A, referred to E_{BA}, is then, using values of current for the motor from Example 52,

$$\begin{aligned} \dot{I}_{Aa_{BA}} &= 90.9 - j68.1 + 90.9 + j0 \text{ amp} \\ &= 181.8 - j68.1 \text{ amp} \\ &= 194\underline{/-20^\circ 48'} \text{ amp.} \end{aligned}$$

This current referred to E_{BC} is

$$\begin{aligned}\dot{I}_{Aa_{BC}} &= 194\underline{/-20^\circ 48'}\underline{/90^\circ}\text{ amp}\\ &= 194\underline{/69^\circ 12'}\text{ amp}\\ &= 68.1 + j181.8\text{ amp.}\end{aligned}$$

The common return current is then

$$\begin{aligned}\dot{I}_{bB_{BC}} &= 68.1 + j181.8 + 90.9 - j68.1\text{ amp}\\ &= 159 + j113.7\text{ amp}\\ &= 196\underline{/35^\circ 27'}\text{ amp.}\end{aligned}$$

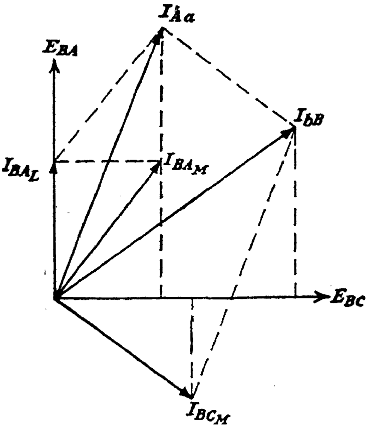


FIG. 147.—Vector diagram for the three-wire two-phase system with unbalanced loads.

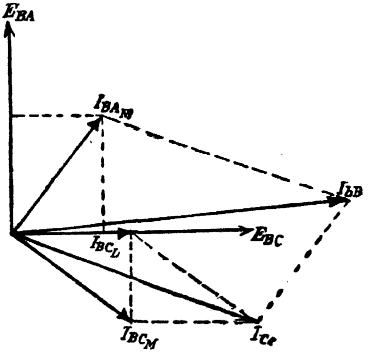


FIG. 148.—Vector diagram for a three-wire two-phase system with unbalanced loads.

The current in line BC is the same as in Example 52, or

$$I_{BC} = 113.5\text{ amp.}$$

These relations are illustrated in Fig. 147.

Example 54

A lighting load of 20 kw is connected across phase B instead of phase A of Example 52; find the currents in each of the wires.

From Example 52,

$$\begin{aligned}\dot{I}_{BA_{BA}} &= 90.9 - j68.1\text{ amp}\\ &= 113.5\underline{/-36^\circ 52'}\text{ amp.}\end{aligned}$$

The current I_{Cc} is found in a similar manner to that for I_{Aa} in Example 53, as

$$\begin{aligned}\dot{I}_{Cc_{BC}} &= 181.8 - j68.1\text{ amp}\\ &= 194\underline{/-20^\circ 48'}\text{ amp.}\end{aligned}$$

Then, since

$$\begin{aligned} \dot{I}_{BA_{BC}} &= 68.1 + j90.9 \text{ amp} \\ \dot{I}_{bB_{BC}} &= 249.9 + j22.8 \text{ amp} \\ &= 250.7\underline{/5°12'} \text{ amp,} \end{aligned}$$

as illustrated in Fig. 148.

90. Five-wire Two-phase System.—The five-wire two-phase system is formed by connecting the middle points of the windings of each of the phases of a four-wire two-phase system and then bringing out a neutral wire from this common point as the fifth wire, as shown in Fig. 149. Single-phase loads for lighting or small motors are connected between the neutral wire N and each of the other wires in such a manner as to balance, so far as possible, the line wire currents. Motor loads for the larger motors are connected to the wires 1, 2, 3, and 4, as in the ordinary four-wire two-phase system.

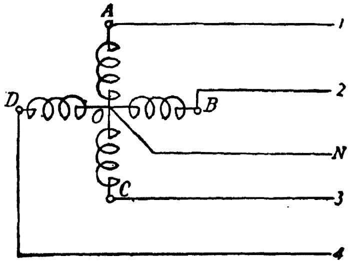


FIG. 149.—Five-wire two-phase system.

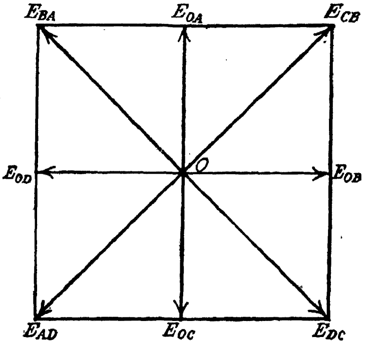


FIG. 150.—Vector diagram of emfs in a five-wire two-phase system.

Voltage Relations.—The vector diagram of emfs is shown in Fig. 150. The emfs between phases are

$$\dot{E}_{CA} = \dot{E}_{OA} - \dot{E}_{OC} = 2\dot{E}_{OA} \text{ volts} \tag{515}$$

and

$$\dot{E}_{DB} = \dot{E}_{OB} - \dot{E}_{OD} = 2\dot{E}_{OB} \text{ volts.} \tag{516}$$

The emfs across phases, as E_{BA}, E_{AD}, etc., are shown as the vector differences of the emfs E_{OB} and E_{OA}, etc., in Fig. 150, or as forming a square with E_{CA} and E_{DB} as diagonals, as in

Fig. 151. Hence, since

$$\dot{E}_{BA} = \dot{E}_{OA} - \dot{E}_{OB} \text{ volts} \tag{517}$$

then

$$E_{BA} = \sqrt{2}E_{OA} \text{ volts.} \tag{518}$$

With standard voltages between lines 1, 2, 3, 4, and the neutral, the voltages between adjacent wires as 1 and 2, 2 and 3, etc., will not be standard.

Current Relations.—The current for each wire is found by adding the single-phase load current from that line to neutral to the balanced polyphase load current of the line. The current in each coil of the generator is the current in the line connected to that coil. The neutral current is found by applying Kirchhoff's law to the point O, as

$$\dot{I}_{NO} = \dot{I}_{OA} + \dot{I}_{OB} + \dot{I}_{OC} + \dot{I}_{OD} \text{ amp,} \tag{519}$$

all currents being referred to the same axis before adding.

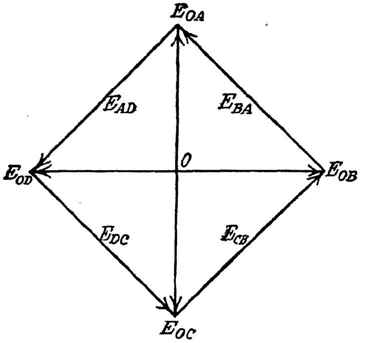


FIG. 151.—Vector diagram of emfs in a five-wire two-phase system.

In this country, this system is seldom used for power transmission or distribution. A further discussion of the system is usually given in the consideration of the balancer transformers for obtaining the neutral wire for a four-slip-ring three-wire d-c generator and for the three-wire d-c connection of a two-phase synchronous converter.

91. Star-connected Loads on a Two-phase System. *Neutral Conductor.—A neutral conductor is that conductor of an electrical supply system which is connected to a neutral point of that system.*

Neutral Point.—The neutral point of a system is that point which has the same potential as the point of junction of a group of nonreactive resistances if connected at their free ends to the appropriate main terminals or lines of the system.

Star-connected Circuit.—A star-connected circuit is a polyphase circuit in which all the current paths within the region that delimits *the circuit extend from each of the points of entry of the line conductors to a common conductor (which may be the neutral conductor).*

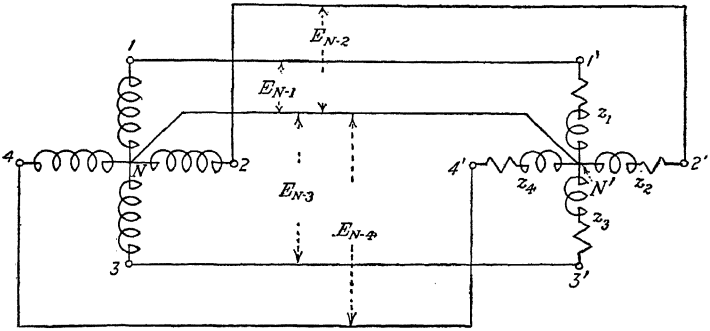


FIG. 152.—Inductive loads connected in star to a five-wire two-phase system.

Four impedances (Fig. 152)

$$\dot{z}_1 = r_1 + jx_1 \text{ ohms},$$
$$\dot{z}_2 = r_2 + jx_2 \text{ ohms},$$
$$\dot{z}_3 = r_3 + jx_3 \text{ ohms},$$

and

$$\dot{z}_4 = r_4 + jx_4 \text{ ohms},$$

connected between the neutral wire and the lines 1, 2, 3, and 4 of a five-wire two-phase system form a star-connected two-phase load. This is illustrated in the following example.

Example 55

Let the emfs to neutral of a five-wire two-phase system be 220 volts, and let the load impedances be

$$\dot{Z}_1 = \dot{Z}_2 = \dot{Z}_3 = \dot{Z}_4 = 6 + j8 = 10\underline{/53^\circ 8'} \text{ ohms}.$$

The currents, each load referred to its own voltage drop as a reference, will be

$$\dot{I}_{N-1} = \dot{I}_{N-2} = \dot{I}_{N-3} = \dot{I}_{N-4} = \frac{220}{10\underline{/53^\circ 8'}} \text{ amp}$$
$$= 22\underline{/-53^\circ 8'} = 13.2 - j17.6 \text{ amp}.$$

Using E_{N-2} as the common reference voltage for all the currents,

$$\dot{I}_{N-1_{N-2}} = 22\underline{/-53^\circ 8'}\underline{/90^\circ} \text{ amp}$$
$$= 22\underline{/36^\circ 52'} = 17.6 + j13.2 \text{ amp},$$
$$\dot{I}_{N-3_{N-2}} = 22\underline{/-53^\circ 8'}\underline{/270^\circ} \text{ amp}$$
$$= 22\underline{/216^\circ 52'} = -17.6 - j13.2 \text{ amp},$$

and

$$_{V-4_{N-2}} = 22\underline{/-53°8'}\,\underline{/180°}\text{ amp}$$
$$= 22\underline{/126°52'} = -13.2 + j17.6\text{ amp.}$$

Hence, combining these four currents,

$$\dot{I}_{N'N_{N-2}} = \dot{I}_{N-1_{N-2}} + \dot{I}_{N-2} + \dot{I}_{N-3_{N-2}} + \dot{I}_{N-4_{N-2}}\text{ amp} \qquad (520)$$
$$= 17.6 + j13.2 + 13.2 - j17.6 - 17.6 - j13.2 - 13.2 + j17.6\text{ amp}$$
$$= 0\text{ amp.}$$

The vector diagram is shown in Fig. 153.

92. Mesh-connected Loads on Two-phase Systems. *Completely Mesh-connected Circuit.—A completely mesh-connected circuit is a polyphase circuit in which, within the region that delimits the circuit, a current path extends directly from each point of entry to every other point of entry.*

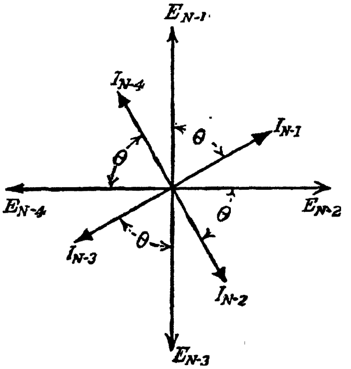


FIG. 153.—Vector diagram of inductive loads connected in star to a five-wire two-phase system.

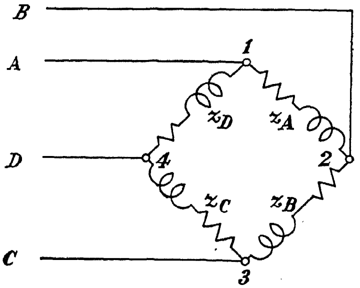


FIG. 154.—Mesh connection of loads to a four-wire two-phase system.

Simply Mesh-connected Circuit.—*A simply mesh-connected circuit is a polyphase circuit in which, within the region that delimits the circuit, a current path extends from every point of entry of each line conductor only to the points of entry of those other line conductors that are nearest to it in effective potential. In a three-phase circuit, a simply mesh-connected circuit is often called a "delta-connected circuit."*

Four equal impedances connected to the four outside wires of a two-phase four-wire generator, as shown in Fig. 154, form a simply mesh-connected load. Let the voltages to the geometric neutral be

$$E_{0-1} = E_{0-2} = E_{0-3} = E_{0-4} = E \text{ volts}; \tag{521}$$

then

$$E_{3-1} = E_{4-2} = 2E \text{ volts} \tag{522}$$

and

$$E_{2-1} = E_{1-4} = E_{4-3} = E_{3-2} = \sqrt{2}E \text{ volts.} \tag{523}$$

The expressions for the vectors representing the four line voltages are as follows, using E_{0-2}, Fig. 155, as the reference,

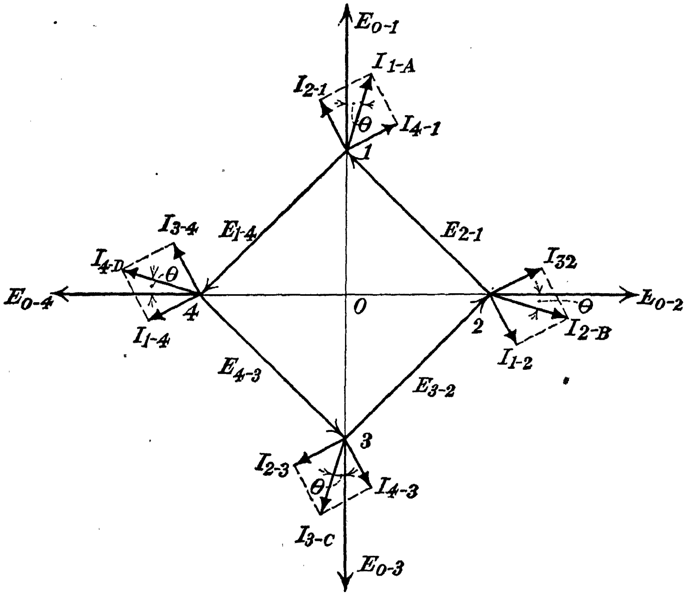


FIG. 155.—Vector diagram of balanced inductive loads connected in mesh to a four-wire two-phase circuit.

where O is the intersection of the diagonals of the square 1-2-3-4:

$$\dot{E}_{2-1} = \sqrt{2}E\underline{/135^\circ} \text{ volts,} \tag{524}$$

$$\dot{E}_{1-4} = \sqrt{2}E\underline{/225^\circ} \text{ volts,} \tag{525}$$

$$\dot{E}_{4-3} = \sqrt{2}E\underline{/-45^\circ} \text{ volts,} \tag{526}$$

$$\dot{E}_{3-2} = \sqrt{2}E\underline{/45^\circ} \text{ volts.} \tag{527}$$

The currents flowing in each impedance are independent of the currents in the other impedances, or

$$\dot{I}_{2-1} = \frac{\dot{E}_{2-1}}{\dot{Z}_{2-1}} = \frac{\sqrt{2}E\underline{/135^\circ}}{Z\underline{/\theta}} = I_{2-1}\underline{/135^\circ - \theta} \text{ amp,} \qquad (528)$$

$$\dot{I}_{1-4} = \frac{\dot{E}_{1-4}}{\dot{Z}_{1-4}} = \frac{\sqrt{2}E\underline{/225^\circ}}{Z\underline{/\theta}} = I_{1-4}\underline{/225^\circ - \theta} \text{ amp,} \qquad (529)$$

$$\dot{I}_{4-3} = \frac{\dot{E}_{4-3}}{\dot{Z}_{4-3}} = \frac{\sqrt{2}E\underline{/-45^\circ}}{Z\underline{/\theta}} = I_{4-3}\underline{/-45^\circ - \theta} \text{ amp,} \qquad (530)$$

and

$$\dot{I}_{3-2} = \frac{\dot{E}_{3-2}}{\dot{Z}_{3-2}} = \frac{\sqrt{2}E\underline{/45^\circ}}{Z\underline{/\theta}} = I_{3-2}\underline{/45^\circ - \theta} \text{ amp.} \qquad (531)$$

The currents at the corner of any mesh must add to zero; hence, the line current will be the combination of the currents for the loads meeting at the corner to which the line is connected. Thus, Fig. 155,

$$\dot{I}_{1-A} = \dot{I}_{2-1} + \dot{I}_{4-1} \text{ amp} \qquad (532)$$

$$= I_{2-1}\underline{/135^\circ - \theta} + I_{4-1}\underline{/45^\circ - \theta} \text{ amp} \qquad (533)$$

$$= \sqrt{2}I_{2-1}\underline{/90^\circ - \theta} \text{ amp.} \qquad (534)$$

The magnitude of this line current is

$$I_{1-A} = \sqrt{2}\,I_{2-1} = \sqrt{2}\,I_{1-4} \text{ amp.} \qquad (535)$$

This current is θ deg. out of phase with the emf E_{0-1}. Similar relations may be found for the other line currents. Thus it is seen that the line current is $\sqrt{2}$ times the load current in magnitude and is displaced from its neutral voltage by the same angle θ that the load currents are displaced from their load voltages. The vector relations are shown in Fig. 156 for a nonreactive load and in Fig. 155 for an inductive load.

Example 56

Consider four resistances of 10 ohms each connected in mesh to a 220-volt two-phase system. Find the emfs and the currents in the system.

The four equal resistances are assumed connected, as shown in Fig. 154, with the emfs E_{3-1} and E_{4-2} maintained at 220 volts at 90 deg. with each other by the impressed voltages of the two-phase system. Draw a square, as shown in Fig. 156, having diagonals 3-1 and 4-2 equal to 220 volts to scale.

These diagonals will be at 90 deg. with each other and will intersect at the point 0, the neutral point of the square, giving E_{0-1}, E_{0-2}, E_{0-3}, and E_{0-4} each equal to one-half the diagonal, or 110 volts. Then

$$\dot{E}_{3-1} = \dot{E}_{0-1} - \dot{E}_{0-3} = 220 \text{ volts and}$$
$$\dot{E}_{4-2} = \dot{E}_{0-2} - \dot{E}_{0-4} = 220 \text{ volts.}$$

The sides of the square will then represent the voltages impressed across the mesh-connected resistances. With E_{0-2} as the reference, these line voltages may be expressed as

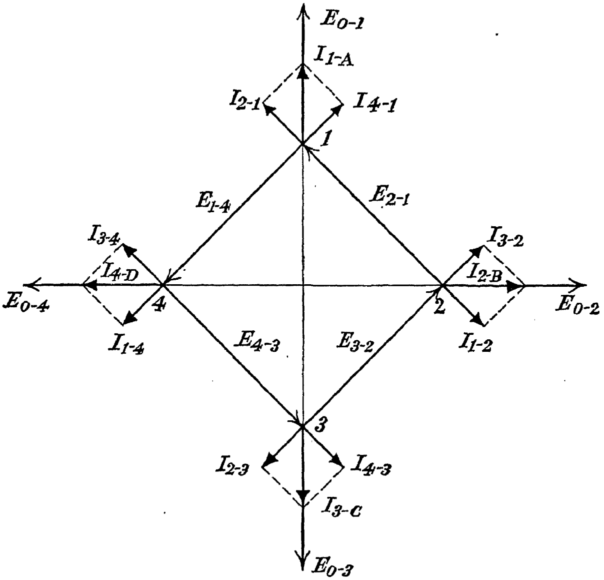


FIG. 156.—Vector diagram for balanced noninductive loads connected in mesh to a four-wire two-phase circuit.

$$\begin{aligned}
\dot{E}_{2-1} &= \dot{E}_{0-1} - \dot{E}_{0-2} \text{ volts}\\
&= -110 + j110 = 155.5\underline{/135^\circ} \text{ volts,}\\
\dot{E}_{1-4} &= \dot{E}_{0-4} - \dot{E}_{0-1} \text{ volts}\\
&= -110 - j110 = 155.5\underline{/-135^\circ} \text{ volts,}\\
\dot{E}_{4-3} &= \dot{E}_{0-3} - \dot{E}_{0-4} \text{ volts}\\
&= 110 - j110 = 155.5\underline{/-45^\circ} \text{ volts,}\\
\dot{E}_{3-2} &= \dot{E}_{0-2} - \dot{E}_{0-3} \text{ volts}\\
&= 110 + j110 = 155.5\underline{/45^\circ} \text{ volts.}
\end{aligned}$$

These vectors form a square with sides equal to 155.5 volts. Since these voltages represent the drops across the loads, the load currents become

$$\begin{aligned}
\dot{I}_{2-1_{0-2}} &= \frac{\dot{E}_{2-1_{0-2}}}{\dot{Z}_{2-1}} = \frac{155.5\underline{/135^\circ}}{10\underline{/0^\circ}} \text{ amp.}\\
&= 15.55\underline{/135^\circ} = -11 + j11 \text{ amp,}\\
\dot{I}_{1-4_{0-2}} &= \frac{\dot{E}_{1-4_{0-2}}}{\dot{Z}_{1-4}} = \frac{155.5\underline{/-135^\circ}}{10\underline{/0^\circ}} \text{ amp}\\
&= 15.55\underline{/-135^\circ} = -11 - j11 \text{ amp,}\\
\dot{I}_{4-3_{0-2}} &= \frac{\dot{E}_{4-3_{0-2}}}{\dot{Z}_{4-3}} = \frac{155.5\underline{/-45^\circ}}{10\underline{/0^\circ}} \text{ amp}\\
&= 15.55\underline{/-45^\circ} = 11 - j11 \text{ amp,}\\
\dot{I}_{3-2_{0-2}} &= \frac{\dot{E}_{3-2_{0-2}}}{\dot{Z}_{3-2}} = \frac{155.5\underline{/45^\circ}}{10\underline{/0^\circ}} \text{ amp}\\
&= 15.55\underline{/45^\circ} = 11 + j11 \text{ amp.}
\end{aligned}$$

These currents are in phase with their respective emfs.

The line currents are found by combining these load currents in the proper order. Then

$$\begin{aligned}
\dot{I}_2 &= \dot{I}_{3-2} - \dot{I}_{2-1} = 11 + j11 + 11 - j11 \text{ amp}\\
&= 22 + j0 = 22 \text{ amp.}
\end{aligned}$$

This current is in phase with the voltage E_{0-2} which was used as a reference. In a similar manner, the currents at the other corners of the square can be found by combining the proper currents at each corner. These currents will be found to be 22 amp. and each in phase with its corresponding neutral voltage.

Example 57

The resistances of Example 56 are replaced by impedances $\dot{Z} = 6 + j8$ connected in mesh. The emf relations are again the same in value and phase position as given in Example 56.

The load currents become

$$\begin{aligned}
\dot{I}_{2-1_{0-2}} &= \frac{\dot{E}_{2-1}}{\dot{Z}_{2-1}} = \frac{155.5\underline{/135^\circ}}{10\underline{/53^\circ 8'}} \text{ amp}\\
&= 15.55\underline{/81^\circ 52'} = 2.20 + j15.38 \text{ amp,}\\
\dot{I}_{3-2_{0-2}} &= \frac{\dot{E}_{3-2_{0-2}}}{\dot{Z}_{3-2}} = \frac{155.5\underline{/45^\circ}}{10\underline{/53^\circ 8'}} \text{ amp}\\
&= 15.55\underline{/-8^\circ 8'} = 15.38 - j2.20 \text{ amp.}
\end{aligned}$$

The current in line wire 2 then becomes

$$\begin{aligned}
\dot{I}_2 &= \dot{I}_{3-2} - \dot{I}_{2-1} = 15.38 - j2.20 - 2.20 - j15.38 \text{ amp}\\
&= 13.18 - j17.58 = 22\underline{/-53^\circ 8'} \text{ amp.}
\end{aligned}$$

This current is seen to be at the same angle behind the corresponding neutral voltage that the load currents are behind their voltage drops. The other load and line currents are calculated in a similar manner.

93. Three-phase Systems.—In the three-phase system, the coils on the armature and the vectors representing the emf waves are 120 deg. apart. In Fig. 159, the three coils AA', BB', and CC' will generate three emf waves 120 deg. apart if they are rotated at constant velocity in a magnetic field. If the field is considered to be uniform, so that sine waves of emf are generated, these waves will have the phase relation shown in Fig. 160. The vector diagram illustrating the vectors for the emf waves of Fig. 160 is shown in Fig. 161. When the zero position for α is taken as the time when coil $A'A$ is crossing the line perpendicular to the axis through the poles, the emf waves will progress in the order shown in Fig. 160 for counterclockwise rotation of the coils. For clockwise rotation of the coils the order of the emf waves will be as shown in Fig. 162 for the same coils and connections as shown in Fig. 159. This precession of the waves, or order of the emfs, is known as "phase sequence." It is seen that phase sequence may be changed in direction by reversing the direction of rotation of the generator; it may also be reversed by changing the order of connection of the coils for a given rotation. Furthermore, changing the direction of rotation of the generator or the phase sequence of the emfs in any manner will change the direction of rotation of any polyphase motors connected to the system. A careful consideration will show that reversal of the polarity of the fields of the generator will not affect the phase sequence of the emfs. The vector diagram illustrating the vectors that

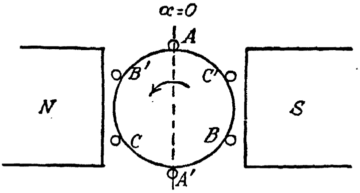


FIG. 159.—Three coils at 120 deg. rotating in a magnetic field.

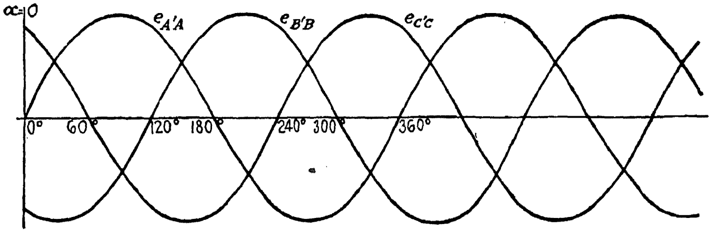


FIG. 160.—Waves of emf generated in the coils of Fig. 159.

represent the waves of Fig. 162 is shown in Fig. 163. When Fig. 163 is compared with Fig. 161, it is seen that the order or or sequence in which the vectors occur is opposite in the two cases, vector rotation being always in a counterclockwise direction.

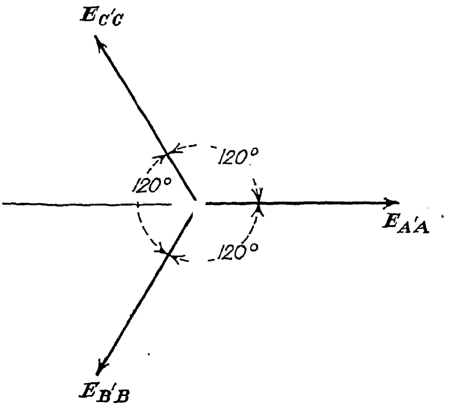


FIG. 161.—Vector diagram representing emf waves of Fig. 160.

The six coil ends of a three-phase machine are seldom brought out for three separate single-phase circuits, as is done for the two separate circuits of the four-wire two-phase system. This connection would require six separate wires for the distribution of the energy and also double the number of insulators, etc. It may also be shown that this method of distribution would require an excessive amount of copper as compared with that required for three- and four-wire three-phase systems or for two-phase systems for the same amount of power and the same distance of

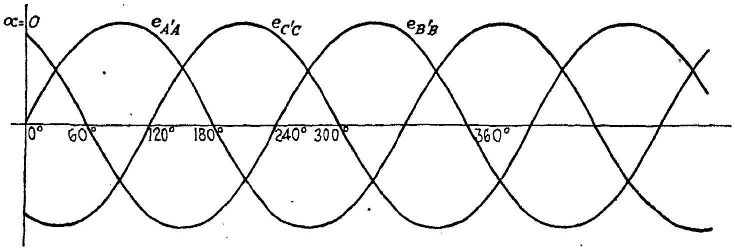


FIG. 162.—Waves of emf generated in the coils of Fig. 159 with the opposite phase sequence to that in Fig. 160.

transmission. Such a connection is, however, used to obtain one form of the six-phase system. There are two common connections of the three groups of coils of a three-phase generator: the star or Y connection and the mesh or delta connection. In the star or Y connection, either three or four wires may be brought out to the outside distributing circuit; in the mesh connection, only three wires are used. The three-wire three-phase system is the more common for either the star or the mesh connection of the generator.

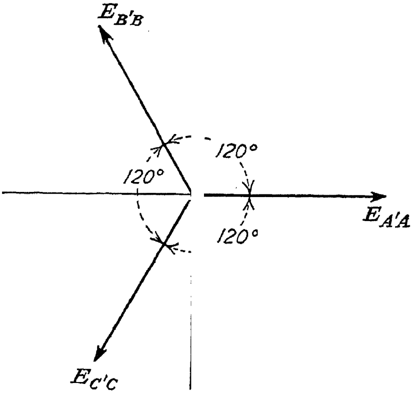


FIG. 163.—Vector diagram representing emf waves of Fig. 162.

94. Four-wire Three-phase Star or Y Connection.

—Connect the three terminals A', B', and C' of the coils in Fig. 159 at a common point, as shown in Fig. 164. The three emf waves for the individual coils, will be as shown in Fig. 160, and their equations will be

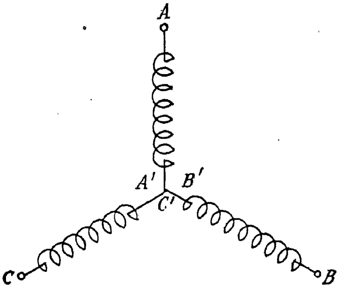


FIG. 164.—Coils of Fig. 159 connected in Y, or star, three-phase.

$$e_{A'A} = E_m \sin \alpha \text{ volts,} \tag{20}$$

$$e_{B'B} = E_m \sin (\alpha + 240°) \text{ volts,} \tag{536}$$

and

$$e_{C'C} = E_m \sin (\alpha + 120°) \text{ volts.} \tag{537}$$

The vector relations are as shown in Fig. 161, the corresponding equations being

$$\dot{E}_{A'A} = E_{A'A}\underline{/0^\circ} \text{ volts,} \tag{538}$$

$$\dot{E}_{B'B} = E_{B'B}\underline{/240^\circ} \text{ volts,} \tag{539}$$

and

$$\dot{E}_{C'C} = E_{C'C}\underline{/120^\circ} \text{ volts.} \tag{540}$$

Bringing out three wires 1, 2, and 3 from the three terminals A, B, and C of the coils and a fourth wire from the common point or neutral A', B', C', as shown in Fig. 165, gives a four-wire three-phase system. If the loads are connected between each of the other wires and the neutral wire N, this wire will carry the vector sum of the currents in the other three wires, since

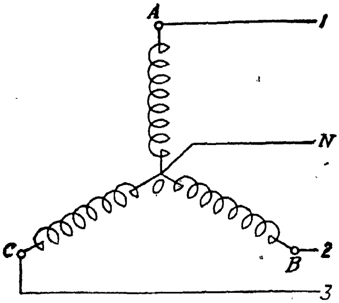


FIG. 165.—Three coils in Y supplying a four-wire three-phase circuit.

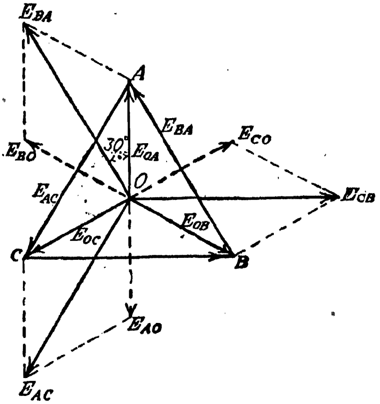


FIG. 166.—Vector diagram of the emfs in a Y-connected three-phase circuit.

$$\dot{I}_{NO} + \dot{I}_{AO} + \dot{I}_{BO} + \dot{I}_{CO} = 0 \text{ amp,} \tag{541}$$

or

$$\dot{I}_{NO} = \dot{I}_{OA} + \dot{I}_{OB} + \dot{I}_{OC} \text{ amp.} \tag{542}$$

The relative value of these currents will be considered later for special cases.

95. Current and Emf Relations in Four-wire Three-phase Circuits.—The emfs E_{OA}, E_{OB}, and E_{OC}, Fig. 165, are equal and 120 deg. apart, as shown in Fig. 166. The vector for these emfs corresponding to Fig. 166 are

$$\dot{E}_{OA} = E_{OA}\underline{/90^\circ} \text{ volts,} \tag{543}$$

$$\dot{E}_{OB} = \tag{544}$$

and

$$\dot{E}_{OC} = E_{OC}/210^\circ \text{ volts.} \tag{545}$$

The voltages between lines can be obtained by applying Kirchhoff's law as

$$\dot{E}_{BA} = \dot{E}_{BO} + \dot{E}_{OA} \text{ volts.} \tag{546}$$

If the neutral voltages form a symmetrical three-phase system so that

$$E_{OA} = E_{OB} = E_{OC} = E_N \text{ volts,} \tag{547}$$

Eq. (546) becomes

$$\dot{E}_{BA} = E_{BO}\underline{/150^\circ} + E_{OA}\underline{/90^\circ} \text{ volts} \tag{548}$$

$$= E_N\underline{/150^\circ} + E_N\underline{/90^\circ} \text{ volts} \tag{549}$$

$$= \sqrt{3}E_N\underline{/120^\circ} \text{ volts.} \tag{550}$$

In a similar manner,

$$\dot{E}_{AC} = \dot{E}_{AO} + \dot{E}_{OC} \text{ volts} \tag{551}$$

$$= E_N\underline{/270^\circ} + E_N\underline{/210^\circ} \text{ volts} \tag{552}$$

$$= \sqrt{3}E_N\underline{/240^\circ} \text{ volts.} \tag{553}$$

$$\dot{E}_{CB} = \dot{E}_{CO} + \dot{E}_{OB} \text{ volts} \tag{554}$$

$$= E_N\underline{/30^\circ} + E_N\underline{/-30^\circ} \text{ volts} \tag{555}$$

$$= \sqrt{3}E_N\underline{/0^\circ} \text{ volts.} \tag{556}$$

The wave forms of these emfs corresponding to Fig. 166 are shown in Fig. 167. In drawing these waves, e_{BA} is found by reversing e_{OB} and adding this wave to e_{OA}. In the same manner, e_{CB} is found by reversing e_{OC} and adding to e_{OB}, and e_{AC} is found by reversing e_{OA} and adding to e_{OC}. From the geometry of Fig. 166 and from the equations of the vectors, it is seen that vectors representing the line voltages when drawn in consecutive order form an equilateral triangle; that the neutral point is at the center of the triangle, *i.e.*, at the intersection of the medians; and that the line voltage is equal to $\sqrt{3}$ times the voltage to neutral, or conversely, the neutral voltage is equal to the line voltage divided by $\sqrt{3}$. This will later be found to hold for any symmetrical three-phase system.

The current relations for the four-wire three-phase system may be found from the principles developed for three-wire a-c circuits. If the loads *A*, *B*, and *C* are connected between the lines 1, 2, and 3 and the neutral wire *N* of the four-wire three-phase system, as shown in Fig. 168, the current in each wire 1, 2, and 3 is found to be the current in each of the loads *A*, *B*, and *C*, respectively. The current in the neutral wire is the vector sum of the currents in the other three wires.

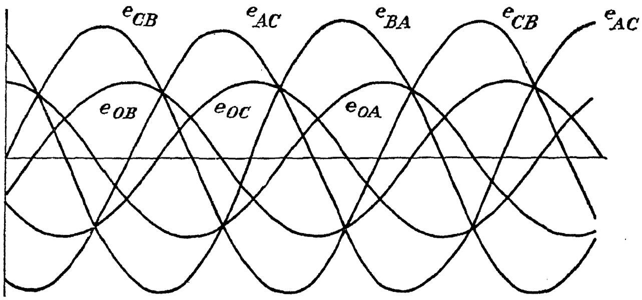


FIG. 167.—Emf waves corresponding to the vector diagram of Fig. 166.

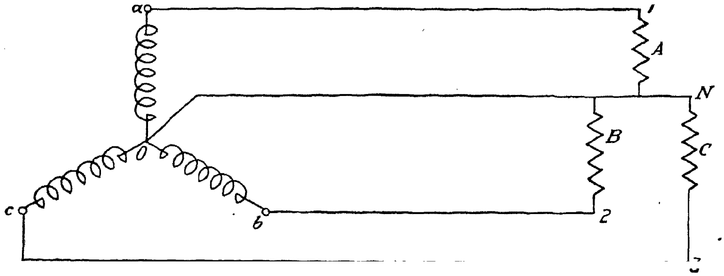


FIG. 168.—Nonreactive loads connected to a four-wire three-phase circuit.

Example 58

Nonreactive loads of 10 kw each are connected between the neutral and each of the lines of a four-wire three-phase system having 100 volts between lines and neutral, Fig. 168. Find the current in each line and the neutral current.

The current in each line wire is

$$I_{oa} = I_{ob} = I_{oc} = \frac{10 \times 1{,}000}{100} = 100 \text{ amp.}$$

These currents are in phase with the generator voltages E_{oa}, E_{ob}, and E_{oc}, as shown in Fig. 169.
From Eq. (542),

$$\dot{I}_{No} = \dot{I}_{ca} + \dot{I}_{ob} + \dot{I}_{oc} \text{ amp} \qquad (542)$$

Choosing E_{ob} as the reference voltage, and since

$$\dot{I}_{oa} = 100 + j0 \text{ amp,}$$

then

$$\dot{I}_{oa_{ob}} = 100\underline{/120°} = -50 + j86.6 \text{ amp,}$$

and

$$\dot{I}_{oc_{ob}} = 100\underline{/-120°} = -50 - j86.6 \text{ amp.}$$

Applying these values in Eq. (516),

$$\dot{I}_{No_{ob}} = 100 - 50 + j86.6 - 50 - j86.6 \text{ amp,}$$

$$I_{No} = 0 \text{ amp.}$$

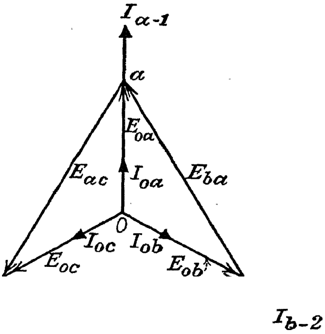


FIG. 169.—Vector diagram of balanced nonreactive loads on a four-wire three-phase circuit.

Hence, in a four-wire three-phase circuit, with balanced nonreactive loads between neutral and the three wires, there is no current in the neutral wire. The line currents are in phase with the neutral voltages since they are in phase with the voltage drop across each load.

Example 59

Inductive loads of 10 kva each, at 0.80 power factor, are substituted for the three loads of Example 58; calculate the line and neutral currents.

The currents are

$$\dot{I}_{oa} = \dot{I}_{ob} = \dot{I} = 80 - j60 = 100\underline{/-36°52'} \text{ amp.}$$

Using E_{ob} as the reference,

$$\dot{I}_{ob} = 80 - j60 \text{ amp,}$$

$$\dot{I}_{oa_{ob}} = 100\underline{/-36°52'}\underline{/120°} \text{ amp} = 100\underline{/83°52'} = 12 + j99.3 \text{ amp,}$$

and

$$\dot{I}_{oc_{ob}} = 100\underline{/-36°52'}\underline{/-120°} \text{ amp} = 100\underline{/-156°52'} = -92 - j39.3 \text{ amp.}$$

Substituting these values of current into Eq. (542),

$$\dot{I}_{No_{ob}} = 80 - j60 + 12 + j99.3 - 92 - j39.3 \text{ amp,}$$

and

$$I_{No} = 0 \text{ amp.}$$

Hence, in a four-wire three-phase circuit with balanced loads, the current in the neutral is zero for sine waves of emf and sine waves of current. The three currents in the line wires are at an angle with the neutral voltages determined by the power factor of these single-phase loads, as shown in Fig. 170.

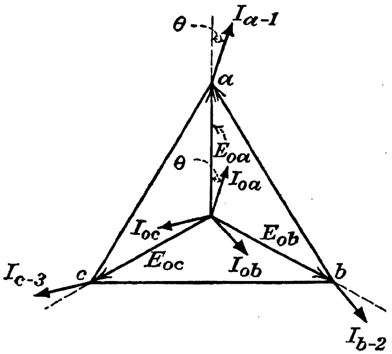


FIG. 170.—Vector diagram of balanced inductive loads on a four-wire three-phase circuit.

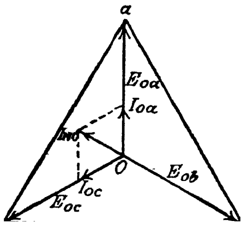


FIG. 171.—Vector diagram for unbalanced nonreactive loads on a four-wire three-phase circuit.

Example 60

Omit one load of Example 58. Let $I_{ob} = 0$ amp.

Hence,

$$\dot{I}_{No_{ob}} = -50 + j86.6 - 50 - j86.6 \text{ amp}$$
$$= -100 + j0 \text{ amp,}$$

and

$$I_{No} = 100 \text{ amp.}$$

A current flows in the neutral equal and opposite to the current I_{ob} of Example 58, or 180 deg. out of phase with E_{ob}, as illustrated in Fig. 171.

Example 61

59

With the same circuit as for Example 49, but with the following loads and power factors:

Load A = 10 kva at 0.80 power factor, inductive;
Load B = 10 kva at 1.00 power factor;
Load C = 10 kva at 0.80 power factor, capacitive.

Hence,

$$\dot{I}_{oa} = 80 - j60 = 100\underline{/-36^\circ52'} \text{ amp,}$$
$$\dot{I}_{ob} = 100 + j0 = 100\underline{/0^\circ} \text{ amp}$$

and

$$\dot{I}_{oc} = 80 + j60 = 100\underline{/36^\circ52'} \text{ amp.}$$

Referring these currents to an axis in phase with E_{ob},

$$\dot{I}_{oa_{ob}} = 100\underline{/-36^\circ52'}\underline{/120^\circ} \text{ amp}$$
$$= 100\underline{/83^\circ58'} = 12 + j99.3 \text{ amp,}$$
$$\dot{I}_{ob_{ob}} = 100 + j0 \text{ amp,}$$

and

$$\dot{I}_{oc_{ob}} = 100\underline{/36^\circ52'}\underline{/-120^\circ} \text{ amp}$$
$$= 100\underline{/-83^\circ8'} = 12 - j99.3 \text{ amp.}$$

Then

$$\dot{I}_{No_{ob}} = 100 + 12 + j99.3 + 12 - j99.3 \text{ amp}$$
$$= 124 + j0 \text{ amp.}$$

Hence,

$$I_{No} = 124 \text{ amp,}$$

as illustrated in Fig. 172.

96. Three-wire Y-connected Three-phase Systems. *a. Balanced Loads.*—In the balanced four-wire three-phase system, it is seen (Figs. 169 and 170) that there is no current in the neutral wire and that the current in each line wire bears the same phase relation to its neutral emf as the load current does to its emf drop. The currents in the line wires are out of phase with the emfs between lines by angles differing from the phase angle of the loads by 30 deg. If the neutral wire is disconnected from the loads, as shown in Fig. 173, no change will be made in the currents or voltages for balanced-load condition and there is produced a

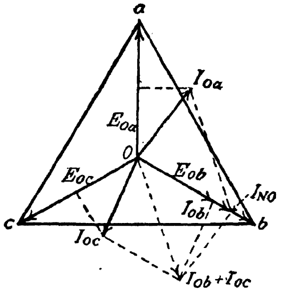


Fig. 172.—Vector diagram for unbalanced loads on a four-wire three-phase circuit.

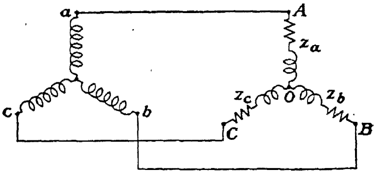


Fig. 173.—Y-connected loads on a three-wire three-phase circuit.

three-wire three-phase system with a Y-connected load. The vector diagram for this type of load with the current lagging the voltage in each phase by the angle θ is shown in Fig. 174. As shown in this vector diagram, the line currents lag the neutral voltages by the same angle θ.

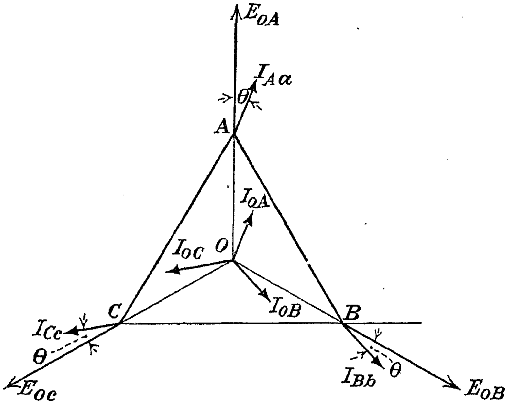


FIG. 174.—Vector diagram for balanced inductive loads connected as shown in Fig. 173.

b. Unbalanced Loads.—In the connection, Fig. 173, for single-phase loads, when the loads are unbalanced, the point O is displaced from the center of the triangle ABC of Fig. 175. As the impedance of OC approaches zero, the point O approaches C, and as the impedance OC approaches infinite values, the point O approaches the side BA of Fig. 175. In the same manner, by varying each load, the point O may be made to lie at any point within or without the triangle ABC, but in any case the vector sum of I_{OA}, I_{OB}, and I_{OC} must always be zero. Owing to this unstable voltage condition, the Y connection of separate single-phase loads is seldom used in a three-wire three-phase system.

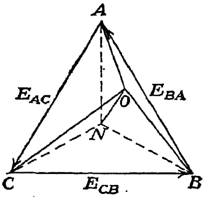


FIG. 175.—Vector diagram for unbalanced loads connected as shown in Fig. 173.

The point O will be displaced from the neutral of the supply system by some voltage E_{NO} (Fig. 175). In order to calculate the current flowing in each impedance, the voltages E_{OA}, E_{OB}, and E_{OC} must be known. The voltages to neutral E_{AN}, E_{BN}, and E_{CN} are known; hence, if E_{NO} can be determined, the voltage drop across each impedance may be obtained. The application of Kirchhoff's laws to this problem gives

$$\dot{E}_{AN} + \dot{E}_{NO} + \dot{E}_{OA} = 0 \text{ volts,} \tag{557}$$
$$\dot{E}_{BN} + \dot{E}_{NO} + \dot{E}_{OB} = 0 \text{ volts,} \tag{558}$$
$$\dot{E}_{CN} + \dot{E}_{NO} + \dot{E}_{OC} = 0 \text{ volts,} \tag{559}$$

or, solving for the load voltages,

$$\dot{E}_{OA} = \dot{E}_{NA} - \dot{E}_{NO} \text{ volts,} \tag{560}$$
$$\dot{E}_{OB} = \dot{E}_{NB} - \dot{E}_{NO} \text{ volts,} \tag{561}$$
$$\dot{E}_{OC} = \dot{E}_{NC} - \dot{E}_{NO} \text{ volts.} \tag{562}$$

The currents are found from the load voltages and the admittances, as

$$\dot{I}_{OA} = \dot{E}_{OA}\dot{Y}_a \text{ amp} \tag{563}$$
$$= \dot{E}_{NA}\dot{Y}_a - \dot{E}_{NO}\dot{Y}_a \text{ amp,} \tag{564}$$
$$\dot{I}_{OB} = \dot{E}_{OB}\dot{Y}_b \text{ amp} \tag{565}$$
$$= \dot{E}_{NB}\dot{Y}_b - \dot{E}_{NO}\dot{Y}_b \text{ amp,} \tag{566}$$
$$\dot{I}_{OC} = \dot{E}_{OC}\dot{Y}_c \text{ amp} \tag{567}$$
$$= \dot{E}_{NC}\dot{Y}_c - \dot{E}_{NO}\dot{Y}_c \text{ amp.} \tag{568}$$

Since with no neutral return the sum of the line currents must be zero,

$$\dot{I}_{OA} + \dot{I}_{OB} + \dot{I}_{OC} = 0 \text{ amp,}$$
$$\dot{E}_{NA}\dot{Y}_a + \dot{E}_{NB}\dot{Y}_b + \dot{E}_{NC}\dot{Y}_c - \dot{E}_{NO}(\dot{Y}_a + \dot{Y}_b + \dot{Y}_c) = 0 \text{ amp,}$$

or

$$\dot{E}_{NO} = \frac{\dot{E}_{NA}\dot{Y}_a + \dot{E}_{NB}\dot{Y}_b + \dot{E}_{NC}\dot{Y}_c}{\dot{Y}_o} \text{ volts,} \tag{569}$$

where

$$\dot{Y}_o = \dot{Y}_a + \dot{Y}_b + \dot{Y}_c \text{ mhos.}$$

Substituting Eq. (569) in Eqs. (560) to (563), the voltages across the load are obtained, and from Eqs. (563), (565), and (567) the three currents are determined.

Example 62

Three impedances $\dot{Z}_a = 20 - j40$, $\dot{Z}_b = 62.5 + j62.5$, and $\dot{Z}_c = 13.5 + j81$, Fig. 176, are connected in Y to a 220-volt three-wire three-phase system. What is the voltage drop across each impedance and the current in each line?

Taking E_{NA}, Fig. 177, as the reference, the neutral voltages are

$$\begin{aligned}\dot{E}_{NA} &= 127 + j0 = 127\underline{/0^\circ} \text{ volts},\\ \dot{E}_{NB} &= -63.5 - j110 = 127\underline{/-120^\circ} \text{ volts},\\ \dot{E}_{NC} &= -63.5 + j110 = 127\underline{/120^\circ} \text{ volts}.\end{aligned}$$

Expressing the three impedances in the polar form,

$$\begin{aligned}\dot{Z}_a &= 44.7\underline{/-63.4^\circ} \text{ ohms},\\ \dot{Z}_b &= 88.4\underline{/45^\circ} \text{ ohms},\\ \dot{Z}_c &= 82.2/80.6^\circ \text{ ohms},\end{aligned}$$

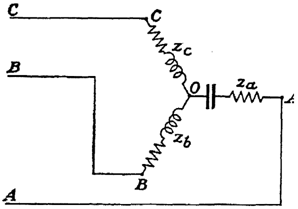


FIG. 176.—Unbalanced Y-connected loads on a three-wire three-phase circuit for Example 62.

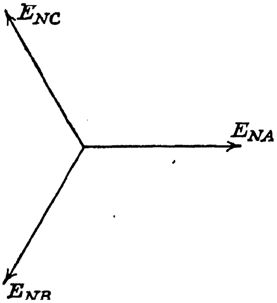


FIG. 177.—Vector diagram for Example 62.

having the admittances,

$$\begin{aligned}\dot{Y}_a &= 0.02235\underline{/63.4^\circ} = 0.01 + j0.02 \text{ mhos},\\ \dot{Y}_b &= 0.01131\underline{/-45^\circ} = 0.008 - j0.008 \text{ mhos},\\ \dot{Y}_c &= 0.01220\underline{/-80.6^\circ} = 0.002 - j0.012 \text{ mhos},\end{aligned}$$

from which Y_o is found as

$$\begin{aligned}\dot{Y}_o &= (0.01 + j0.02) + (0.008 - j0.008) + (0.002 - j0.012) \text{ mhos},\\ &= 0.02 + j0 \text{ mhos},\end{aligned}$$

the currents becoming

$$\begin{aligned}\dot{E}_{NA}\dot{Y}_a &= 127\underline{/0^\circ} \times 0.02235\underline{/63.4^\circ} \text{ amp}\\ &= 2.838\underline{/63.4^\circ} = 1.27 + j2.535 \text{ amp},\\ \dot{E}_{NB}\dot{Y}_b &= 127\underline{/-120^\circ} \times 0.01131\underline{/-45^\circ} \text{ amp}\\ &= 1.436\underline{/-165^\circ} \text{ amp}\\ &= -1.386 - j0.372 \text{ amp},\\ \dot{E}_{NC}\dot{Y}_c &= 127\underline{/120^\circ} \times 0.0122\underline{/-80.6^\circ} \text{ amp}\\ &= 1.548\underline{/39.4^\circ} = 1.196 + j0.982 \text{ amp},\end{aligned}$$

from which

$$\begin{aligned}
\dot{E}_{NO} &= \frac{(1.27 + j2.535) + (-1.386 - j0.372) + (1.196 + j0.982)}{0.02 + j0} \text{ volts} \\
&= \frac{1.08 + j3.145}{0.02 + j0} = 54 + j157.3 \text{ volts.} \\
\dot{E}_{OA} &= \dot{E}_{NA} - \dot{E}_{NO} \text{ volts} \\
&= 73 - j157.3 \text{ volts,} \\
\dot{E}_{OB} &= \dot{E}_{NB} - \dot{E}_{NO} \text{ volts} \\
&= -117.5 - j267.3 \text{ volts,} \\
\dot{E}_{OC} &= \dot{E}_{NC} - \dot{E}_{NO} \text{ volts} \\
&= -117.5 - j47.3 \text{ volts,} \\
\dot{I}_{OB} &= \dot{E}_{OB}\dot{Y}_b = (-117.5 - j267.3)(0.008 - j0.008) \text{ amp} \\
&= -3.078 - j1.198 \text{ amp,} \\
\dot{I}_{OC} &= \dot{E}_{OC}\dot{Y}_c = (-117.5 - j47.3)(0.002 - j0.012) \text{ amp} \\
&= -0.802 + j1.313 \text{ amp,} \\
\dot{I}_{OA} &= \dot{E}_{OA}\dot{Y}_a = (73 - j157.3)(0.01 + j0.02) \text{ amp} \\
&= 3.88 - j0.113 \text{ amp.}
\end{aligned}$$

These relations are shown in the vector diagram (Fig. 178). It is seen that the point O falls outside the line-voltage triangle ABC and that the voltage E_{OB} is considerably greater than the line voltage. This is due to a resonant effect between the inductances and the capacitance in the loads.

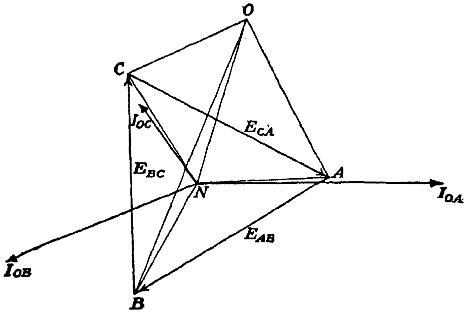


FIG. 178.—Vector diagram for Example 62.

97. Delta or Mesh Connection for Three-phase Systems.—Connect the coils of the generator in Fig. 159 so that A is connected to B', B is connected to C', and C is connected to A', as shown in Fig. 179, and with vectors as shown in Fig. 180. This connection is usually illustrated as shown in Fig. 181 and is called the "delta" or "mesh" connection of the generator, the line wires being connected to the points A, B, and C. The emf between wires in the delta-connected system is seen to be the coil voltage of the machine. The vectors in Fig. 182 are 120 deg. apart and, when drawn in a consecutive order, form an equilateral triangle, a true three-phase system similar to that shown in Fig. 166 for the Y connection. There is no point to which to connect a fourth wire, or neutral; hence this connection may be used only for three-wire three-phase systems.

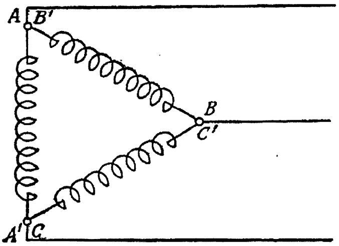


FIG. 179.—Coils of Fig. 159 connected three-phase delta or mesh.

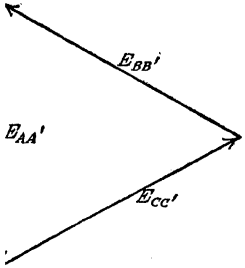


FIG. 180.—Vector diagram for coils of Fig. 179.

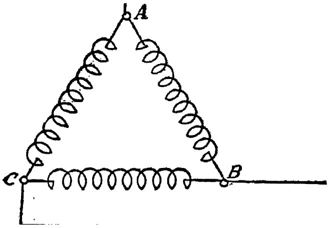


FIG. 181.—A three-wire three-phase circuit supplied by delta-connected coils.

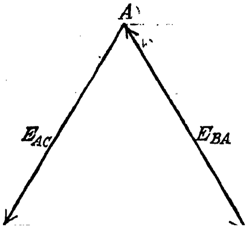


FIG. 182.—Vector diagram for a delta-connected three-wire three-phase circuit.

The equations for the emfs that form a symmetrical three-phase system are, as before,

$$e_{CB} = E_m \sin \alpha \text{ volts,} \tag{20}$$

$$e_{AC} = E_m \sin (\alpha + 240°) \text{ volts,} \tag{536}$$

and

$$e_{BA} = E_m \sin (\alpha + 120°) \text{ volts.} \tag{537}$$

These three emfs are connected in series, and the total emf acting to send a current around the delta will be

$$= e_{AC} + e_{BA} + e_{CB} \text{ volts} \tag{570}$$

$$= E_m \sin \alpha + E_m \sin (\alpha + 240°) + E_m \sin (\alpha + 120°) \text{ volts}$$

$$= E_m[\sin \alpha + \sin (\alpha + 240°) + \sin (\alpha + 120°)] \text{ volts,} \tag{571}$$

or

$$e_o \quad 0.$$

Hence, there is no emf (unbalanced) acting about the winding of the delta-connected generator, and no circulating current will flow for balanced conditions owing to these fundamental voltages. Each emf, however, operates to send a current in the pair of line wires connected to that coil.

The vector equations for the emfs corresponding to Fig. 182 are

$$\dot{E}_{CB} = E_{CB}\underline{/0°} \text{ volts,} \tag{572}$$

$$\dot{E}_{AC} = E_{AC}\underline{/240°} \text{ volts,} \tag{573}$$

$$\dot{E}_{BA} = E_{BA}\underline{/120°} \text{ volts.} \tag{574}$$

The total emf operating upon the circuit consisting of the three windings in series is the vector sum of the induced emfs, *i.e.*,

$$\dot{E}_o = \dot{E}_{CB} + 1 \qquad \dot{E}_{AC} \text{ volts,} \tag{575}$$

$$E_o = 0 \text{ volts.} \tag{576}$$

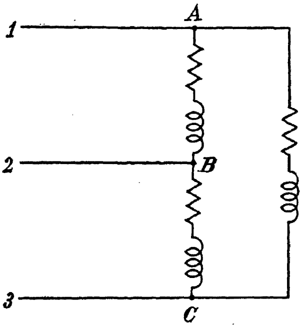


FIG. 183.—Delta-connected inductive loads on a three-phase circuit.

98. Current Relations in Delta-connected Three-phase Loads.

a. Balanced Loads.—Connect three equal impedances between the line wires *A*, *B*, and *C* of any three-wire three-phase system, as shown in Fig. 183. The current in each load will be out of phase with the emf drop across it by the angle θ, as shown in Fig. 184. These line voltages form a symmetrical three-phase system, so that

$$E_{CB} = E_{BA} = E_{AC} = E_L \text{ volts.} \tag{577}$$

The vector equations for these emfs, corresponding to Fig. 184, are

$$\dot{E}_{CB} = E_{CB}\underline{/0°} = E_L\underline{/0°} \text{ volts,} \tag{578}$$

$$\dot{E}_{BA} = E_{BA}\underline{/120°} = E_L\underline{/120°} \text{ volts,} \tag{579}$$

$$\dot{E}_{AC} = E_{AC}\underline{/240°} = E_L\underline{/240°} \text{ volts.} \tag{580}$$

The equations of the load currents are

$$\dot{I}_{CB} = \frac{\dot{E}_{CB}}{\dot{Z}} = \frac{E_{CB}\underline{/0^\circ}}{Z\underline{/\theta}} = I_{CB}\underline{/-\theta} \text{ amp,} \tag{581}$$

$$\dot{I}_{BA} = \frac{\dot{E}_{BA}}{\dot{Z}} = \frac{E_{BA}\underline{/120^\circ}}{Z\underline{/\theta}} = I_{BA}\underline{/120^\circ - \theta} \text{ amp,} \tag{582}$$

$$\dot{I}_{AC} = \frac{\dot{E}_{AC}}{\dot{Z}} = \frac{E_{AC}\underline{/240^\circ}}{Z\underline{/\theta}} = I_{AC}\underline{/240^\circ - \theta} \text{ amp.} \tag{583}$$

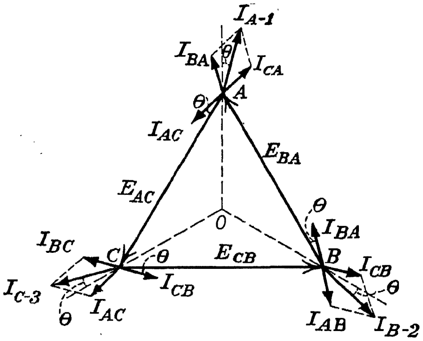


FIG. 184.—Vector diagram of balanced delta-connected inductive loads on a three-phase circuit.

The line currents are found by combining these load currents. At the point B, Fig. 184,

$$\dot{I}_{B-2} = \dot{I}_{CB} + \dot{I}_{AB} \text{ amp} \tag{584}$$
$$= I_{CB}\underline{/-\theta} + I_{AB}\underline{/300^\circ - \theta} \text{ amp} \tag{585}$$
$$\underline{/-30^\circ - \theta} \text{ amp.} \tag{586}$$

At the point A,

$$\dot{I}_{A-1} = \dot{I}_{BA} + \dot{I}_{CA} \text{ amp} \tag{587}$$
$$= I_{BA}\underline{/120^\circ - \theta} + I_{CA}\underline{/60^\circ - \theta} \text{ amp} \tag{588}$$
$$= \sqrt{3}I_{BA}\underline{/90^\circ - \theta} \text{ amp.} \tag{589}$$

At the point C,

$$\dot{I}_{C-3} = \dot{I}_{AC} + \dot{I}_{BC} \text{ amp} \tag{590}$$
$$= I_{AC}\underline{/240^\circ - \theta} + I_{BC}\underline{/180^\circ - \theta} \text{ amp.} \tag{591}$$

For a balanced delta-connected load the line currents are equal to $\sqrt{3}$ times the load currents and are 30 deg. out of phase with each of the adjacent load currents. This may be seen from Fig. 184, as the currents I_{AB} and I_{CB} are 60 deg. apart and I_{B-2}, the diagonal of the parallelogram, bisects this 60-deg. angle, making I_{B-2} 30 deg. ahead of I_{AB} and 30 deg. behind I_{CB}.

When Eq. (544) for the neutral emf E_{OB} is compared with Eq. (586) for the line current in line B, it is seen that the line current in B differs by the angle θ from the neutral voltage E_{OB}. This is true at each corner of the delta, as indicated in Fig. 184, *the line currents in the three-phase balanced delta-connected system making the same angle with the neutral emfs as the load currents do with their load emfs.*

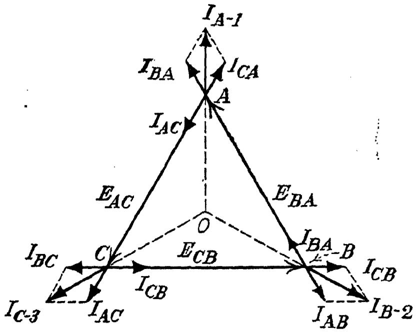


FIG. 185.—Vector diagram for balanced delta-connected noninductive loads on a three-phase circuit.

For nonreactive loads the angle θ becomes zero, as shown in Fig. 185, and the load currents are in phase with the line voltages. The combinations of these load currents at each corner of the delta give the line currents in phase with the neutral voltages E_{OA}, E_{OB}, and E_{OC} at the points A, B, and C, as shown in Fig. 185.

Comparing these results with those found for a star-connected load in Art. 95, *it is found that, in general, in any balanced three-phase load the line currents are out of phase with the neutral voltages by the angle θ where $\cos \theta$ is the power factor of the load* and that they are out of phase with the two adjacent line voltages by $(30° + \theta)$ and $(30° - \theta)$. These relations have an important bearing on power measurements in three-phase circuits.

b. Unbalanced Loads.—For unbalanced impedances placed between the line wires of a three-phase system, the current for each load is calculated, and the line currents are the sum of the two load currents where they branch out from their respective line wires. These line currents for the unbalanced loads are not equal to $\sqrt{3}$ times the load currents, and all line currents may make different angles with the neutral voltages. The current and voltage relations for balanced load impedances are illustrated in Examples 63 and 64, and those for unbalanced load impedances are shown in Examples 65 and 66.

Example 63

Assume a three-wire three-phase system with 100 volts between wires and with three resistances of 10 ohms each connected between these wires, similar to the loads shown in Fig. 183. Find the line currents and their phase relations to the line voltages.

From the conditions of the problems,

$$\dot{I}_{AC} = \dot{I}_{CB} = \dot{I}_{BA} = \frac{100}{10} = 10 + j0 \text{ amp.}$$

At the corner B, Fig. 183, using E_{CB} as the reference,

$$\begin{aligned} \dot{I}_{B-2} &= \dot{I}_{CB_{CB}} - \dot{I}_{BA_{CB}} \text{ amp,} \\ \dot{I}_{BA_{CB}} &= 10\underline{/120^\circ} = -5 + j8.66 \text{ amp,} \end{aligned}$$

and

$$\begin{aligned} \dot{I}_{B-2_{CB}} &= 10 - (-5 + j8.66) \text{ amp} \\ &= 15 - j8.66 = 17.3\underline{/-30^\circ} \text{ amp,} \end{aligned}$$

which is 30 deg. behind E_{CB} and 30 deg. ahead of E_{AB}. In a similar manner, at A, using E_{BA} as the reference,

$$\begin{aligned} \dot{I}_{A-1_{BA}} &= \dot{I}_{BA} - \dot{I}_{AC_{BA}} \text{ amp} \\ &= 15 - j8.66 = 17.3\underline{/-30^\circ} \text{ amp.} \end{aligned}$$

Also at C, using E_{AC} as the reference,

$$\dot{I}_{C-3_{AC}} = 15 - j8.66 = 17.3\underline{/-30^\circ} \text{ amp.}$$

Example 64

Replace the three loads of Example 63 with three impedances, as shown in Fig. 183, each impedance having a value of $\dot{Z} = 8 + j6 = 10\underline{/53^\circ 8'}$ ohms.

Then

$$\begin{aligned} \dot{I}_{AC} = \dot{I}_{CB} = \dot{I}_{BA} &= \frac{100\underline{/0^\circ}}{10\underline{/53^\circ 8'}} \text{ amp} \\ &= 10\underline{/-53^\circ 8'} = 8 - j6 \text{ amp.} \end{aligned}$$

At the corner B, Fig. 184,

$$\dot{I}_{B-2_{CB}} = \dot{I}_{CB} - \dot{I}_{BA_{CB}} \text{ amp,}$$

but

$$\dot{I}_{BA_{CB}} = 10\underline{/-5} \quad \text{amp}$$
$$2' = 1.2 + j9.93 \text{ amp},$$

$$\dot{I}_{B-2_{CB}} = 8 - j6 - 1.2 - j9.93 \text{ amp}$$
$$= 6.8 - j15.93 = 17.3\underline{/-83°8'} \text{ amp}.$$

The currents in lines A and C may be found in a similar manner.

Example 65

Assume the three impedances,

$$\dot{Z}_{BA} = 8 - j6 = 10\underline{/-36°52'} \text{ ohms},$$
$$\dot{Z}_{CB} = 8 + j6 = 10\underline{/36°52'} \text{ ohms},$$

and

$$\dot{Z}_{AC} = 10 + j0 = 10\underline{/0°} \text{ ohms},$$

connected in mesh to the 100-volt three-wire three-phase circuit of Example 63, as shown in Fig. 186.

The load currents in the three impedances will be

$$\dot{I}_{BA} = \frac{100}{10\underline{/-36°52'}} = 10\underline{/36°52'} \text{ amp}$$
$$= 8 + j6 \text{ amp},$$

$$\dot{I}_{CB} = \frac{100}{10\underline{/36°52'}} = 10\underline{/-36°52'} \text{ amp}$$
$$= 8 - j6 \text{ amp},$$

and

$$\dot{I}_{AC} = \frac{100}{10\underline{/0°}} = 10 + j0 \text{ amp}.$$

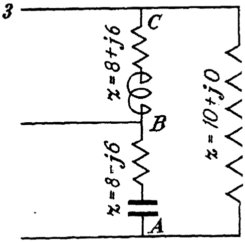


FIG. 186.—Unbalanced delta-connected loads for Example 65.

At the corner A, or line wire A, and using E_{BA} as the reference,

$$\dot{I}_{A-1} = \dot{I}_{BA} - \dot{I}_{AC} \text{ amp},$$

but

$$\dot{I}_{AC_{BA}} = 10\underline{/120°} = -5 + j8.66 \text{ amp},$$

and

$$\dot{I}_{A-1_{BA}} = 8 + j6 + 5 - j8.66 \text{ amp}$$
$$= 13 - j2.66 = 13.27\underline{/-11°32'} \text{ amp}.$$

At the corner C, using E_{BA} as the reference,

$$\dot{I}_{C-3} = \dot{I}_{AC} - \dot{I}_{CB} \text{ amp},$$

but

$$\dot{I}_{CB_{BA}} = 10\underline{/-36°52'}\underline{/-120°} \text{ amp}$$
$$= 10\underline{/-156°52'} = -9.2 - j3.93 \text{ amp},$$
$$\dot{I}_{C-3_{BA}} = -5 + j8.66 + 9.2 + j3.93 \text{ amp}$$
$$= 42 + j12.59 = 13.27\underline{/71°35'} \text{ amp}.$$

At the corner B, using $\dot{E}_{BA}$ as the reference,

$$\dot{I}_{B-2} = \dot{I}_{CB} - \dot{I}_{BA} \text{ amp,}$$

or

$$\dot{I}_{B-2_{BA}} = -9.2 - j3.93 - 8 - j6 \text{ amp}$$
$$= -17.2 - j9.93 = 19.8\underline{/-150^\circ} \text{ amp.}$$

The line-wire currents as well as the load currents are unbalanced in amount as well as in phase position. The solution is illustrated in Fig. 187.

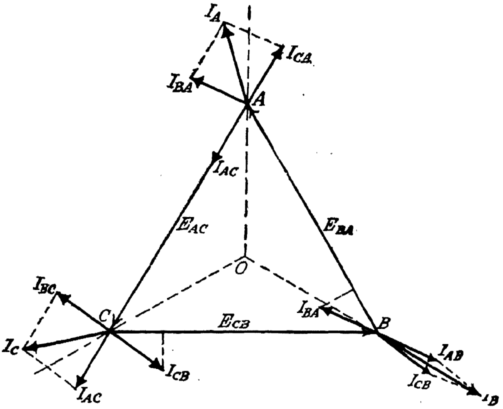


FIG. 187.—Vector diagram for unbalanced delta-connected loads for Example 65.

Example 66

Change the order of connection of the impedances of Example 65 to agree with that illustrated in Fig. 188, the values and phase sequence of the emfs remaining as before.

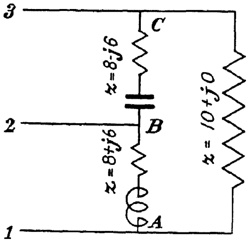


FIG. 188.—Unbalanced delta-connected loads for Example 66.

The new load currents will be

$$\dot{I}_{BA} = 8 - j6 = 10\underline{/-36^\circ 52'} \text{ amp,}$$
$$\dot{I}_{CB} = 8 + j6 = 10\underline{/36^\circ 52'} \text{ amp,}$$

and

$$\dot{I}_{AC} = 10 + j0 = 10\underline{/0^\circ} \text{ amp.}$$

At the corner A, or line wire A,

$$\dot{I}_{AC_{BA}} = 10\underline{/120^\circ} = -5 + j8.66 \text{ amp,}$$

or

$$\dot{I}_{A-1_{BA}} = 8 - j6 + 5 - j8.66 \text{ amp}$$
$$= 13 - j14.66 = 19.6\underline{/-48^\circ 10'} \text{ amp.}$$

At the corner B, or line wire B, using $\dot{E}_{BA}$ as the reference,

$$\dot{I}_{CB_{BA}} = 10\underline{/36^\circ 52'}\underline{/120^\circ} \text{ amp}$$
$$= 10\underline{/-83^\circ 8'} = 1.2 - j9.93 \text{ amp,}$$

or

$$\dot{I}_{B-2_{BA}} = 1.2 - j9.93 - 8 - j6 \text{ amp}$$
$$= -6.8 + j3.93 = 7.86\underline{/150^\circ} \text{ amp.}$$

At the corner C, or line wire C, using E_{BA} as the reference,

$$6 - 1.2 + j9.93 \text{ amp}$$
$$-6.2 + j18.59 = 19.6\underline{/108^\circ 25'} \text{ amp.}$$

These values are illustrated in Fig. 189.

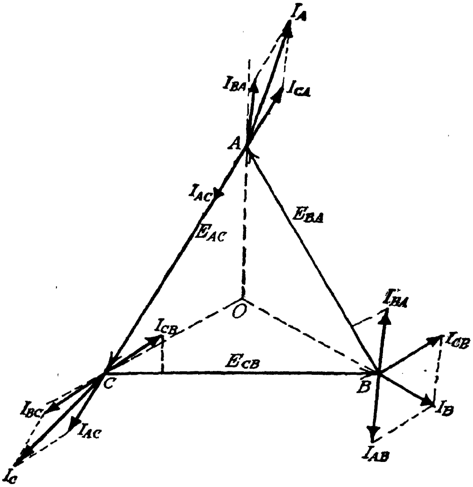


FIG. 189.—Vector diagram for unbalanced delta-connected loads for Example 66.

99. Active Power Consumed in Three-phase Circuits.

—The active power consumed by each load for the balanced Y-connected loads, as shown in Figs. 173 and 174, is, respectively,

$$P_{OA} = E_{OA}I_{OA} \cos \theta \text{ watts,} \tag{595}$$
$$P_{OB} = E_{OB}I_{OB} \cos \theta \text{ watts,} \tag{596}$$

and

$$P_{OC} = E_{OC}I_{OC} \cos \theta \text{ watts.} \tag{597}$$

The total power consumed is

$$\begin{aligned} P_0 &= P_{OA} + P_{OB} + P_{OC} \text{ watts} \\ &= E_{OA}I_{OA} \cos \theta + E_{OB}I_{OB} \cos \theta + E_{OC}I_{OC} \cos \theta \text{ watts} \\ &= 3E_LI_L \cos \theta \text{ watts,} \end{aligned} \tag{598}$$

where E_L is the phase or load voltage and I_L is the load current. If the line currents and line voltages are used instead of the load currents and load voltages, Eq. (598) becomes

$$P_0 = \sqrt{3}EI \cos \theta \text{ watts,} \tag{599}$$

where E is the line voltage and I is the line current, since in the Y-connected system, the load currents are the same as the line currents and the load voltages are equal to the line voltages divided by $\sqrt{3}$.

The power consumed in each load of the balanced delta-connected loads, shown in Figs. 183 and 184, is, respectively,

$$P_{BA} = E_{BA}I_{BA} \cos \theta \text{ watts,} \tag{600}$$
$$P_{AC} = E_{AC}I_{AC} \cos \theta \text{ watts,} \tag{601}$$
$$P_{CB} = E_{CB}I_{CB} \cos \theta \text{ watts.} \tag{602}$$

The total power consumed in terms of load voltages and currents is

$$P_0 = P_{BA} + P_{AC} + P_{CB} \text{ watts} \tag{603}$$
$$= 3E_LI_L \cos \theta \text{ watts.} \tag{598}$$

Since in the delta system the load voltages are equal to the line voltages and the load currents are equal to the line currents divided by $\sqrt{3}$, the expression for total power, in terms of line voltage and current, becomes

$$P_0 = \sqrt{3}EI \cos \theta \text{ watts.} \tag{599}$$

It is seen from Eq. 599 that the expression for power in the balanced three-phase three-wire system is the same for the Y as for the delta connection. If the power consumed and the line voltage are known for a balanced three-phase load, the line current may be found from Eq. (599) as

$$I = \frac{P_0}{\sqrt{3}E \cos \theta} \text{ amp.} \tag{606}$$

100. Reactive Power in Three-phase Circuits with Balanced Loads.—The total reactive power in the Y-connected system of Figs. 173 and 174 is the sum of the reactive powers of the individual phases, or

$$\begin{aligned}
Q_o &= Q_{OA} + Q_{OB} + Q_{OC} \text{ vars} && (607)\\
&= E_{OA}I_{OA} \sin \theta + E_{OB}I_{OB} \sin \theta + E_{OC}I_{OC} \sin \theta \text{ vars} && (608)\\
&= 3E_LI_L \sin \theta \text{ vars.} && (609)
\end{aligned}$$

In terms of line voltage and currents the total reactive power is

$$Q_o = \sqrt{3}EI \sin \theta \text{ vars.} \tag{610}$$

For the delta-connected system of Figs. 183 and 184, the expression for total reactive power in terms of load currents and voltages is

$$\begin{aligned}
Q_o &= Q_{BA} + Q_{AC} + Q_{CB} \text{ vars} && (611)\\
&= E_{BA}I_{BA} \sin \theta + E_{AC}I_{AC} \sin \theta + E_{CB}I_{CB} \sin \theta \text{ vars} && (612)\\
&= 3E_LI_L \sin \theta \text{ vars.} && (609)
\end{aligned}$$

Expressed in terms of line currents and voltages, the total reactive power becomes

$$Q_o = \sqrt{3}EI \sin \theta \text{ vars,} \tag{610}$$

which is the same as for the Y-connected system with balanced loads.

101. Vector and Apparent Power in Three-phase Circuits with Balanced Loads.—The vector and apparent power in the three-phase circuit with balanced loads is equal to the square root of the sum of the squares of the active and reactive power. That is,

$$\begin{aligned}
U_0 = V_0 &= \sqrt{P_O{}^2 + Q_O{}^2} \text{ volt-amp} && (615)\\
&= \sqrt{(\sqrt{3}EI \cos \theta)^2 + \qquad\qquad\qquad} && \\
&= \sqrt{(\sqrt{3}EI)^2(\cos^2 \theta + \sin^2 \theta)} \text{ volt-amp} && (617)\\
&= \sqrt{3}EI \text{ volt-amp.} && (618)
\end{aligned}$$

Example 67

A three-phase system carries an inductive load of 0.80 power factor. The line current was found to be 377 amp and the line voltage 230 volts. Find the active, reactive, and vector power.

Applying the equations just derived,

$$\begin{aligned}
P &= \sqrt{3}EI \cos \theta 10^{-3} \text{ kw}\\
&= \sqrt{3} \times 230 \times 277 \times 0.80 \times 10^{-3} = 120 \text{ kw.}\\
Q &= \sqrt{3}EI \sin \theta 10^{-3} \text{ kvar}\\
&= \sqrt{3} \times 230 \times 377 \times (-0.60) \times 10^{-3} = -90 \text{ kvar,}\\
U = V &= \sqrt{3} \times 230 \times 377 \times 10^{-3} = 150 \text{ kva.}
\end{aligned}$$

Problems

1-8. Two coils of an a-c generator are placed 90 deg. apart on the armature, and each coil generates 2,200 volts. The emf E_{AB} leads E_{CD} by 90 deg., and the middle points of the coils are joined together. (*a*) Write the equations for the wave forms of these two emfs, using E_{CD} as the reference, and draw the corresponding wave forms. (*b*) Express the emfs in vector form, polar notation, and draw the corresponding vector diagram.

2-8. Three emfs E_{ab}, E_{cd}, and E_{fg} form a symmetrical three-phase system, the phase sequence being in the order stated. (*a*) Write the equations for the wave forms, using (1) the wave of e_{ab} as the reference and (2) the wave of e_{cd} as the reference, and draw the corresponding wave forms. (*b*) Express the emfs forming this system in the vector form, polar notation, using (1) E_{ab} as the reference and (2) E_{cd} as the reference, and draw the corresponding vector diagrams.

3-8. A two-phase four-wire system has the emfs $\dot{E}_{FG} = 120 + j0$ and $\dot{E}_{HK} = 0 + j120$. The impedances $\dot{Z}_{FG} = 12 + j4$ and $\dot{Z}_{HK} = 8 - j2$ are connected to the respective phases. What current flows in each wire? Draw the circuit and vector diagrams.

4-8. The emfs E_{12} and E_{34} of a four-wire two-phase system are each 220 volts. An impedance $\dot{Z}_{12} = 10\underline{/60^\circ}$ ohms is connected across one phase and $\dot{Z}_{34} = 8\underline{/-75^\circ}$ ohms across the other phase. What current flows in each wire with E_{12} as the reference? Draw the circuit and vector diagrams.

5-8. Two admittances, each $\dot{Y} = 0.05\underline{/-45^\circ}$, are connected across the two phases of a 550-volt four-wire two-phase system. What current flows in each wire? Draw the circuit and vector diagrams.

6-8. A three-wire two-phase system has the emfs $\dot{E}_{BA} = 110\underline{/0^\circ}$ and $\dot{E}_{BC} = 110\underline{/90^\circ}$. The load impedances are $\dot{Z}_{BA} = 6 + j8$ and $\dot{Z}_{BC} = 4 + j3$. Find the current in each wire. Draw the circuit and vector diagrams.

7-8. A three-wire two-phase system has 440 volts between the common return and each outside wire, the emfs being $\dot{E}_{12} = 440\underline{/0^\circ}$ and $\dot{E}_{13} = 440\underline{/90^\circ}$. An impedance $\dot{Z}_{12} = 22\underline{/40^\circ}$ is connected across one phase and $\dot{Z}_{13} = 11\underline{/-40^\circ}$ across the other phase; what current flows in each wire? Draw the circuit and vector diagrams.

8-8. What active and reactive power are furnished to each of the loads of a two-phase four-wire 220-volt system, the load impedances each being $\dot{Z} = 7 + j2$? What are the total active and reactive power?

9-8. A 2,200-volt three-wire two-phase generator furnishes power to a balanced load of 5,000 kw at 0.75 power factor. What current flows in each wire?

10-8. What active and reactive power are furnished to each of the loads in Prob. 4-8?

11-8. A two-phase three-wire system has the emfs $\dot{E}_{HK} = 120\underline{/0^\circ}$ and $\dot{E}_{HL} = 100\underline{/90^\circ}$. The load on phase *HK* is 2 kw at 0.75 power factor inductive and on phase *KL* is 1.9 kw at 0.90 power factor capacitive. Find the current flowing in each wire and draw the circuit and vector diagrams.

12-8. A 2,200-volt three-wire two-phase system has a two-phase induction motor load taking 300 kva input at 0.60 power factor connected to the three wires and a lighting load taking 50 kw connected from one wire to the common return. Find the currents in each of the three wires. Draw the circuit and vector diagrams.

13-8. A 22,000-volt four-wire two-phase generator has the middle points of its phases connected to form a five-wire two-phase system. (*a*) If four equal impedances, represented by $\dot{Z} = 40 + j50$, are connected between neutral and each wire, what current flows in each wire? (*b*) If one of these impedances is changed to one having $\dot{Z} = 40 - j50$, what current flows in each wire?

14-8. A five-wire two-phase system having 125 volts from neutral to each outside wire has four equal impedances, $\dot{Z} = 5\underline{/50^\circ}$, connected between neutral and each outside wire. (*a*) What is the current in each wire? (*b*) A two-phase induction motor taking a balanced load of 25 kw input at 0.70 power factor is also connected across the outside wires; find the current in each wire, and draw the circuit and complete vector diagrams.

15-8. A five-wire two-phase generator has 250 volts between neutral and each outside wire. Four equal impedances $\dot{Z} = 10\underline{/35^\circ}$ are connected in mesh between the outside wires; what is the current in each wire? What power is consumed? Draw the vector diagram.

16-8. Consider four impedances, each equal to $\dot{Z} = 7 + j8$ ohms, connected in mesh to a 220-volt two-phase system, the voltage being measured across the corners of the square. Find the currents in each load and in each line wire. Draw the vector diagram.

17-8. A three-phase four-wire system has the three emfs to neutral $\dot{E}_{NA} = 127\underline{/0^\circ}$, $\dot{E}_{NB} = 127\underline{/240^\circ}$, and $\dot{E}_{NC} = 127\underline{/120^\circ}$. The three impedances $\dot{Z}_{NA} = 8 - j6$, $\dot{Z}_{NB} = 4 + j2$, and $\dot{Z}_{NC} = 4 + j6$ are connected between the outside wires and the neutral. What is the current in each line wire and the neutral? Draw the circuit and vector diagrams.

18-8. A three-phase four-wire system having 69 volts to neutral has admittances $\dot{Y}_{Na} = 0.4\underline{/20^\circ}$, $\dot{Y}_{Nb} = 0.8\underline{/0^\circ}$, and $\dot{Y}_{Nc} = 1.2\underline{/-30^\circ}$ connected between the lines and neutral. What is the current in each line and in the neutral? Draw the connection and vector diagrams.

19-8. A four-wire three-phase system has the emfs $\dot{E}_{NA} = 138\underline{/30^\circ}$, $\dot{E}_{NB} = 138\underline{/270^\circ}$, and $\dot{E}_{NC} = 138\underline{/150^\circ}$. A load of 2 kva at unity power factor is connected between line *A* and neutral; a load of 4.1 kw at 0.80 power factor inductive is connected between line *B* and neutral; and a load of 3 kva at 0.90 power factor inductive is connected between line *C* and neutral. What are the currents in each line and the neutral? Draw the vector diagram.

20-8. An unsymmetrical three-phase four-wire system has the emfs to neutral $\dot{E}_{NA} = 100\underline{/0^\circ}$, $\dot{E}_{NB} = 120\underline{/265^\circ}$, and $\dot{E}_{NC} = 85\underline{/130^\circ}$. An impedance $\dot{Z} = 5\underline{/20^\circ}$ is connected between each line wire and the neutral. Find the current in each line and in the neutral. Draw the vector diagram.

21-8. Three impedances $\dot{Z}_1 = 10 + j0$, $\dot{Z}_2 = 7 + j6$, and $\dot{Z}_3 = 2 - j5$ are connected between neutral and the three outside wires, respectively, of a four-wire three-phase system having 1,100 volts to neutral. Find the current in each line wire and in the neutral, and draw the complete vector diagram for the circuit.

22-8. In a three-wire three-phase system the line voltages are $\dot{E}_{ab} = 120\underline{/0^\circ}$, $\dot{E}_{fg} = 120\underline{/240^\circ}$, and $\dot{E}_{cd} = 120\underline{/120^\circ}$, the delta being formed by connecting g to a, b to c, and d to f. An impedance $\dot{Z} = 817\underline{/30^\circ}$ is connected between each of the lines to form a balanced delta-connected load. What are the currents in each load and each line? Draw a complete vector diagram showing the emfs and the line currents.

23-8. In a three-wire three-phase system the emfs between the lines A, B, and C are $\dot{E}_{AB} = 220\underline{/0^\circ}$, $\dot{E}_{BC} = 220\underline{/120^\circ}$, and $\dot{E}_{CA} = 220\underline{/240^\circ}$. The impedances $\dot{Z}_{AB} = 8 + j6$, $\dot{Z}_{BC} = 7.6 + j0$, and $\dot{Z}_{CA} = 5.4 - j1.8$ are connected between the lines, forming a mesh-connected load. Find the load and line currents, and draw a complete vector diagram showing the emfs and currents.

24-8. Three loads of 200, 400, and 600 amp, respectively, at unity power factor are connected in delta on a three-phase line. What is the current in each line wire? Draw a complete vector diagram showing the voltages, line currents, and load currents.

25-8. In a three-wire three-phase system the emfs between the lines F, G, and H are $\dot{E}_{FG} = 440\underline{/0^\circ}$, $\dot{E}_{GH} = 440\underline{/240^\circ}$, and $\dot{E}_{HF} = 440\underline{/120^\circ}$. The impedances $\dot{Z}_{FG} = 28.4\underline{/20^\circ}$, $\dot{Z}_{GH} = 15\underline{/0^\circ}$, and $\dot{Z}_{HF} = 31.2\underline{/30^\circ}$ are connected between lines forming a mesh-connected load. Find the line and load currents, and draw the complete vector diagram.

26-8. A symmetrical three-wire three-phase system has both balanced-star and mesh-connected loads. The emfs from neutral to the three lines A, B, and C are $\dot{E}_{NA} = 120\underline{/0^\circ}$, $\dot{E}_{NB} = 120\underline{/420^\circ}$, and $\dot{E}_{NC} = 120\underline{/120^\circ}$. The star-connected impedances are $\dot{Z}_{NA} = \dot{Z}_{NC} = \dot{Z}_{NB} = 12\underline{/15^\circ}$, and the mesh-connected impedances are $\dot{Z}_{AB} = \dot{Z}_{BC} = \dot{Z}_{CA} = 10\underline{/0^\circ}$. What currents are flowing in each load and line and in the neutral? Draw the circuit and complete vector diagrams.

27-8. A four-wire three-phase system has both star- and mesh-connected loads. The emfs between the lines A, B, and C are $\dot{E}_{AB} = 240\underline{/0^\circ}$, $\dot{E}_{BC} = 240\underline{/120^\circ}$, and $\dot{E}_{CA} = 240\underline{/240^\circ}$. The star-connected loads are $\dot{Z}_{NA} = 9.6\underline{/-30^\circ}$, $\dot{Z}_{NB} = 8.4\underline{/20^\circ}$, and $\dot{Z}_{NC} = 4.3\underline{/0^\circ}$. The mesh-connected loads are $\dot{Z}_{AB} = 15.6\underline{/30^\circ}$, $\dot{Z}_{BC} = 12\underline{/0^\circ}$, and $\dot{Z}_{CA} = 8.6\underline{/0^\circ}$. Find the load, line, and neutral currents, and draw the complete vector diagram.

28-8. An unsymmetrical three-wire three-phase system has delta-connected loads. The emf between the wires A, B, and C are $\dot{E}_{AB} = 184 + j0$, $\dot{E}_{BC} = -122 + j168$, and $\dot{E}_{CA} = -62 - j168$. The impedances are $\dot{Z}_{AB} = 8 + j6$, $\dot{Z}_{BC} = 12.6 + j0$, and $\dot{Z}_{CA} = 9.8 + j0$. What are the currents flowing in each load and line wire? Draw a circuit diagram and a complete vector diagram.

29-8. A three-wire three-phase circuit supplies energy at 230 volts between wires to an induction motor taking 100 kw at 0.65 power factor. What current flows in each line wire? What are the reactive and the vector power?

30-8. A 240-volt three-phase induction motor operating at 0.50 power factor takes 120 amp at each terminal. What are the active, reactive, and vector power consumed by the motor?

31-8. The meter readings on a three-phase transmission line delivering power to a substation are: voltmeter 112,600 volts, ammeter 131 amp, wattmeter 18,600 kw. Find the reactive and apparent powers, and the power factor of the line.

32-8. Three impedances, each equal to $\dot{Z} = 4 + j4$, are connected in delta between 230-volt three-phase lines. Find the active, reactive, and vector power delivered to the loads.

33-8. Three impedances, each $\dot{Z} = 12.8\underline{/-30^\circ}$, are connected in Y to a 6,600-volt four-wire three-phase system. Find the active, reactive, and vector power delivered to the loads.

34-8. The vector power of a balanced, 2,300-volt three-wire three-phase system is 976 vector kva and the active power is 832 kw. Calculate the power factor, the reactive power, and the line current.

35-8. In a symmetrical three-wire two-phase system with line wires A and C and common return wire B, $\dot{E}_{BA} = 115\underline{/90^\circ}$ and $\dot{E}_{BC} = 115\underline{/0^\circ}$. The impedances are $\dot{Z}_{BC} = 1 + j2$ and $\dot{Z}_{BA} = 2 - j4$; find the currents in each of the wires and the emf across the outside wires. Draw the circuit and vector diagrams.

36-8. Consider the same loads and voltages as in Prob. 35-8 but with E_{BA} 90 deg. behind E_{BC}, *i.e.*, $\dot{E}_{BC} = 115\underline{/0^\circ}$ and $\dot{E}_{BA} = 115\underline{/270^\circ}$. Find the currents in each of the wires, using E_{BC} as the reference. From the results of these two problems, explain how the phase sequence of a two-phase system may be obtained by laboratory tests.

37-8. An induction motor is connected to a 220-volt four-wire two-phase generator. What is the current in each wire if the motor takes 300 kw at 0.40 power factor?

38-8. Energy is furnished at 2,200 volts three-wire two-phase to a motor load taking 800 kw at 0.70 power factor inductive; what is the current in each line wire and in the common return? A lighting load of 200 kw is now connected between one wire and the common return; what is the current in each wire?

39-8. Let the emfs to neutral of a five-wire two-phase system be 1,000 volts and the impedances from neutral to each outside wire be, respectively, $\dot{Z}_{NA} = 10 + j10$, $\dot{Z}_{NB} = 8 + j1$, $\dot{Z}_{NC} = 0 - j8$, and $\dot{Z}_{ND} = 5 + j10$. There is also a balanced two-phase induction-motor load connected across both phases and taking 200 kva at 0.60 power factor. Using E_{NB} as the reference, find the current in each wire, and draw the complete vector diagram.

40-8. A five-wire two-phase system has 500 volts to neutral and carries a balanced motor load of 500 kw at 0.50 power factor inductive. There is also a single-phase lighting load of 50 kw between the neutral and one outside wire. What is the current in each wire?

41-8. Three impedances, $\dot{Z}_1 = 8 - j6$, $\dot{Z}_2 = 4 + j2$, and $\dot{Z}_3 = 7 - j10$, respectively, are connected in delta between 1,100-volt three-phase lines; what current flows in each line wire?

42-8. Three impedances, $\dot{Z}_1 = 0 - j10$, $\dot{Z}_2 = 10 + j10$, and $\dot{Z}_3 = 8 - j6$, are connected in delta between 220-volt, three-phase lines. Find the values of the line-wire currents in reference to the voltage drop across Z_2.

43-8. Loads of 150 kva each at 0.90 power factor leading are connected between the neutral and each of the outside wires of a four-wire three-phase system having 1,732 volts to neutral. Find the current in each line and the neutral wire.

44-8. Three impedances, $\dot{Z}_1 = 8 - j6$, $\dot{Z}_2 = 5 + j0$, and $\dot{Z}_3 = 4 + j2$, are connected in star. The neutral of this load is connected to a four-wire three-phase generator having 2,200 volts to neutral. Find the magnitudes of the currents in each line wire and in the neutral.

45-8. In a three-wire three-phase system, the voltage between wires is 2,200 volts, and the loads are the impedances $\dot{Z} = 20 - j20$ connected in delta. Find the currents and their phase relations to each of the line voltages.

46-8. A symmetrical three-wire three-phase system is supplied to three unequal impedances connected in star. The emfs of the system with lines A, B, and C are $\dot{E}_{AB} = 220\underline{/0^\circ}$, $\dot{E}_{BC} = 220\underline{/120^\circ}$, and $\dot{E}_{CA} = 220\underline{/240^\circ}$. The impedances are $\dot{Z}_{OA} = 6 - j8$, $\dot{Z}_{OB} = 4 + j2$, and $\dot{Z}_{OC} = 8.6 + j0$. Find the voltage drop across each impedance and the currents in each line.

47-8. Three loads are connected in delta to a 440-volt three-wire three-phase system. The load between A and B is 4 kva at 0.80 power factor inductive, the load between B and C is 6.3 kva at 0.90 power factor capacitive, and the load between C and A is 5.5 kva at unity power factor. Find the current in each load and line. Draw the complete vector diagram.

48-8. In a three-phase four-wire system with lines A, B, and C, the emfs to neutral are $\dot{E}_{NA} = 190\underline{/0^\circ}$, $\dot{E}_{NB} = 200\underline{/240^\circ}$, and $\dot{E}_{NC} = 240\underline{/120^\circ}$. The load impedances are $\dot{Z}_{NA} = 9.6\underline{/45^\circ}$, $\dot{Z}_{NB} = 12\underline{/-30^\circ}$, and $\dot{Z}_{NC} = 3.8\underline{/0^\circ}$. Find the current in each line and in the neutral. Draw the complete vector diagram.

CHAPTER IX

POWER MEASUREMENTS IN POLYPHASE CIRCUITS

Instantaneous Power of a Polyphase Circuit.—The instantaneous power at the points of entry of a polyphase circuit is equal to the algebraic sum of the products obtained by multiplying the instantaneous current at each point of entry by the potential difference between that point of entry and some arbitrarily selected base point (which may be the neutral point of entry) from which all the potential differences are measured. The value of the instantaneous power is given in watts when the currents are in amperes and the potential differences are in volts.

Active Power of a Polyphase Circuit.—The active power at the points of entry of a polyphase circuit is the time average of the values of the instantaneous power at the points of entry, the average being taken over a complete cycle of the alternating current. From the foregoing it follows that the active power at the points of entry of a polyphase circuit is equal to the algebraic sum of the active powers for the individual points of entry when the potential differences are all determined with respect to the same arbitrarily selected base point (which may be the neutral point of entry). The value of active power is given in watts when the effective currents are in amperes and the effective potential differences are in volts.

Reactive Power of a Polyphase Circuit.—The reactive power at the points of entry of a polyphase circuit is the algebraic sum of the reactive powers of the individual points of entry when the potential differences are all determined with respect to the same base point (which may be the neutral point of entry). The magnitude and sign of the reactive power of a polyphase circuit is independent of the point that is chosen as the base point provided that the same convention as regards sign is used for all points of entry. The convention used in these definitions gives a positive sign to the reactive power of a harmonic component at any point of entry when the same phase relation of current and potential difference is that which exists if, within the delimited region, a capacitive circuit connects the point *of entry to the base point. The value of reactive power is given in vars where the effective currents are in amperes and the effective potential differences are in volts.*

Vector Power of a Polyphase Circuit.—Vector power at the points of entry of a polyphase circuit is equal to the square root of the sum of the squares of the active power and the reactive power. The unit of vector power is the vector volt-ampere.

Arithmetic Apparent Power of a Polyphase Circuit.—The arithmetic apparent power at the points of entry of a polyphase circuit is equal to the arithmetic sum of the products obtained by multiplying the effective current at each point of entry by the effective potential difference between the point of entry and the neutral point of entry. The unit of arithmetic apparent power is the volt-ampere.

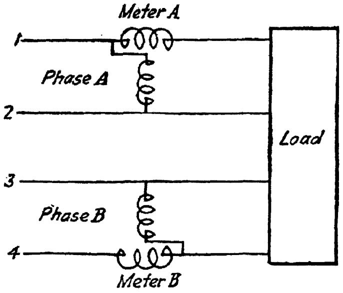


FIG. 190.—Two-wattmeter method of metering the load on a four-wire two-phase circuit.

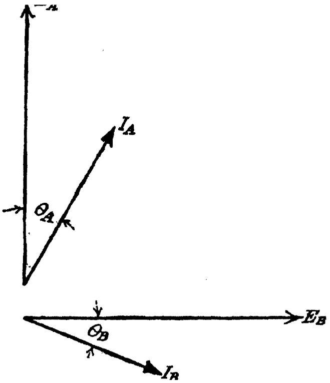


FIG. 191.—Vector diagram for the currents and emfs of Fig. 190.

102. Power Measurements in Four-wire Two-phase Circuits. In Chap. VIII, it has been shown that a four-wire two-phase circuit consists of two single-phase circuits associated by being supplied from the same generator and by being connected to the same polyphase motors. Figure 190 shows the method of connection of two wattmeters for indication of power in such a circuit, and Fig. 191 shows the vector diagram. Meter A will indicate

$$P_A = E_A I_A (P.F._A) \text{ watts}$$

or, for sine waves of current and voltage,

$$P_A = E_A I_A \cos \theta_A \text{ watts,} \tag{51}$$

and meter B will indicate

$$P_B = E_B I_B \cos \theta_B \text{ watts.} \tag{51}$$

The total power taken by the load is then

$$\begin{aligned} P_O &= P_A + P_B \text{ watts} \\ &= E_A I_A \cos \theta_A + E_B I_B \cos \theta_B \text{ watts.} \end{aligned} \tag{619}$$

Most systems are symmetrical, and the voltages E_A and E_B are the same in value. For balanced loads,

$$I_A = I_B = I \text{ amp}$$

and

$$\theta_A = \theta_B = \theta \text{ deg.}$$

Hence,

$$P_O = P_A + P_B = 2EI \cos \theta \text{ watts,} \tag{510}$$

which is the total power in the circuit. For polyphase motor loads, the two sets of meter coils are combined in one case and operate on the same indicating system, giving, on one dial, the total power consumed by the load. Such an instrument is called a "polyphase wattmeter."

Instead of balancing the meter torque against a spring, as is done in an indicating instrument, it may be utilized to turn a disk mounted on a shaft. This shaft actuates a recording mechanism, and the disk is retarded by a torque which is proportional to the speed of the disk. Such a meter, known as a "watt-hour meter," records the product of power and time, or the energy used by the circuit. The meter integrates the energy used over any interval of time chosen and is used to determine the total energy used during an interval of time such as one month.

When two meters are used, if by accident or otherwise the potential coils of the two meters are interchanged so that in meter A the current I_A is associated with E_B and in meter B the current I_B is associated with E_A, meter A will indicate

$$\begin{aligned} Q_A &= E_B I_A \cos (90^\circ - \theta) \text{ vars} \\ &= E_B I_A \sin \theta \text{ vars,} \end{aligned} \tag{620}$$

and meter B will indicate

$$\begin{aligned} Q_B &= E_A I_B \cos (90° + \theta) \text{ vars} \\ &= -E_A I_B \sin \theta \text{ vars.} \end{aligned} \tag{621}$$

By reversing the connections of the potential coil of meter B, both readings may be made positive, and for balanced loads

$$\begin{aligned} Q_O &= Q_A + Q_B \text{ vars} \\ &= 2EI \sin \theta \text{ vars.} \end{aligned} \tag{511}$$

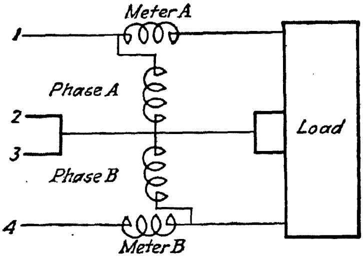


Fig. 192.—Two-wattmeter method of metering the load on a three-wire two-phase circuit.

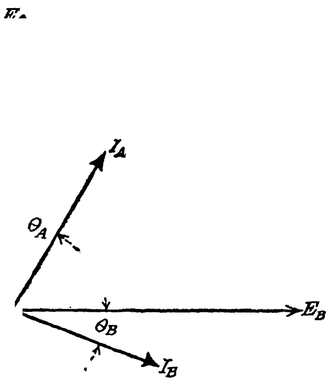


Fig. 193.—Vector diagram for Fig. 192.

Since the reactive power in a balanced polyphase system is the sum of the reactive power in each of the phases, two meters so connected or a polyphase meter connected in this manner will indicate the total reactive power in the circuit instead of the total active power.

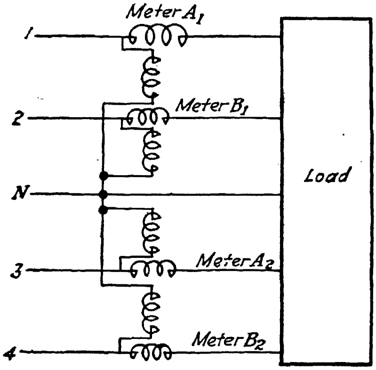


Fig. 194.—Four wattmeters connected to measure the power in a five-wire two-phase circuit.

103. Power Measurements in Three-wire Two-phase Circuits.—For the three-wire two-phase system, the two meters are connected as shown in Fig. 192 with vector diagram as shown in Fig. 193. The same equations hold as for the four-wire two-phase system.

104. Power Measurements in Five-wire Two-phase Circuits.—For the five-wire two-phase system, four wattmeters are used with their current coils connected in the four line wires, and each of the potential coils is connected between the neutral and the same wire in which its respective current coil is connected, as shown in Fig. 194. The sum of these meter readings will give the total power consumed in such a system. The vector diagrams are shown in Figs. 150 and 151.

105. Power Measurements in Three-phase Circuits.—Let E_{ab}, E_{bc}, and E_{ca} (Fig. 195) represent the three voltages between the line wires of any three-phase circuit in which the voltages may be either equal or unequal. Suppose that the currents taken by the three single-phase loads are I_{ab}, I_{bc}, and I_{ca} at angles α', β', and γ', respectively, with their emfs. The power taken by each load is then

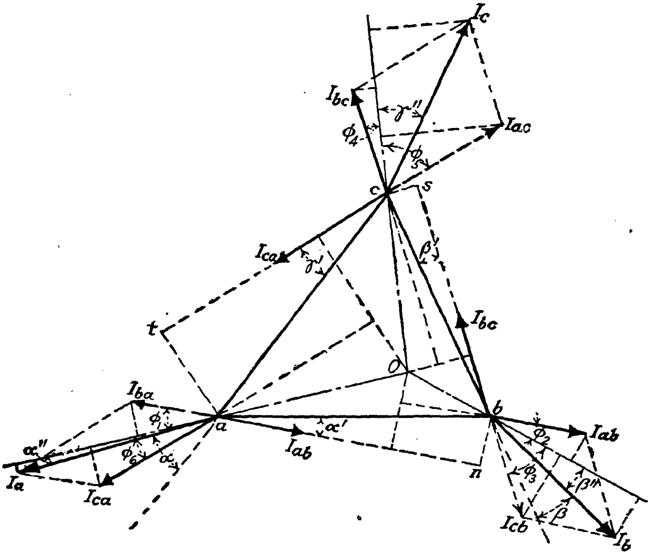


FIG. 195.—Vector relations of currents and emfs in an unsymmetrical unbalanced three-phase system.

$$P_{ab} = E_{ab}I_{ab} \cos \alpha' \text{ watts,} \tag{623}$$

$$P_{bc} = E_{bc}I_{bc} \cos \beta' \text{ watts,} \tag{624}$$

and

$$P_{ca} = E_{ca}I_{ca} \cos \gamma' \text{ watts,} \tag{625}$$

and the total power is given by

$$P_o = P_{ab} + P_{bc} + P_{ca} \text{ watts} \tag{626}$$
$$= E_{ab}I_{ab} \cos \alpha' + E_{bc}I_{bc} \cos \beta' + E_{ca}I_{ca} \cos \gamma' \text{ watts.} \tag{627}$$

Choose any point O within or without the triangle abc and connect O with a, b, and c, giving the voltages E_{oa}, E_{ob}, and E_{oc}. Extend the lines I_{ab}, I_{bc}, and I_{ca}, projecting O on each of these lines. Project the point a on I_{ca} at t, the point b on I_{ab} at n, and the point c on I_{bc} at s. Then

$$E_{an} = E_{ab} \cos \alpha' \text{ volts,} \tag{628}$$
$$E_{bs} = E_{bc} \cos \beta' \text{ volts,} \tag{629}$$

and

$$E_{ct} = E_{ca} \cos \gamma' \text{ volts.} \tag{630}$$

Substituting the values from Eqs. (628) to (630) into Eq. (627),

$$P_o = E_{an}I_{ab} + E_{bs}I_{bc} + E_{ct}I_{ca} \text{ watts.} \tag{631}$$

From projections of E_{oa}, E_{ob}, and E_{oc} on the current lines and from the relations of emfs in a polyphase circuit,

$$E_{an} = \qquad \cos \phi_2 - E_{oa} \cos \phi_1 \text{ volts,} \tag{632}$$
$$= E_{o} \ \cos \phi_4 - E_{ob} \cos \phi_3 \text{ volts,} \tag{633}$$

and

$$E_{ct} = E_{oa} \cos \phi_6 - E_{oc} \cos \qquad ; \text{ volts.} \tag{634}$$

Substituting these values into Eq. (631),

$$P_o = (E_{ob} \cos \phi_2 - E_{oa} \cos \phi_1)I_{ab} + (E_{oc} \cos \phi_4 - E_{ob} \cos \phi_3) I_{bc} + (E_{oa} \cos \phi_6 - E_{oc} \cos \phi_5)I_{ca} \text{ watts.} \tag{635}$$

Rearranging Eq. (635),

$$P_o = E_{oa}(I_{ca} \cos \phi_6 - I_{ab} \cos \phi_1) + E_{ob}(I_{ab} \cos \phi_2 - I_{bc} \cos \phi_3) + E_{oc}(I_{bc} \cos \phi_4 - I_{ca} \cos \phi_5) \text{ watts.} \tag{636}$$

From an inspection of Fig. 195 and Eq. (636), it is seen that

$$I_{ca} \cos \phi_6 - I_{ab} \cos \phi_1 = I_a \cos \alpha'' \text{ amp} \tag{637}$$

when I_{ca}, $-I_{ab}$, and I_a are all projected on E_{oa} produced,

$$I_{ab} \cos \phi_2 - I_{bc} \cos \phi_3 = I_b \cos \beta'' \text{ amp} \tag{638}$$

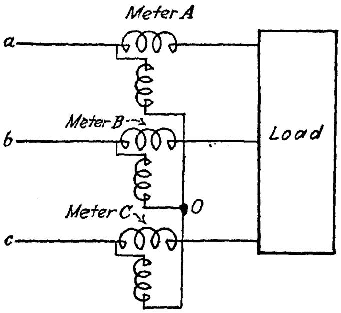
when I_{ab}, $-I_{bc}$, and I_b are all projected on E_{ob} produced, and

$$I_{bc} \cos \phi_4 - I_{ca} \cos \phi_5 = I_c \cos \gamma'' \text{ amp} \tag{639}$$

when I_{bc}, $-I_{ca}$, and I_c are all projected on E_{oc} produced. Hence,

$$P_o = E_{oa}I_a \cos \alpha'' + E_{ob}I_b \cos \beta'' + E_{oc}I_c \cos \gamma'' \text{ watts.} \quad (640)$$

It is seen that I_a, I_b, and I_c are the line wire currents due to the delta-connected loads and that E_{oa}, E_{ob}, and E_{oc} are the emfs from some common point within the triangle to the corners of the triangle *abc*. In other words, the power in any three-wire three-phase circuit may be measured by three wattmeters connected with the current coils in the three line wires *a*, *b*, and *c* and with their potential coils connected between a common point and the three line wires, respectively, as shown in Fig. 196. The meter coils may have like or unlike resistances, the point *O* being established as the point of equilibrium of the voltage drop across the potential-coil circuits, as explained under Y-connected loads in three-wire three-phase circuits in Chap. VIII. It is understood in this connection that the voltages and currents are of such values as can be used directly on a meter; for higher voltages and currents, potential and current transformers may have to be used. Each of the meters in this connection may not indicate the power used per phase, but the sum of their indications is the total power consumed by the three-wire three-phase circuit with any unbalancing of the currents and voltages.



106. Power Measurements in Four-wire Three-phase Circuits.—For the four-wire three-phase circuit, the common connection *O* of the meters must be connected to the neutral wire and the three current coils of the meters in the three outside wires, as shown in Fig. 197. In this case, the point *O* becomes the intersection of the medians of the voltage triangle *abc*, Fig. 198. Three meters must be used in the four-wire three-phase system since current may flow in any wire and return by the neutral independent of the other two wires when unbalanced single-phase loads are connected to the system. In this case, each of the meters reads the power consumed by the phase to which its current coil is connected. The sum of the meter readings is at all times the total power consumed by the loads beyond the meters.

107. Two-wattmeter Method for Power Measurements in Three-wire Three-phase Circuits.—To meter a three-wire three-phase load using only two wattmeters. move the point O

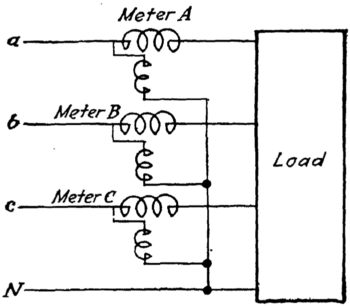


FIG. 197.—Three-wattmeter connection for measuring the power in a four-wire three-phase circuit.

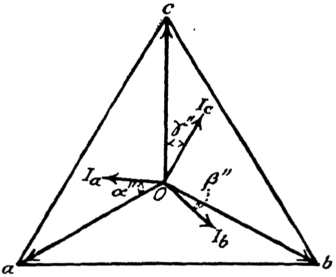


FIG. 198.—Vector diagram for Fig. 197.

to any line wire as c (Fig. 195). Since the preceding proof holds for the point O at any place within the triangle abc, it must hold for O moved to the point c. Meter C (Fig. 196) has its potential coil short circuited, and the meter may be removed since it can give no deflection. From the foregoing proof the sum of the readings of the other two meters must give the total power. It is seen that meters A and B have the full line potential across their potential-coil circuits, as shown in Fig. 199, and that the current coils carry the current in two of the line wires. Referring to Fig. 199 and to the vector diagram in Fig. 195, it is seen that the current in the current coil of meter A is I_a and the potential across the potential coil is E_{ca}, which is α degrees ahead of I_a, the meter reading being

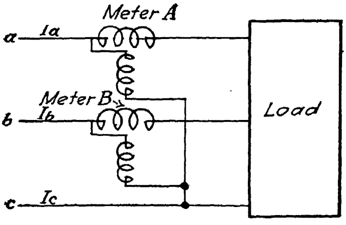


FIG. 199.—Two-wattmeter method of measuring the power in a three-wire three-phase circuit.

$$P_a = E_{ca}I_a \cos \alpha \text{ watts.} \tag{641}$$

In the same manner, the current in the current coil of meter B is I_b, and the potential-coil circuit has the emf E_{cb} across it β deg.

behind I_b, giving the reading

$$P_b = E_{cb}I_b \cos \beta \text{ watts.} \tag{642}$$

The total power, being the sum of the two meter readings, becomes

$$P_o = P_a + P_b = E_{ca}I_a \cos \alpha + E_{cb}I_b \cos \beta \text{ watts.} \tag{643}$$

In the above proof, unbalanced conditions of emf, current, and phase angles have been assumed. Hence, the two-wattmeter method for the measurement of power in a three-wire three-phase circuit may be used for unbalanced as well as for balanced conditions. In this method of metering the load neither wattmeter indicates the power for any particular load or phase, it being merely the sum of the indications of the two meters, which is important in determining the total power in the circuit.

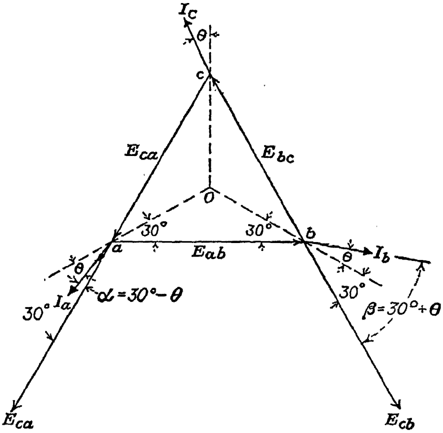


FIG. 200.—Vector diagram for Fig. 199, capacitive load.

108. Balanced Loads on Three-wire Three-phase Circuits.—In three-wire three-phase circuits with two meters, as illustrated in Fig. 199, the vector diagram will be as shown in Fig. 200 for a capacitive load and in Fig. 201 for an inductive load. In either case, the two meters will indicate

$$P_a = E_{ca}I_a \cos \alpha = E_{ca}I_a \cos (30° - \theta) \text{ watts} \tag{644}$$

$$= EI \cos (30° - \theta) \text{ watts,} \tag{645}$$

$$P_b = E_{cb}I_b \cos \beta = E_{cb}I_b \cos (30° + \theta) \text{ watts} \tag{646}$$

$$= EI \cos (30° + \theta) \text{ watts,} \tag{647}$$

where E is the emf between lines;
I is the line current; and
θ is the phase angle of the load, and is positive for capacitive loads and negative for inductive loads.

Expanding Eqs. (645) and (647),

$$P_a = EI \cos 30^\circ \cos \theta + EI \sin 30^\circ \sin \theta \text{ watts,} \tag{648}$$
$$P_b = EI \cos 30^\circ \cos \theta - EI \sin 30^\circ \sin \theta \text{ watts.} \tag{649}$$

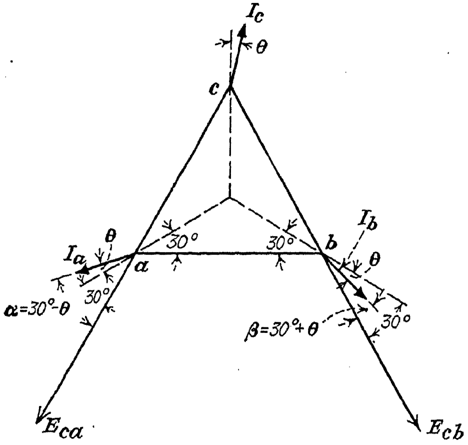


FIG. 201.—Vector diagram for Fig. 199, inductive load.

Adding Eqs. (648) and (649),

$$\begin{aligned} P_a + P_b &= 2EI \cos 30^\circ \cos \theta \text{ watts} \\ &= \sqrt{3}EI \cos \theta \text{ watts,} \end{aligned} \tag{599}$$

which was shown to be the expression for total power in a three-wire three-phase circuit in Art. 105. This same expression was also derived in Art. 99.

Since the two-wattmeter method is the practical one for metering power in three-wire three-phase systems, several special cases with balanced loads of different power factors will be considered.

Nonreactive load, $\theta =$

$$P_a = EI \cos (30^\circ) = \frac{\sqrt{3}}{2} EI \text{ watts,}$$

$$P_b = EI \cos (30^\circ) = \frac{\sqrt{3}}{2} EI \text{ watts,}$$

and

$$P_o = \sqrt{3} EI \cos \theta = 2\frac{\sqrt{3}}{2} EI = \sqrt{3} EI \text{ watts.}$$

The meters read alike, and each indicates one-half the total power.

Inductive load, $\theta = -30^\circ$:

$$P_a = EI \cos 60^\circ = \tfrac{1}{2} EI \text{ watts,}$$
$$P_b = EI \cos 0^\circ = EI \text{ watts,}$$

and

$$P_o = \tfrac{1}{2} EI + EI = \tfrac{3}{2} EI = \qquad\qquad \cos \theta \text{ watts,}$$

the total power in the circuit being

$$P_o = \sqrt{3} EI \cos (-30^\circ) = \qquad\qquad = \frac{3}{2} EI \text{ watts.}$$

It is seen that meter B indicates two-thirds the total power and meter A the other one-third. For 30-deg. lead this condition is reversed, meter A giving the greater indication.

Inductive load, $\theta = -60^\circ$:

$$P_a = EI \cos (30^\circ + 60^\circ) = 0 \text{ watts,}$$
$$P_b = EI \cos (30^\circ - 60^\circ) = \frac{\sqrt{3}}{2} EI \text{ watts,}$$
$$P_o = \frac{\sqrt{3}}{2} EI \text{ watts,}$$

but the total power in the circuit is

$$P_o = \qquad\qquad \cos (-60^\circ) = \frac{\sqrt{3}}{2} EI \text{ watts.}$$

In this case, meter B indicates the entire power, and meter A indicates zero although each meter has the same amount of current flowing in its current coil and the same value of emf impressed across its potential coil. For capacitive loads of 0.5 power factor, meter A indicates the entire load and meter B indicates zero.

Loads of Less Than 0.5 *Power Factor.*—For loads having power factors less than 0.5, one meter will indicate positive and the other negative, the algebraic sum of their indications being the total power.

From Eqs. (644) and (646) and the four special cases considered, it is seen that, while in every case the sum of the two meter readings gives the total power consumed, individual meter readings have no definite meaning since the division of the total power between the two meters varies with the values of the phase angle θ. It has been shown also that, when $\theta = \pm 60$ deg. so that $30° + \theta = 90°$, or $-30° - \theta = -90°$, one meter will indicate zero, and that for larger values of θ the numerical reading of one meter must be considered as negative and its reading subtracted from that of the other meter to obtain the total power.

Owing to this fact certain precautions must be taken when using the two-wattmeter method to measure the power in a three-wire three-phase circuit when the power factor of the circuit is not known. If the proper connection is known for the current and potential coils of the meters to obtain positive readings in a single-phase circuit and these connections are carefully followed, then a deflection of the needle over the scale means a positive reading. These connections, however, are not always known or carefully followed, and it becomes necessary to make some test to determine the proper sign to prefix to the readings of each meter. The simplest test is based on the fact already shown that, in any three-phase system, the line currents are displaced from the two adjacent line voltages by $(30° + \theta)$ and $(30° - \theta)$. Thus, in Figs. 200 and 201, I_b is displaced from E_{cb} by $(30° + \theta)$ and is displaced from E_{ab} by $(30° - \theta)$. Meter B (Fig. 199) indicates

$$P_b = EI \cos (30° + \theta) \text{ watts,} \qquad (647)$$

when connected as shown. If the potential coil end is moved from line c to line a, the meter will indicate

$$P_b' = EI \cos (30° - \theta) \text{ watts.} \qquad (645)$$

If θ is greater than 60 deg., these two readings will be in opposite directions or the needle will be deflected backward in one case. If θ is less than 60 deg., the two readings will have different magnitudes, but they will both have the same sign. The test is, therefore, to shift the potential coil of one meter from one line to the other so as to associate the line current with the two adjacent line voltages. If the meter indicates in the same direction, either positive or negative, in the two cases, the power factor is greater than 0.5 and both meters should be connected to indicate positively. If, however, the two readings are in opposite directions, then the power factor is less than 0.5 and the meter having the smaller reading must be connected to read positively, but a negative sign must be given to its numerical readings.

109. Power Factor from the Two-wattmeter Readings in a Balanced Three-wire Three-phase Circuit.—For balanced loads and two wattmeters, the power factor of the load may be determined from the wattmeter readings, without the use of the voltmeter or ammeter readings, with greater accuracy than by the use of the ratio of total power to total volt-amperes. If Eqs. (645) and (647) are expanded,

$$P_a = EI\,[\cos 30^\circ \cos\theta + \sin 30^\circ \sin\theta] \text{ watts,} \tag{648}$$

$$P_b = EI\,[\cos 30^\circ \cos\theta - \sin 30^\circ \sin\theta] \text{ watts.} \tag{649}$$

Hence,

$$\begin{aligned} P_a + P_b &= 2EI \cos 30^\circ \cos\theta \text{ watts} \\ &= \sqrt{3}EI \cos\theta \text{ watts;} \end{aligned} \tag{650}$$

also

$$\begin{aligned} P_a - P_b &= 2EI \sin 30^\circ \sin\theta \text{ vars} \\ &= EI \sin\theta \text{ vars.} \end{aligned} \tag{651}$$

Dividing Eq. (651) by Eq. (650),

$$\frac{P_a - P_b}{P_a + P_b} = \frac{EI \sin\theta}{\sqrt{3}EI \cos\theta} \tag{652}$$

or

$$\tan\theta = \sqrt{3}\frac{P_a - P_b}{P_a + P_b}, \tag{653}$$

i.e., the tangent of the angle of lead or lag is proportional to the ratio of the difference of the meter readings to the sum of the meter readings. From this relation, the power factor may be determined for balanced loads without knowing the voltage or the current in the circuit.

$$\text{Power factor} = \cos\tan^{-1}\sqrt{3}\frac{P_a - P_b}{P_a + P_b} \tag{654}$$

$$= \frac{P_a + P_b}{\sqrt{(P_a + P_b)^2 + 3(P_a - P_b)^2}}. \tag{655}$$

Owing to the great difference between the meter readings in the two-wattmeter method of measurement of power, two separate watt-hour meters are seldom used for metering three-phase energy. Instead, the two distinct electrical and magnetic systems operate on the same integrating system in a polyphase watt-hour meter, as explained in the consideration of two-phase metering. In this manner many questions, difficult to explain to a nontechnical customer, are easily avoided. To illustrate the two-wattmeter method consider the following examples.

Example 68

Assume a three-wire three-phase system (Example 63) with 100 volts between wires and with three resistances of 10 ohms each connected in delta between these lines. Connect the current coils of each of two wattmeters into lines B and C and with their potential coils connected to line A. Find the reading of each of these instruments and the total power consumed

$$\begin{aligned}\dot I_{B_{AB}} &= 17.3\underline{/30^\circ}\text{ amp}\\ &= 15 + j8.66\text{ amp.}\\ \dot I_{C_{AC}} &= 17.3\underline{/-30^\circ}\text{ amp}\\ &= 15 - j8.66\text{ amp.}\end{aligned}$$

The readings of the wattmeters will be

$$\begin{aligned}P_B &= 100 \times 15 = 1{,}500\text{ watts,}\\ P_C &= 100 \times 15 = 1{,}500\text{ watts,}\end{aligned}$$

the total power being

$$P_0 = P_B + P_C = 3{,}000\text{ watts.}$$

Example 69

Replace the three resistances in Example 68 with three impedances, each of value $\dot Z = 8 + j6 = 10\underline{/36^\circ 52'}$ ohms. Find the readings of each of the instruments and the total power consumed by the load.

$$\begin{aligned}\dot I_{B_{AB}} &= 17.3\underline{/-36^\circ 52'}\underline{/30^\circ}\text{ amp}\\ &= 17.3\underline{/-6^\circ 52'} = 17.2 - j2.08\text{ amp,}\\ \dot I_{C_{AC}} &= 17.3\underline{/-36^\circ 52'}\underline{/-30^\circ}\text{ amp}\\ &= 17.3\underline{/-66^\circ 52'} = 6.8 - j15.93\text{ amp.}\end{aligned}$$

The readings of each of the instruments will be

$$\begin{aligned}P_B &= 100 \times 17.2 = 1{,}720\text{ watts,}\\ P_C &= 100 \times 6.8 = 680\text{ watts,}\end{aligned}$$

and

$$P_0 = 1{,}720 + 680 = 2{,}400\text{ watts,}$$

also

$$P_0 = \sqrt{3} \times 100 \times 17.3 \times 0.80 = 2{,}400\text{ watts.}$$

Example 70

Consider three load impedances $\dot{Z}_{BA} = 8 + j6$, $\dot{Z}_{BC} = 8 - j6$, and $\dot{Z}_{CA} = 10 + j0$ connected to the lines of Example 69 and Example 66. Place the current coils of each of two wattmeters in the lines A and C and connect their potential coils to line B. Find the reading of each instrument and the total power taken by the loads.

From Example 66, the currents are

$$\dot{I}_{C_{AC}} = 19.2 - j3.93 \text{ amp};$$

then

$$\begin{aligned} \dot{I}_{C_{BC}} &= (19.2 - j3.93)\underline{/60^\circ} \text{ amp} \\ &= 13.0 + j14.74 \text{ amp}, \\ \dot{I}_{A_{BA}} &= 13.0 - j14.66 \text{ amp}. \end{aligned}$$

The meters will indicate

$$\begin{aligned} P_C &= 100 \times 13 = 1{,}300 \text{ watts}, \\ P_A &= 100 \times 13 = 1{,}300 \text{ watts}, \end{aligned}$$

giving the total power used of

$$P_0 = P_C + P_A = 2{,}600 \text{ watts}.$$

It is accidental that these instruments indicate the same value when the loads are unbalanced in this manner.

Example 71

Find the readings of each of two wattmeters used to measure the power input to an induction motor taking 100 kw at 0.80 power factor.

$$P_O = \sqrt{3}EI \cos\theta = 100 \text{ kw},$$

$$EI = \frac{100}{\sqrt{3} \times 0.80} = 72.3 \text{ kva},$$

hence,

$$\begin{aligned} P_1 &= EI\,[\cos 30^\circ \cos\theta - \sin 30^\circ \sin\theta] \text{ kw}, \\ P_2 &= EI\,[\cos 30^\circ \cos\theta + \sin 30^\circ \sin\theta] \text{ kw}, \end{aligned}$$

or

$$\begin{aligned} P_1 &= 72.3[(0.866 \times 0.8) - (0.5 \times (-0.6))] = 71.5 \text{ kw}, \\ P_2 &= 72.3[(0.866 \times 0.8) + (0.5 \times (-0.6))] = 28.5 \text{ kw}, \end{aligned}$$

and

$$P_0 = 71.5 + 28.5 = 100 \text{ kw}.$$

110. One-wattmeter Methods in Balanced Three-wire Three-phase Circuits.—In three-phase circuits, where only approximate values of the total power are desired, a reduction in the cost of instruments and a saving in space on the switchboard may sometimes be made by the use of one of the common one-wattmeter methods of connection. The first of these methods involves the use of a "Y box." The Y box consists of two resistances each equal to the resistance of the potential coil circuit of the wattmeter to be used. These two resistances and the potential coil of the wattmeter are connected in Y to the three line wires, thus establishing an artificial neutral point for the system, and the current coil of the wattmeter is placed in the line wire to which its potential coil is connected. The wattmeter indicates one-third of the total power consumed and the scale may be calibrated to give the total power, or a factor of three may be used with a standard meter scale.

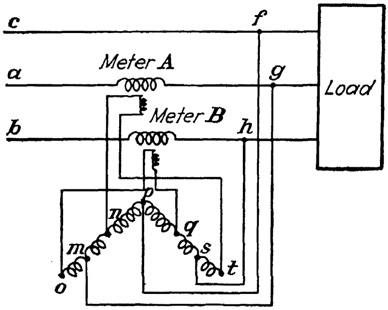


FIG. 202.—Two wattmeters connected to measure the reactive power in a three-wire three-phase circuit.

Another one-wattmeter method is obtained by connecting the current coil of a single-phase wattmeter in one line of the three-phase circuit and one end of the potential coil of the wattmeter to that line. The other end of the potential coil is connected first to one of the adjacent line wires of the circuit and then to the other, readings being taken for each connection. The sum of the two readings gives the total power on a balanced circuit. This is, in effect, a modification of the two-wattmeter method, and the same equations apply as were derived for that method. The power factor may be determined from the relation of the sum and difference of these readings as was explained in Art. 109.

111. Two-wattmeter Method of Measurement of Reactive Power in Three-wire Three-phase Circuits.—The reactive power in a three-wire three-phase circuit may be measured by using two meters and two phasing transformers. The two *phasing* *transformers* are used to shift the phases of the two potentials applied to the potential coils of the meters and to change the magnitudes of these potentials so that the meter readings may be used directly without the use of multiplying factors. The connections are shown in Fig. 202 and the corresponding vector diagram in Fig. 203. The *phasing transformers* consist of two *autotransformers* op and tp, Fig. 202. The points m, p, and s are connected across the lines, as shown. The potential coil of meter A is connected to n and t, and that of meter B to q and o. The emfs E_{po} and E_{pt} are $2/\sqrt{3}$ times the line emfs E_{pm} and E_{ps}.

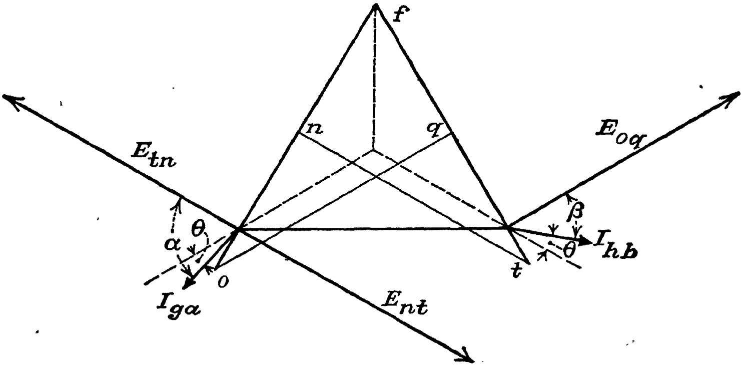


FIG. 203.—Vector diagram for Fig. 202.

The point n is tapped at the mid-point of po and m at the mid-point of pt, so that for balanced loads of unity power factor the voltage applied to the potential coil of each meter is 90 deg. out of phase with the current in its respective current coil.

The emf E_{nt} is

$$E_{nt} = 0.866E_t = \frac{2}{\sqrt{3}}E_t \text{ volts}$$

$$= \frac{\sqrt{3}}{2} \times \frac{2}{\sqrt{3}}E = E \text{ volts.} \tag{656}$$

The indication of meter A is the product of the potential and current applied to the coils multiplied by the cosine of the angle between them, or

$$Q_a = E_{tn}I_{ga} \cos \alpha \text{ vars} \tag{657}$$

$$= EI \cos (60° - \theta) \text{ vars,} \tag{658}$$

where θ is negative for inductive loads. The indication of meter B is

$$Q_b = E_{oq}I_{hb}\cos\beta \text{ vars} \tag{659}$$

$$= EI\cos(60^\circ + \theta) \text{ vars.} \tag{660}$$

Expanding Eqs. (658) and (660),

$$Q_a = EI\cos 60^\circ \cos\theta + EI\sin 60^\circ \sin\theta \text{ vars,} \tag{661}$$

$$Q_b = EI\cos 60^\circ \cos\theta - EI\sin 60^\circ \sin\theta \text{ vars.} \tag{662}$$

Subtracting Eq. (662) from Eq. (661),

$$Q_a - Q_b = 2EI\sin 60^\circ \sin\theta \text{ vars}$$

$$= \sqrt{3}EI\sin\theta \text{ vars.} \tag{610}$$

This was shown to be the total reactive power in a three-phase system with a balanced load in Art. 100, Chap. VIII, so that the difference in the readings of the two meters, as connected, is the reactive power in a balanced three-phase system. The reactive power thus measured is correct for any three-phase system in which the voltages are balanced, even if the currents are unsymmetrical. If both the currents and the voltages are unbalanced, the correct value of reactive power will not be indicated, the error being increased with the degree of unbalance.

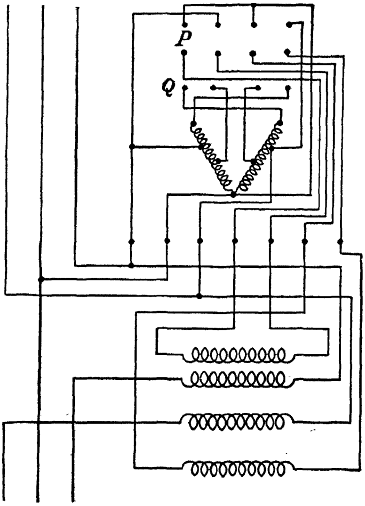


FIG. 204.—Connection diagram for a polyphase wattmeter connected so as to indicate active or reactive power.

The meter elements are usually combined into one meter to form a polyphase reactive-power meter. In this case, the connection from one of the potential coils is reversed so that the meter torque will be subtractive, which is equivalent to subtracting the individual meter readings. Instruments are made commercially that will give either power or reactive power by closing a reversing switch in either of two directions. The connection diagram for such an instrument is shown in Fig. 204. From this instrument the active power, reactive power, and power factor of the circuit are quickly obtainable.

Example 72

A polyphase meter is used to obtain the reading of active and reactive power in a three-phase 230-volt circuit supplying an induction-motor load. If the active-power reading is found to be 296 kw and the reactive power 187 kvar, find the vector power, the current flowing in the circuit, and the power factor of the load.

The vector power is

$$V = \sqrt{P^2 + Q^2} = \sqrt{296^2 + 187^2} = 350 \text{ vector kva.}$$

$$P.F. = \frac{296}{350} \times 100 = 84.6 \text{ per cent,}$$

$$I = \frac{350{,}000}{\sqrt{3} \times 230} = 879 \text{ amp.}$$

112. Calculation of Wattmeter Readings for Unbalanced Loads in a Three-wire Three-phase Circuit.—In determining the readings of each of two wattmeters used to indicate the power consumed by an unbalanced load on a three-wire three-phase circuit, two general methods may be used. The first method consists in the determination of the total current in each meter coil and its phase relation with the potential coil voltage, as was done in Example 70. The second method makes use of the theorem of superposition by finding the indication of each meter separately, first for the balanced portion of the load, and then for the single-phase unbalanced portion. The total indication is the algebraic sum of these partial readings. This method of superposition is used in the polyphase wattmeter and watt-hour meter and in the totalizing watt-hour meters where the torque exerted by each separate load or current is used to actuate the meters. This second method will be illustrated by the use of several examples.

Example 73

An induction motor draws 300 kw at 0.80 power factor inductive from the lines *ABC* of a 220-volt three-wire three-phase circuit. A lighting load of 50 kw at unity power factor is taken from lines *B* and *C*. Two wattmeters are connected with current coils in wires *B* and *C* and with potential coils connected from *B* to *A* and from *C* to *A*, respectively; find the indication of each meter and the total power.

For the motor load,

$$P_m = \sqrt{3}EI \cos\theta = 300 \text{ kw, and}$$
$$EI = \frac{300}{\sqrt{3} \times 0.80} = 216.5 \text{ kva.}$$

Hence, for the motor alone,

$$P_{C1} = EI[\cos 30° \cos\theta + \sin 30° \sin\theta] \text{ kw,}$$
$$P_{B1} = EI[\cos 30° \cos\theta - \sin 30° \sin\theta] \text{ kw,}$$

or

$$\begin{aligned} P_{C1} &= 216.5[(0.866 \times 0.80) + (0.50)(-0.60)] \\ &= 85.5 \text{ kw,} \\ P_{B1} &= 216.5[(0.866 \times 0.80) - (0.50)(-0.60)] \\ &= 214.5 \text{ kw.} \end{aligned}$$

For the lighting load alone, the current taken will be in phase with E_{CB} and with the value

$$I_C = I_B = \frac{50 \times 1{,}000}{220} = 227.2 \text{ amp.}$$

This current will lead E_{AB} by 60 deg. and lag E_{AC} by 60 deg., so that for this load

$$P_{C2} = 227.2 \times 220 \cos 60° = 25 \text{ kw,}$$
$$P_{B2} = 227.2 \times 220 \cos 60° = 25 \text{ kw.}$$

The total reading of each meter with both loads will be

$$\begin{aligned} P_C &= P_{C1} + P_{C2} = 85.5 + 25 = 110.5 \text{ kw,} \\ P_B &= P_{B1} + P_{B2} = 214.5 + 25 = 239.5 \text{ kw,} \\ P_0 &= P_C + P_B = 110.5 + 239.5 = 350 \text{ kw, or} \\ P_0 &= 300 + 50 = 350 \text{ kw.} \end{aligned}$$

Example 74

Assume that the lighting load had been connected between A and C instead of B and C in Example 73. Find the indications of each of the meters and the total power.

The motor portion of the meter indications will not be changed. For the lighting load, $I_B = 0$, $I_C = 227.2$ amp. in phase with E_{AC}, so that

$$P_{B2} = 0 \text{ kw,}$$
$$P_{C2} = \frac{220 \times 227.2}{1{,}000} = 50 \text{ kw.}$$

Then

$$\begin{aligned} P_B &= 214.5 \text{ kw,} \\ P_C &= 85.5 + 50 = 135.5 \text{ kw,} \\ P_0 &= 214.5 + 135.5 = 350 \text{ kw.} \end{aligned}$$

Example 75

Assume three single-phase loads connected in delta to the lines *A*, *B*, and *C* of Example 73. Let $\dot{I}_{AB_{AB}} = 10 + j0$ amp, $\dot{I}_{BC_{BC}} = 8 - j6$ amp, and $\dot{I}_{CA_{CA}} = 8 + j6$ amp. With meters connected as in Example 73, find the indication of each meter.

For meter in line *B* there are two currents and one potential, the reference being E_{AB}. For the current $\dot{I}_{AB_{AB}} = 10 + j0$ amp, the reading of the meter in line *B* will be

$$P_{B1} = 220 \times 10 = 2{,}200 \text{ watts.}$$

For the current in the load *CB*,

$$\dot{I}_{CB_{CB}} = 8 - j6 = 10\underline{/-36^\circ 52'} \text{ amp.}$$

Referring this current to E_{AB},

$$\begin{aligned}\dot{I}_{CB_{AB}} &= 10\underline{/-36^\circ 52'}\,\underline{/60^\circ} \text{ amp} \\ &= 10\underline{/23^\circ 8'} = 9.2 + j3.93 \text{ amp,}\end{aligned}$$

and the reading of the meter for this current would be

$$P_{B2} = 220 \times 9.2 = 2{,}025 \text{ watts.}$$

With both currents flowing,

$$P_{B0} = P_{B1} + P_{B2} = 2{,}200 + 2{,}025 = 4{,}225 \text{ watts.}$$

For the meter in line *C* and current for load *CA* and potential E_{AC},

$$\dot{I}_{AC_{AC}} = 8 + j6 \text{ amp,}$$

and for this current,

$$P_{C1} = 220 \times 8 = 1{,}760 \text{ watts.}$$

For the current in load *BC*,

$$\dot{I}_{BC_{BC}} = 8 - j6 = 10\underline{/-36^\circ 52'} \text{ amp.}$$

Referring this current to E_{AC},

$$\begin{aligned}\dot{I}_{BC_{AC}} &= 10\underline{/-36^\circ 52'}\,\underline{/-60^\circ} \text{ amp} \\ &= 10\underline{/-96^\circ 52'} = -1.2 + j9.93 \text{ amp,}\end{aligned}$$

and the reading of the meter for this current alone would be

$$P_{C2} = -1.2 \times 220 = -265 \text{ watts.}$$

With both currents flowing,

$$P_{C0} = 1{,}760 - 265 = 1{,}495 \text{ watts.}$$

The total power used by the circuit is

$$P_0 = P_{B0} + P_{C0} = 4{,}225 + 1{,}495 = 5{,}720 \text{ watts.}$$

As a check, the total power used by the three loads is

$$\begin{aligned} P_{AB} &= 220 \times 10 = 2{,}200 \text{ watts,} \\ P_{BC} &= 220 \times 8 = 1{,}760 \text{ watts,} \\ P_{CA} &= 220 \times 8 = 1{,}760 \text{ watts,} \end{aligned}$$

and

$$2{,}200 + 1{,}760 + 1{,}760 = 5{,}720 \text{ watts.}$$

It should be noted that meter B would indicate positive with either current alone whereas meter C would indicate positive for current I_{AC} and negative for current I_{BC} alone.

Problems

1-9. Each of two wattmeters used to measure the power delivered to an inductive load in a 220-volt four-wire, two-phase system indicates 25 kw. The line current is 131 amp. What are the active, reactive and vector power in each phase? What are the total active, reactive, and vector power delivered to the load.

2-9. The following balanced loads are connected to a 2,200-volt four-wire two-phase system: (*a*) a lighting load of 200 kw; (*b*) an induction-motor load having an input of 400 kw at 0.70 power factor; (*c*) a synchronous-motor load of 200 kw input at 0.90 power factor leading. What are the active, reactive, and vector power delivered to these loads? What is the current taken by each load? What is the total current? What would be the reading of each of two wattmeters used to measure the total load? What is the power factor of the total load?

3-9. The following balanced loads are connected to a four-wire two-phase system with 33,000 volts across phases: (*a*) a lamp load of 1,000 kw; (*b*) an induction-motor load of 1,200 kva input at 0.85 power factor; (*c*) a synchronous-motor load of 400 kw input at 0.40 power factor leading. What are the total active, reactive, and vector power? Find the indication of each of two wattmeters used to measure the total load.

4-9. A balanced load of 500 kva at 0.87 power factor is carried by a 440-volt three-wire two-phase system. What is the current in each of the three wires? What is the reading of each of two wattmeters used to measure the power delivered to the load?

5-9. A five-wire two-phase system with 110 volts to neutral carries a balanced load of 39.4 kva at 0.74 power factor. What is the indication of each of the four wattmeters used to measure the power delivered to this load? What is the current in each line wire?

6-9. A 4,000-volt four-wire three-phase feeder carries a balanced lighting load of 400 kva and a balanced induction-motor load of 500 kw input at 0.85 power factor. What is the total power taken by the combined loads? What is the power factor? What is the line current? What is the reading of each of the three wattmeters used to measure this power?

7-9. Two single-phase wattmeters are connected to indicate the power taken by an induction motor drawing a balanced load of 400 kva at 0.45 power factor inductive from a 220-volt three-phase line. What is the reading of each meter?

8-9. A four-wire three-phase system carries a balanced induction-motor load of 800 kva at 0.75 power factor, a balanced synchronous-motor load of 700 kva input at 0.75 power factor, current leading, and a single-phase lighting load of 100 kw from one line to neutral. Draw a circuit diagram showing the wattmeters to be used to indicate the total load, and give the reading of each meter.

9-9. A 230-volt three-wire three-phase system carries a balanced lighting load of 25 kw and a balanced induction-motor load of 50 kw input at 0.50 power factor. What are the total active, reactive and apparent power? What is the total current in each line wire? If the power is measured by the two-wattmeter method, what is the reading of each meter?

10-9. A three-phase balanced delta-connected load is taking 100 kw at 0.707 power factor inductive at 440 volts. Find the line currents and the current in each phase of the load. Find the reading of each of two wattmeters used to indicate the power taken by the load.

11-9. Two wattmeters are used to indicate the load on a 2,300-volt three-wire three-phase system supplying a balanced lighting load of 200 kw and a balanced induction-motor load of 250 kva input at 0.60 power factor. What is the reading of each meter? What is the line current?

12-9. A balanced three-phase delta-connected load is taking 2,000 kw at 0.70 power factor inductive at 2,200 volts. Find the current in each phase of the load and the reading of each of two wattmeters used to indicate the power. Find the reactive and apparent power.

13-9. A 200-volt three-wire three-phase balanced system has its power input measured by two wattmeters. One meter indicates 73 kw and the other 26 kw; what are the active, reactive, and apparent power delivered to the load? What is the line current?

14-9. A 44,000-volt three-wire three-phase balanced system has its power input measured by two wattmeters. One meter indicates 3,000 kw and the other 6,000 kw. What are the active, reactive, and apparent power delivered to the load? What is the line current?

15-9. A 110-volt three-wire three-phase system with a balanced load has its power input measured by the two-wattmeter method. The reading of one wattmeter is 36 kw and of the other −19 kw. What are the active and reactive power delivered to the load? What is the power factor? What is the line current?

16-9. Two wattmeters connected to indicate the power of a balanced three-phase load indicate 1,500 and −375 kw, respectively. What are the total power input and the power factor of the load?

17-9. The active and reactive power delivered from a 110-000-volt three-phase transmission line are measured by a polyphase wattmeter and reactive meter. The wattmeter reading is 37,000 kw, and the reactive meter reading is 31,500 kvar inductive. What is the line current? What is the power factor of the load on the transmission line?

18-9. The meters on two 66,000-volt three-phase transmission lines delivering power from a station indicate as follows: wattmeter on line *A*, 18,000 kw; reactive meter on line *A*, 13,400 kvar inductive; wattmeter on line *B*, 23,000 kw; reactive meter on line *B*, 7,600 kvar inductive. The power comes into the station on a 110,000-volt three-phase line. What are the active, reactive, and apparent power delivered by the 110,000-volt line? What is the current in the line wire of each transmission line?

19-9. Three impedances each represented by $\dot{Z} = 10 + j17.32$ are connected in delta to a 1,100-volt three-phase system, and two wattmeters are used to indicate the power; what is the reading of each meter? The fuse blows in the line to which the potential coils of the meters are connected between the point of connection and the loads. What is now the reading of each meter?

20-9. A wattmeter is connected into a three-wire two-phase system with its current coil in the common return and with its potential coil across the outside wires. The emfs from line to middle wire are each 220 volts, and the load impedances are represented by $\dot{Z} = 10 + j10$ ohms. Find the current in each of the three wires and the reading of the wattmeter.

21-9. Two wattmeters are used to indicate the power on a 2,300-volt three-wire three-phase system supplying a balanced lighting load of 100 kw and a balanced induction-motor load of 250 kva input at 0.70 power factor. What is the reading of each meter?

22-9. A 22,000-volt three-wire three-phase system with a balanced load has its power input measured by two wattmeters. The indication of one meter is 5,000 kw and of the other is 2,500 kw. What is the power factor of the load? What are the active, reactive, and apparent power of the load? What is the line current?

23-9. Three unequal impedances, $\dot{Z}_{OA} = 5 + j6$, $\dot{Z}_{OB} = 2 - j7$, and $\dot{Z}_{OC} = 10 + j0$, are connected in star to a balanced three-wire three-phase 440-volt system whose phase sequence is E_{AB}, E_{BC}, E_{CA}. (*a*) Find the voltages E_{OA}, E_{OB}, and E_{OC}, the line currents, and the total power. (*b*) Three wattmeters, having potential coils of equal resistance, are connected with their current coils in lines *A*, *B*, and *C* and their potential coils in *Y*; what is the indication of each meter? (*c*) The common point of connection of the potential coils is moved to *C*; what is the reading of each meter?

24-9. In a three-wire three-phase system, the phase sequence is E_{BC}, E_{AB}, E_{CA}. The line voltages are 2,300 volts. The load on *AB* is a capacitive load of 40 kva at 0.90 power factor; the load on *BC* is an inductive load of 20 kva at 0.85 power factor; and the load on *AC* is a lighting load of 15 kw. Show how to connect two wattmeters into the circuit correctly to indicate the total power, and give the reading of each of these meters.

25-9. A 22,000-volt three-wire three-phase system has a balanced motor load of 400 kva at 0.70 power factor inductive across the line wires *A*, *B*, and *C* and a lighting load of 200 kw input connected between lines *A* and *B*. What is the reading of each of the two wattmeters used to indicate the total power if their current coils are connected in lines *B* and *C*?

CHAPTER X

CONSIDERATION OF UNBALANCED POLYPHASE CIRCUITS BY MEANS OF SYMMETRICAL PHASE COMPONENTS

A very convenient method of calculating the currents flowing in polyphase systems having unbalanced loads, especially in such extreme cases as unsymmetrical short circuits and grounds, is by the use of symmetrical phase components. Since the field of application of this method has been constantly increasing, a brief explanation of its fundamental principles will be given here, and the method will be applied to the solution of simple unbalanced loads on three-phase circuits.

113. Degrees of Freedom for a System of Vectors.—By degrees of freedom in any system of forces is meant the number of independent ways in which a quantity may vary or the number of independent parameters required completely to specify the quantity. Thus, a scalar quantity can vary only in magnitude and has, therefore, but 1 degree of freedom. A vector within a plane can vary in magnitude and in direction within that plane and has, therefore, 2 degrees of freedom. A vector in space can vary in magnitude and also in direction from two planes perpendicular to each other and has 3 degrees of freedom.

A system of three vectors in a plane can be described only by specifying each vector separately, requiring six parameters, and the system has, in the general case, 6 degrees of freedom. If the system is symmetrical, however, the six parameters are not independent; the nature of symmetry expressing the relation between the various parameters. A system of three symmetrical vectors has, consequently, but 2 degrees of freedom. This suggests the possibility of defining any system of three coplanar unsymmetrical vectors, having 6 degrees of freedom, in terms of three coplanar symmetrical systems of three vectors, each having 2 degrees of freedom.

114. Synthesis of a System of Vector Components to Form Balanced or Unbalanced Three-phase Vector Systems.—It has been shown by C. L. Fortescue[1] that any system of unbalanced unsymmetrical three-phase currents, or voltages, may be represented by three symmetrical-component systems: one having the same phase sequence, or order, as the original system, one having the opposite, or reversed, phase sequence, and one having a zero phase sequence. A balanced system of vector components having the same phase sequence as the original system of vectors is shown in Fig. 205. This is designated as the first order, or the *positive-sequence*, system of components, and it is composed of the three vectors I_{A1}, I_{B1}, and I_{C1} which are equal and at 120 deg. with each other, since the system is balanced. A second balanced system of component vectors with the reversed sequence is shown in Fig. 206, these vectors I_{A2}, I_{B2}, and I_{C2} being equal, at 120 deg. with each other, and designated as the second order, or *negative sequence*, of vectors. In Fig. 207 is shown a third system of vectors I_{A0}, I_{B0}, and I_{C0} which are identical in magnitude and in phase position and which form a *zero-sequence* system of component vectors.

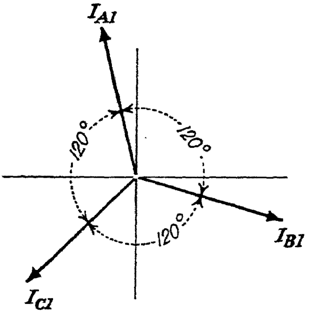


FIG. 205.—Balanced vector system forming positive-sequence components.

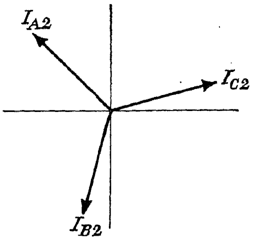


FIG. 206.—Balanced vector system forming negative-sequence components.

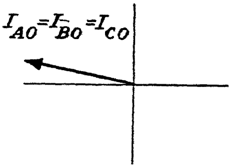


FIG. 207.—System of zero-sequence components.

Symmetrical Components of a Polyphase Set of Potential Differences.—The symmetrical components of an unsymmetrical polyphase set of n sinusoidal potential differences are the n sym-

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[1] Method of Symmetrical Coordinates Applied to the Solution of Polyphase Networks, *Trans. A.I.E.E.*, Vol. 37, 1918, p. 1047.

metrical polyphase sets into which the unsymmetrical set may be uniquely resolved, each component set consisting of n potential differences and each having a characteristic angular phase difference that increases from set to set by $2\pi/n$ radians. The components are called the first, or positive-sequence, component; the second component; the kth component; and the nth, or zero-sequence, component. This method of resolution is also applicable to polyphase sets of currents and to sets of impedances and admittances in a polyphase circuit when treated as vectors.

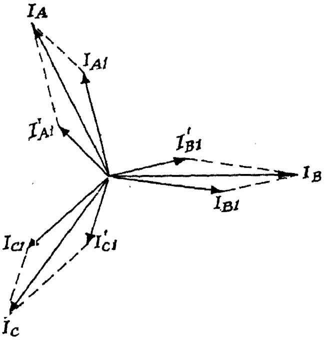

FIG. 208.—Combination of the currents in two balanced three-phase systems having the same phase sequence.

If two balanced systems of component vectors of the same phase sequence are combined, the resultant system is also balanced, as shown in Fig. 208, where a balanced-component system I_{A1}, I_{B1}, I_{C1} is added to a second balanced-component system I_{A1}', I_{B1}', I_{C1}', to form a third system I_A, I_B, I_C. It will be noted that the new system of vectors is symmetrical and balanced like each of the two component systems but that the new vectors are in general different in length and phase position from the original systems. If two balanced systems of components of opposite phase sequence are combined, an unbalanced system of vectors results, the degree of unbalance depending upon the relative magnitudes and phase positions of the two component systems. An addition of this kind is shown in Fig. 209, where the balanced system I_{A1}, I_{B1}, I_{C1} is added to the balanced system of opposite sequence I_{A2}, I_{B2}, I_{C2} to form the new unbalanced system I_A, I_B, I_C. It is to be noted that the sum of the vectors of the new unsymmetrical system so formed is always equal to zero, since the sum of each of the balanced com-

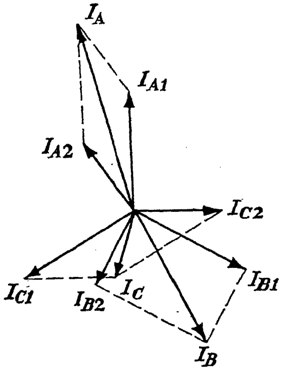

FIG. 209.—Combination of the currents in two balanced three-phase systems having opposite phase sequence.

ponent systems is equal to zero. If, however, the three systems of vector components as shown in Figs. 205, 206, and 207, are combined by addition, as shown in Fig. 210, the resultant vector system is unbalanced and the sum of the resulting vectors is not equal to zero.

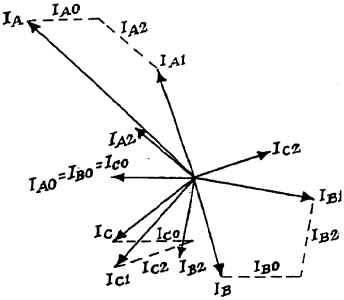

FIG. 210.—Currents in the three phases of an unbalanced three-phase system made up of positive-, negative-, and zero-sequence components.

Expressing these relations analytically the following equations result:

$$\dot{I}_A = \dot{I}_{A0} + \dot{I}_{A1} + \dot{I}_{A2} \text{ amp}, \tag{663}$$
$$\dot{I}_B = \dot{I}_{B0} + \dot{I}_{B1} + \dot{I}_{B2} \text{ amp}, \tag{664}$$
$$\dot{I}_C = \dot{I}_{C0} + \dot{I}_{C1} + \dot{I}_{C2} \text{ amp}. \tag{665}$$

Since all the vectors in a given sequence are of the same magnitude and they are related by angles that are multiples of 120 deg., they may be expressed as follows:

$$\dot{I}_{B0} = \dot{I}_{A0} = \dot{I}_{C0} \text{ amp}, \tag{666}$$
$$\dot{I}_{B1} = \dot{I}_{A1}\underline{/-120^\circ} \text{ amp}, \tag{667}$$
$$\dot{I}_{C1} = \dot{I}_{A1}\underline{/120^\circ} \text{ amp}, \tag{668}$$
$$\dot{I}_{B2} = \dot{I}_{A2}\underline{/120^\circ} \text{ amp}, \tag{669}$$
$$\dot{I}_{C2} = \dot{I}_{A2}\underline{/-120^\circ} \text{ amp}. \tag{670}$$

For convenience let

$$a = 1\underline{/120^\circ} = -0.50 + j0.866 \tag{671}$$

represent a unit operator producing a shift of 120 deg. in a counterclockwise direction. Then,

$$a^2 = 1\underline{/-120^\circ} = -0.50 - j0.866, \tag{672}$$
$$a^3 = 1\underline{/360^\circ} = 1.00 + j0, \tag{673}$$

and

$$a^4 = 1\underline{/120^\circ} = a. \tag{674}$$

Using this operator, Eqs. (666) to (670) may be rewritten

$$\dot{I}_{B1} = a^2\dot{I}_{A1} \text{ amp}, \tag{675}$$
$$\dot{I}_{B2} = a\dot{I}_{A2} \text{ amp}, \tag{676}$$

and the components of I_C in terms of I_A are

$$\dot{I}_{C1} = a\dot{I}_{A1} \text{ amp,} \tag{677}$$
$$\dot{I}_{C2} = a^2\dot{I}_{A2} \text{ amp.} \tag{678}$$

Then it follows that Eqs. (663) to (665) may be written

$$\dot{I}_A = \dot{I}_{A0} + \dot{I}_{A1} + \dot{I}_{A2} \text{ amp,} \tag{663}$$
$$\dot{I}_B = \dot{I}_{A0} + a^2\dot{I}_{A1} + a\dot{I}_{A2} \text{ amp,} \tag{679}$$
$$\dot{I}_C = \dot{I}_{A0} + a\dot{I}_{A1} + a^2\dot{I}_{A2} \text{ amp.} \tag{680}$$

115. Analytical Determination of the Symmetrical Phase Components.—In Art. 114, it has been shown that the three systems of symmetrical phase components may be combined to give systems of unsymmetrical vectors. In Eqs. (663) to (665), the three vectors on the left of the signs of equality are expressed in terms of nine related vectors on their right. Of these nine vectors the groups of three in the horizontal rows add to form the original vectors and the groups of three in the vertical columns form the balanced-vector component systems, the first column being the zero-sequence system, the second the positive-sequence system, and the third the negative-sequence system. These are again associated in terms of three vectors in Eqs. (663), (679), and (680).

Add the three equations (663), (679), and (680). It should be noted that the sum of the three vectors in each of the second and third columns will be zero, since the vectors are always equal and at 120 degrees from each other. Then

$$\dot{I}_A + \dot{I}_B + \dot{I}_C = 3\dot{I}_{A0} \text{ amp,} \tag{681}$$

or

$$\dot{I}_{A0} = \genfrac{}{}{0pt}{}{\dot{I}_A + \dot{I}_B + \dot{I}_C}{} \text{ amp,} \tag{682}$$

or the zero sequence vector is one third the sum of the three original vectors.

Multiply Eq. (679) by a and Eq. (680) by a^2, and assemble with Eq. (663) to give

$$\dot{I}_A = \dot{I}_{A0} + \dot{I}_{A1} + \dot{I}_{A2} \text{ amp,} \tag{683}$$
$$a\dot{I}_B = a\dot{I}_{A0} + \dot{I}_{A1} + a^2\dot{I}_{A2} \text{ amp,} \tag{684}$$
$$a^2\dot{I}_C = a^2\dot{I}_{A0} + \dot{I}_{A1} + a\dot{I}_{A2} \text{ amp.} \tag{685}$$

It is seen that the first and third columns of the second members form two vector systems, each of equal vectors and at 120 deg., so that they will add to zero. Although the vectors in the first member, or column on the left, appear to be at 120 deg., the original magnitudes were different, and they were neither in phase nor at 120 deg. before the multiplication by the unit vectors. Adding these three equations will determine I_{A1} as

$$\dot{I}_{A1} = \frac{\dot{I}_A + a\dot{I}_B + a^2\dot{I}_C}{3} \text{ amp.} \tag{686}$$

Multiply Eq. (679) by a^2 and Eq. (680) by a, and assemble with Eq. (663) to give

$$\dot{I}_A = \dot{I}_{A0} + \dot{I}_{A1} + \dot{I}_{A2} \text{ amp,} \tag{663}$$
$$a^2\dot{I}_B = a^2\dot{I}_{A0} + a\dot{I}_{A1} + \dot{I}_{A2} \text{ amp,} \tag{687}$$
$$a\dot{I}_C = a\dot{I}_{A0} + a^2\dot{I}_{A1} + \dot{I}_{A2} \text{ amp.} \tag{688}$$

Adding these equations and solving for I_{A2} gives

$$\dot{I}_{A2} = \frac{\dot{I}_A + a^2\dot{I}_B + a\dot{I}_C}{3} \text{ amp.} \tag{689}$$

By means of Eqs. (682), (686), and (689) the three components of the vector I_A may be determined, from which, in turn, the complete positive-, negative-, and zero-sequence component systems may be derived by the use of Eqs. (666) to (670).

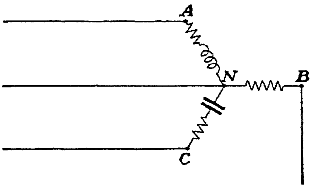

FIG. 211.—Three unbalanced loads connected in Y to a symmetrical four-wire three-phase system, Example 76.

Example 76

In the four-wire three-phase system shown in Fig. 211, the load currents are $\dot{I}_{AN} = 60 + j0$, $\dot{I}_{BN} = 45 - j75$, and $\dot{I}_{CN} = -21 + j120$. What are the symmetrical components of the currents?

From the statement of the problem

$$I_{AN} = 60 + j0 = 60\underline{/0^\circ} \text{ amp}$$
$$I_{BN} = 45 - j75 = 87.3\underline{/-59^\circ} \text{ amp}$$
$$I_{CN} = -21 + j120 = 121.8\underline{/100^\circ} \text{ amp.}$$

The zero-sequence component is

$$\dot{I}_{AN0} = \frac{60 + j0 + 45 - j75 - 21 + j120}{3} \text{ amp}$$
$$= 28 + j15 \text{ amp.}$$

The positive-sequence component is

$$\dot{I}_{AN1} = 60\underline{/0^\circ} + 87.3\underline{/-59^\circ}\underline{\underline{/120^\circ}} + 121.8\underline{/100^\circ}\underline{\underline{/-120^\circ}}$$

$$= 72.3 + j11.6 = 73.2\underline{/9^\circ 8'} \text{ amp.}$$

The negative-sequence component is

$$\dot{I}_{AN2} = \frac{60\underline{/0^\circ} + 87.3\underline{/-59^\circ}\underline{/-120^\circ} + 121.8\underline{/100^\circ}/120^\circ}{3} \text{ amp}$$

$$= -40.3 - j26.6 = 48.7\underline{/-146^\circ 50'} \text{ amp.}$$

The symmetrical components for the B phase are

$$\begin{aligned}
\dot{I}_{BN0} &= \dot{I}_{AN0} = 28 + j15 \text{ amp,} \\
\dot{I}_{BN1} &= a^2\dot{I}_{AN1} = 73.2\underline{/9^\circ 8'}\ \underline{/-120^\circ} = 73.2\underline{/-110^\circ 52'} \\
&= -26.2 - j68.4 \text{ amp,} \\
\dot{I}_{BN2} &= a\dot{I}_{AN2} = 48.7\underline{/-146^\circ 50'}\ \underline{/120^\circ} \text{ amp} \\
&= 48.7\underline{/-26^\circ 50'} = 43.2 - j21.6 \text{ amp.}
\end{aligned}$$

For the C phase, they are

$$\begin{aligned}
\dot{I}_{CN0} &= \dot{I}_{AN0} = 28 + j15 \text{ amp} \\
\dot{I}_{CN1} &= a\dot{I}_{AN1} = 73.2\underline{/9^\circ 8'}\ \underline{/120^\circ} \\
&= 73.2\underline{/129^\circ 8'} = -46.1 + j56.8 \text{ amp} \\
\dot{I}_{CN2} &= a^2\dot{I}_{AN2} = 48.7\underline{/-146^\circ 50'}\ \underline{/-120^\circ} \\
&= 48.7\underline{/94^\circ 10'} = -2.9 + j48.2 \text{ amp.}
\end{aligned}$$

The symmetrical components of the vector I_A may also be obtained by performing the operations graphically which are indicated by Eqs. (682), (686), and (689). This method is illustrated in Fig. 212, which gives a graphical solution for I_{A1}, the positive sequence component of I_A. The vector I_A is plotted in its original position, vector I_B is rotated 120 deg. to give $a\dot{I}_B$ and is then added to $\dot{I}_A$; and the vector I_C is rotated 240 deg. to give $a^2\dot{I}_C$, which is then added to the sum of $\dot{I}_A$ and $a\dot{I}_B$. The vector component I_{A1} is one-third of the resultant vector.

116. Some General Relations in Three-phase Systems.—The following general conclusions for the different types of three-phase systems may be deduced from the fundamental relations just derived.

a. In any three-phase system, no zero-sequence component can exist in the line voltage since their sum is equal to zero.

b. In three-wire three-phase systems, no zero-sequence component can exist in the line currents since their sum is equal to zero.

c. In a three-wire three-phase system with delta-connected loads, a zero-sequence component can exist in the load currents since their sum is in general not equal to zero.

d. In a three-wire three-phase system with unbalanced star-connected loads, a zero-sequence component may exist in the load voltages since their sum is in general not equal to zero.

e. In a four-wire three-phase system, a zero-sequence component may exist in the line currents since their sum is in general not equal to zero. In this system, the zero-sequence components I_{A0}, I_{B0}, and I_{C0}, being in phase, all flow along the lines toward the neutral, and the neutral wire carries the sum of the three currents, or the entire residual current. If the neutral currents return through the ground, as is sometimes the case in transmission lines with grounded neutrals at the transformers, the zero-sequence components flow over the line wires in parallel and in phase and return through the ground connection. This produces interference by induction with adjacent telephone lines which is sometimes difficult to eliminate.

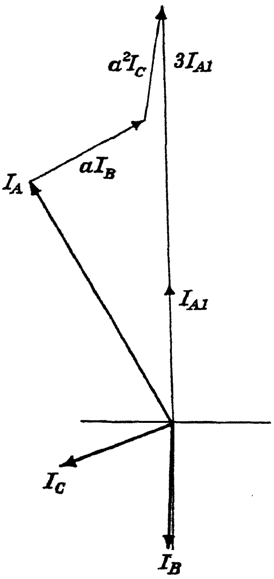

FIG. 212.—Graphical solution for positive-sequence component I_{A1}.

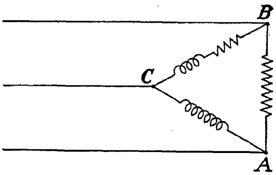

FIG. 213.—Three unbalanced loads connected in delta to a three-wire three-phase system.

117. Unbalanced Voltages Impressed on Unbalanced Mesh-connected Loads.—Consider three unbalanced loads connected to an unbalanced three-phase system, as shown in Fig. 213, the three voltages and the three load impedances being given. The sequence components of the load currents may be derived in terms of the sequence components of the line voltages combined with the given load admittances. The symmetrical components of the line current may be found in terms of the symmetrical components of the load currents. The three line voltages may be resolved into their symmetrical components in terms of one voltage as E_{AB}, in the following manner:

$$\dot{E}_{AB0} = \tfrac{1}{3}(\dot{E}_{AB} + \dot{E}_{BC} + \dot{E}_{CA}) \text{ volts,} \qquad (690)$$

$$\dot{E}_{AB1} = \tfrac{1}{3}(\dot{E}_{AB} + a\dot{E}_{BC} + a^2\dot{E}_{CA}) \text{ volts,} \qquad (691)$$

$$\dot{E}_{AB2} = \tfrac{1}{3}(\dot{E}_{AB} + a^2\dot{E}_{BC} + a\dot{E}_{CA}) \text{ volts.} \qquad (692)$$

Since the line voltages form a closed triangle, their vector sum must be zero, and there can be no zero-sequence components in the line voltages, or $E_{AB0} = 0$. The load voltages are

$$E_{AB} = E_{AB1} + E_{AB2} \text{ volts,} \qquad (693)$$

$$\dot{E}_{BC} = a^2\dot{E}_{AB1} + a\dot{E}_{AB2} \text{ volts,} \qquad (694)$$

$$\dot{E}_{CA} = a\dot{E}_{AB1} + a^2\dot{E}_{AB2} \text{ volts.} \qquad (695)$$

The load currents are

$$\dot{I}_{AB} = \dot{E}_{AB}\dot{Y}_{AB} \text{ amp} \qquad (696)$$

$$= \dot{E}_{AB1}\dot{Y}_{AB} + \dot{E}_{AB2}\dot{Y}_{AB} \text{ amp,} \qquad (697)$$

$$\dot{I}_{BC} = \dot{E}_{AB1}a^2\dot{Y}_{BC} + \dot{E}_{AB2}a\dot{Y}_{BC} \text{ amp,} \qquad (698)$$

$$\dot{I}_{CA} = \dot{E}_{AB1}a\dot{Y}_{CA} + \dot{E}_{AB2}a^2\dot{Y}_{CA} \text{ amp.} \qquad (699)$$

Expressing these currents in terms of their symmetrical components,

$$\begin{aligned}\dot{I}_{AB0} &= \tfrac{1}{3}(\dot{I}_{AB} + \dot{I}_{BC} + \dot{I}_{CA}) \text{ amp}\\ &= \dot{E}_{AB1}\frac{\dot{Y}_{AB} + a^2\dot{Y}_{BC} + a\dot{Y}_{CA}}{3} + \\ &\qquad \dot{E}_{AB2}\frac{\dot{Y}_{AB} + a\dot{Y}_{BC} + a^2\dot{Y}_{CA}}{3} \text{ amp,} \qquad (700)\end{aligned}$$

$$\begin{aligned}\dot{I}_{AB1} &= \tfrac{1}{3}(\dot{I}_{AB} + a\dot{I}_{BC} + a^2\dot{I}_{CA}) \text{ amp} \qquad (686)\\ &= \dot{E}_{AB1}\frac{\dot{Y}_{AB} + a^3\dot{Y}_{BC} + a^3\dot{Y}_{CA}}{3} + \\ &\qquad \dot{E}_{AB2}\frac{\dot{Y}_{AB} + a^2\dot{Y}_{BC} + a^4\dot{Y}_{CA}}{3} \text{ amp} \qquad (701)\\ &= \dot{E}_{AB1}\frac{\dot{Y}_{AB} + \dot{Y}_{BC} + \dot{Y}_{CA}}{3} + \\ &\qquad \dot{E}_{AB2}\frac{\dot{Y}_{AB} + a^2\dot{Y}_{BC} + a\dot{Y}_{CA}}{3} \text{ amp,} \qquad (702)\end{aligned}$$

$$\begin{aligned}\dot{I}_{AB2} &= \tfrac{1}{3}(\dot{I}_{AB} + a^2\dot{I}_{BC} + a\dot{I}_{CA}) \text{ amp} \qquad (689)\\ &= \dot{E}_{AB1}\frac{\dot{Y}_{AB} + a^4\dot{Y}_{BC} + a^2\dot{Y}_{CA}}{3} + \\ &\qquad \dot{E}_{AB2}\frac{\dot{Y}_{AB} + a^3\dot{Y}_{BC} + a^3\dot{Y}_{CA}}{3} \text{ amp} \qquad (703)\\ &= \dot{E}_{AB1}\frac{\dot{Y}_{AB} + a\dot{Y}_{BC} + a^2\dot{Y}_{CA}}{3} + \\ &\qquad \dot{E}_{AB2}\frac{\dot{Y}_{AB} + \dot{Y}_{BC} + \dot{Y}_{CA}}{3} \text{ amp.} \qquad (704)\end{aligned}$$

By similarity to Eqs. (690) to (692) for the symmetrical components of voltage and current,

$$\dot{Y}_{AB0} = \tfrac{1}{3}(\dot{Y}_{AB} + \dot{Y}_{BC} + \dot{Y}_{CA}) \text{ mhos,} \tag{705}$$
$$\dot{Y}_{AB1} = \tfrac{1}{3}(\dot{Y}_{AB} + a\dot{Y}_{BC} + a^2\dot{Y}_{CA}) \text{ mhos,} \tag{706}$$
$$\dot{Y}_{AB2} = \tfrac{1}{3}(\dot{Y}_{AB} + a^2\dot{Y}_{BC} + a\dot{Y}_{CA}) \text{ mhos,} \tag{707}$$

and Eqs. (700) to (704) may be rewritten in terms of these new admittances,

$$\dot{I}_{AB0} = \dot{E}_{AB1}\dot{Y}_{AB2} + \dot{E}_{AB2}\dot{Y}_{AB1} \text{ amp,} \tag{708}$$
$$\dot{I}_{AB1} = \dot{E}_{AB1}\dot{Y}_{AB0} + \dot{E}_{AB2}\dot{Y}_{AB2} \text{ amp,} \tag{709}$$
$$\dot{I}_{AB2} = \dot{E}_{AB1}\dot{Y}_{AB1} + \dot{E}_{AB2}\dot{Y}_{AB0} \text{ amp.} \tag{710}$$

Equations (708) to (710) give the symmetrical components of the load current I_{AB} in terms of known values. The addition of these components gives the current I_{AB}. The symmetrical components of the other currents may be obtained from those of I_{AB}, and the other two currents may then be found.

The components of the line current may be obtained from those of the load currents as follows:

$$\begin{aligned}\dot{I}_A &= \dot{I}_{AB} - \dot{I}_{CA} \text{ amp}\\ &= (1 - a)\dot{I}_{AB1} + (1 - a^2)\dot{I}_{AB2} \text{ amp;}\end{aligned} \tag{711}$$

but

$$\begin{aligned}(1 - a^2) &= (1.5 + j0.866)\\ &= \sqrt{3}\underline{/30^\circ},\end{aligned} \tag{712}$$

and

$$\begin{aligned}(1 - a) &= (0.866 - j0.5)\sqrt{3}\\ &= \sqrt{3}\underline{/-30^\circ},\end{aligned} \tag{713}$$

so that

$$\dot{I}_A = \sqrt{3}\dot{I}_{AB1}\underline{/-30^\circ} + \sqrt{3}\dot{I}_{AB2}\underline{/30^\circ} \text{ amp.} \tag{714}$$

Since there can be no zero-sequence components in the line currents,

$$\dot{I}_A = \dot{I}_{A1} + \dot{I}_{A2} \text{ amp,}$$

and from Eq. (714),

$$\dot{I}_{A1} = \sqrt{3}\dot{I}_{AB1}\underline{/-30^\circ} \text{ amp,} \qquad (715)$$

$$\dot{I}_{A2} = \sqrt{3}\dot{I}_{AB2}\underline{/30^\circ} \text{ amp.} \qquad (716)$$

The relations, as to magnitude and phase, between the positive-sequence components of line and load currents and between the negative-sequence components of line and load currents are the same as for any balanced symmetrical mesh-connected system. The zero-sequence components of the load currents, *i.e.*, I_{AB0}, I_{BC0}, and I_{CA0}, are all in phase and hence circulate in the mesh-connected load and do not appear in the line currents.

It should be noted that the expressions Eqs. (705) to (707) are each used in connection with the various sequence components of voltage; they are not, therefore, the admittances of the circuit to zero-, positive-, and negative-sequence components of voltages, respectively, but are so designated because of the similarity in form to the expressions for the corresponding components of voltage.

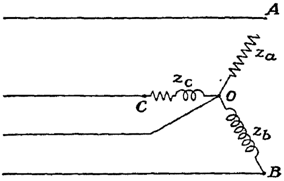

FIG. 214.—Three unbalanced loads connected in Y to a three-phase system.

118. General Case of Unbalanced Impedances Connected in Star with Unbalanced Current Flowing in Them. Three impedances Z_a, Z_b, and Z_c are connected in star in Fig. 214 with the unbalanced currents I_a, I_b, and I_c, respectively, flowing through them. The symmetrical components of the load voltages may be expressed in terms of the symmetrical components of the currents and the impedances. The load voltages may be expressed in terms of the impedance drops as

$$\dot{E}_{OA} = \dot{I}_a\dot{Z}_a = \dot{I}_{a1}\dot{Z}_a + \dot{I}_{a2}\dot{Z}_a + \dot{I}_{a0}\dot{Z}_a \text{ volts.} \qquad (717)$$

$$\begin{aligned}\dot{E}_{OB} &= \dot{I}_b\dot{Z}_b = \dot{I}_{b1}\dot{Z}_b + \dot{I}_{b2}\dot{Z}_b + \dot{I}_{b0}\dot{Z}_b \\ &= a^2\dot{I}_{a1}\dot{Z}_b + a\dot{I}_{a2}\dot{Z}_b + \dot{I}_{a0}\dot{Z}_b \text{ volts,} \qquad (718)\end{aligned}$$

$$\begin{aligned}\dot{E}_{OC} &= \dot{I}_c\dot{Z}_c = \dot{I}_{c1}\dot{Z}_c + \dot{I}_{c2}\dot{Z}_c + \dot{I}_{c0}\dot{Z}_c \text{ volts} \\ &= a\dot{I}_{a1}\dot{Z}_c + a^2\dot{I}_{c2}\dot{Z}_c + \dot{I}_{a0}\dot{Z}_c \text{ volts.} \qquad (719)\end{aligned}$$

Solving Eqs. (717) to (719) for the positive-sequence component of E_{OA},

$$\dot{E}_{OA1} = \frac{\dot{E}_{OA} + a\dot{E}_{OB} + a^2\dot{E}_{OC}}{3} \text{ volts.} \tag{720}$$

Substituting the values of E_{OA}, E_{OB}, and E_{OC} from Eqs. (717) to (719),

$$\begin{aligned}\dot{E}_{OA1} &= \frac{\dot{I}_{a1}\dot{Z}_a + \dot{I}_{a2}\dot{Z}_a + \dot{I}_{a0}\dot{Z}_a + \dot{I}_{a1}\dot{Z}_b + a^2\dot{I}_{a2}\dot{Z}_b + a\dot{I}_{a0}\dot{Z}_b + \dot{I}_{a1}\dot{Z}_c + a\dot{I}_{a2}\dot{Z}_c + a^2\dot{I}_{a0}\dot{Z}_c}{3} \\ &= \dot{I}_{a1}\left[\frac{\dot{Z}_a + \dot{Z}_b + \dot{Z}_c}{3}\right] + \dot{I}_{a2}\left[\frac{\dot{Z}_a + a^2\dot{Z}_b + a\dot{Z}_c}{3}\right] + \\ &\qquad \dot{I}_{a0}\left[\frac{\dot{Z}_a + a\dot{Z}_b + a^2\dot{Z}_c}{3}\right] \text{ volts.}\end{aligned} \tag{721}$$

It is seen from Eq. (721) that each of the components of current reacts with each of the three impedances to produce the positive-sequence component of voltage. The impedance groupings have the same form as those used in calculating the components of the vector I_a, and the expression may be shortened by using a similar notation for the values of impedance, *i.e.*,

$$\frac{\dot{Z}_a + \dot{Z}_b + \dot{Z}_c}{3} = \dot{Z}_0 \text{ ohms,} \tag{722}$$

$$\frac{\dot{Z}_a + a\dot{Z}_b + a^2\dot{Z}_c}{3} = \dot{Z}_1 \text{ ohms,} \tag{723}$$

$$\frac{\dot{Z}_a + a^2\dot{Z}_b + a\dot{Z}_c}{3} = \dot{Z}_2 \text{ ohms.} \tag{724}$$

These expressions are not the positive-, negative- and zero-sequence impedances of the equipment that make up the circuit, these impedances being different for various kinds of machinery and apparatus. They are, however, useful abbreviations, and the three components of voltage may be expressed in terms of these impedances as follows:

$$\dot{E}_{OA1} = \dot{I}_{A1}\dot{Z}_0 + \dot{I}_{A2}\dot{Z}_2 + \dot{I}_{A0}\dot{Z}_1 \text{ volts,} \tag{725}$$

$$\dot{E}_{OA2} = \dot{I}_{A1}\dot{Z}_1 + \dot{I}_{A2}\dot{Z}_0 + \dot{I}_{A0}\dot{Z}_2 \text{ volts,} \tag{726}$$

$$\dot{E}_{OA0} = \dot{I}_{A1}\dot{Z}_2 + \dot{I}_{A2}\dot{Z}_1 + \dot{I}_{A0}\dot{Z}_0 \text{ volts.} \tag{727}$$

The equations for several types of special cases may be obtained directly from these equations. For example, if the neutral is not connected to a return circuit there can be no zero-sequence component of current and the term involving it drops out of the equations. A very important special case is the one in which the impedances are symmetrical. This case occurs for the line impedance in the calculation of fault currents in power-system networks, an important application of symmetrical phase components. In this case, the three impedances are equal, and

$$\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = \dot{Z} \text{ ohms.} \tag{728}$$

Since the three impedances are equal, the sum of the balanced system of vectors denoted by Z_1 and Z_2 is equal to zero and Eqs. (725), (726), and (727) reduce to

$$\begin{aligned} \dot{E}_{a1} &= \dot{I}_{a1}\dot{Z}_0 = \dot{I}_{a1}\frac{\dot{Z}_a + \dot{Z}_b + \dot{Z}_c}{3} \text{ volts} \\ &= \dot{I}_{a1}\dot{Z} \text{ volts,} && (729) \\ \dot{E}_{a2} &= \dot{I}_{a2}\dot{Z} \text{ volts,} && (730) \\ \dot{E}_{a0} &= \dot{I}_{a0}\dot{Z} \text{ volts.} && (731) \end{aligned}$$

In this case, each component of voltage is made up of the impedance drop due to the corresponding component of current.

119. Unbalanced Voltages Impressed on Unbalanced Star-connected Loads.—Three unequal impedances Z_{OA}, Z_{OB}, and Z_{OC} are connected in star to an unbalanced three-wire three-phase system similar to that shown in Fig. 214 with the neutral connection omitted. The line voltages E_{AB}, E_{BC}, and E_{CA}, being unbalanced, may be expressed in terms of symmetrical components as in Eqs. (690) to (692). Since the line voltages must add to zero,

$$\begin{aligned} \dot{E}_{AB} + \dot{E}_{BC} + \dot{E}_{CA} &= 0 \text{ volts} && (732) \\ \dot{E}_{AB0} &= 0 \text{ volts,} \end{aligned}$$

and

$$\begin{aligned} \dot{E}_{AB} &= \dot{E}_{AB1} + \dot{E}_{AB2} \text{ volts,} && (693) \\ \dot{E}_{BC} &= a^2\dot{E}_{AB1} + a\dot{E}_{AB2} \text{ volts,} && (694) \\ \dot{E}_{CA} &= a\dot{E}_{AB1} + a^2\dot{E}_{AB2} \text{ volts,} && (695) \end{aligned}$$

the unbalanced load voltages E_{AO}, E_{BO}, and E_{CO} will contain all three of the symmetrical components and may be expressed as

$$\dot{E}_{AB} = \dot{E}_{AO} - \dot{E}_{BO} \text{ volts} \tag{733}$$
$$= (1 - a^2)\dot{E}_{AO1} + (1 - a)\dot{E}_{AO2} \text{ volts,} \tag{734}$$
$$\dot{E}_{BC} = \dot{E}_{BO} - \dot{E}_{CO} \text{ volts} \tag{735}$$
$$= (a^2 - a)\dot{E}_{AO1} + (a - a^2)\dot{E}_{AO2} \text{ volts,} \tag{736}$$
$$\dot{E}_{CA} = \dot{E}_{CO} - \dot{E}_{AO} \text{ volts} \tag{737}$$
$$= (a - 1)\dot{E}_{AO1} + (a^2 - 1)\dot{E}_{AO2} \text{ volts.} \tag{738}$$

Combining Eqs. (693) and (694) with Eqs. (734) and (736),

$$\dot{E}_{AB1} + \dot{E}_{AB2} = (1 - a^2)\dot{E}_{AO1} + (1 - a)\dot{E}_{AO2} \text{ volts,} \tag{739}$$
$$a\dot{E}_{AB1} + \dot{E}_{AB2} = (a - 1)\dot{E}_{AO1} + (1 - a)\dot{E}_{AO2} \text{ volts.} \tag{740}$$

Subtracting Eq. (740) from Eq. (739),

$$(1 - a)\dot{E}_{AB1} = (2 - a^2 - a)\dot{E}_{AO1} \text{ volts,} \tag{741}$$

and substituting the values of a and a^2 from Eqs. (671) and (672),

$$\sqrt{3}\dot{E}_{AB1}\underline{/-30^\circ} = 3\dot{E}_{AO1} \text{ volts,}$$

or

$$\dot{E}_{AO1} = \frac{1}{\sqrt{3}}\dot{E}_{AB1}\underline{/-30^\circ} \text{ volts.} \tag{742}$$

Likewise, combining Eqs. (694) and (695) with Eqs. (736) and (738),

$$a\dot{E}_{AB1} + \dot{E}_{AB2} = (a - 1)\dot{E}_{AO1} + (1 - a)\dot{E}_{AO2} \text{ volts,} \tag{743}$$
$$a\dot{E}_{AB1} + a^2\dot{E}_{AB2} = (a - 1)\dot{E}_{AO1} + (a^2 - 1)\dot{E}_{AO2} \text{ volts.} \tag{744}$$

Subtracting Eq. (744) from Eq. (743),

$$(1 - a^2)\dot{E}_{AB2} = (2 - a^2 - a)\dot{E}_{AO2} \text{ volts,} \tag{745}$$

and again substituting the values of a and a^2 from Eqs. (671) and (672),

$$\sqrt{3}\dot{E}_{AB2}\underline{/30^\circ} = 3\dot{E}_{AO2} \text{ volts,}$$

or

$$\dot{E}_{AO2} = \frac{1}{\sqrt{3}}\dot{E}_{AB2}\underline{/30^\circ} \text{ volts.} \tag{746}$$

Since

$$\dot{E}_{AB1} = \tfrac{1}{3}(\dot{E}_{AB} + \dot{E}_{BC}\underline{/120^\circ} + \dot{E}_{CA}\underline{/-120^\circ}) \text{ volts} \tag{691}$$

and

$$\dot{E}_{AB2} = \tfrac{1}{3}(\dot{E}_{AB} + \dot{E}_{BC}\underline{/-120^\circ} + \dot{E}_{CA}\underline{/120^\circ}) \text{ volts,} \tag{692}$$

then it follows that

$$\dot{E}_{AO1} = \frac{1}{3\sqrt{3}}(\qquad \underline{-30^\circ} + \dot{E}_{BC}\underline{/90^\circ} + \dot{E}_{CA}\underline{/-150^\circ}\,)\text{volts}, \tag{747}$$

$$\dot{E}_{AO2} = \qquad\qquad + \dot{E}_{BC}\underline{/-90^\circ} + \dot{E}_{CA}\underline{/150^\circ})\text{ volts}, \tag{748}$$

so that

$$\dot{E}_{AO1} + \dot{E}_{AO2} = \tfrac{1}{3}(\dot{E}_{AB} - \dot{E}_{CA})\text{ volts}. \tag{749}$$

From Fig. 215, it is seen that $\frac{1}{3}(\dot{E}_{AB} - \dot{E}_{CA})$ is the voltage $\dot{E}_{AN}$ from A to the center of gravity of the triangle of line voltages. Hence,

$$\dot{E}_{AO1} + \dot{E}_{AO2} = \dot{E}_{AN}\text{ volts}, \tag{750}$$

and since

$$\dot{E}_{AO} = \dot{E}_{AO0} + \dot{E}_{AO1} + \dot{E}_{AO2}\text{ volts},$$

the voltage E_{AO0} must be the displacement of O, the common point of the load voltages, from N, the center of gravity of the triangle of line voltages. From Eqs. (742) and (746), it is seen that

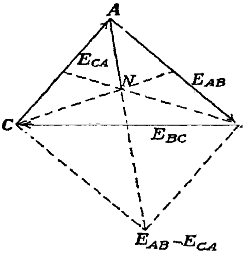

FIG. 215.—Construction for the neutral point for an unbalanced system of three-phase voltages.

$$\dot{E}_{AO1} = \quad {}^{1} \quad \dot{E}_{AB1}\underline{/-30^\circ} = \dot{E}_{AN1}\text{ volts} \tag{751}$$

is the neutral voltage of the positive-sequence components of the line voltages and that

$$\dot{E}_{AO2} = \frac{1}{}\dot{E} \qquad\qquad = \dot{E}_{AN2}\text{ volts} \tag{752}$$

is the neutral voltage of the negative-sequence components of the line voltages.

The voltages E_{AO1} and E_{AO2} may, therefore, be determined from the known sequence components of the line voltages. The zero-sequence component E_{AO0} is indeterminable from the line voltages, as it is a function of the load impedances. It may be determined as follows: The load currents are

$$\dot{I}_{AO} = \dot{E}_{AO}\dot{Y}_a = \dot{E}_{AO0}\dot{Y}_a + \dot{E}_{AO1}\dot{Y}_a + \dot{E}_{AO2}\dot{Y}_a\text{ amp}, \tag{753}$$
$$\dot{I}_{BO} = \dot{E}_{BO}\dot{Y}_b = \dot{E}_{AO0}\dot{Y}_b + \dot{E}_{AO1}a^2\dot{Y}_b + \dot{E}_{AO2}a\dot{Y}_b\text{ amp}, \tag{754}$$
$$\dot{I}_{CO} = \dot{E}_{CO}\dot{Y}_c = \dot{E}_{AO0}\dot{Y}_c + \dot{E}_{AO1}a\dot{Y}_c + \dot{E}_{AO2}a^2\dot{Y}_c\text{ amp}, \tag{755}$$

Then, since

$$\dot{I}_{AO0} = \tfrac{1}{3}(\dot{I}_{AO} + \dot{I}_{BO} + \dot{I}_{CO}) \text{ amp} \qquad (682)$$

$$= \dot{E}_{AO0}\frac{\dot{Y}_a + \dot{Y}_b + \dot{Y}_c}{3} + \dot{E}_{AO1}\frac{\dot{Y}_a + a^2\dot{Y}_b + a\dot{Y}_c}{3} + \dot{E}_{AO2}\frac{\dot{Y}_a + a\dot{Y}_b + a^2\dot{Y}_c}{3} \text{ amp,} \qquad (756)$$

$$\dot{I}_{AO1} = \tfrac{1}{3}(\dot{I}_{AO} + a\dot{I}_{BO} + a^2\dot{I}_{CO}) \text{ amp} \qquad (686)$$

$$= \dot{E}_{AO0}\frac{\dot{Y}_a + a\dot{Y}_b + a^2\dot{Y}_c}{3} + \dot{E}_{AO1}\frac{\dot{Y}_a + \dot{Y}_b + \dot{Y}_c}{3} + \dot{E}_{AO2}\frac{\dot{Y}_a + a^2\dot{Y}_b + a\dot{Y}_c}{3} \text{ amp,} \qquad (757)$$

$$\dot{I}_{AO2} = \tfrac{1}{3}(\dot{I}_{AO} + a^2\dot{I}_{BO} + a\dot{I}_{CO}) \text{ amp} \qquad (689)$$

$$= \dot{E}_{AO0}\frac{\dot{Y}_a + a^2\dot{Y}_b + a\dot{Y}_c}{3} + \dot{E}_{AO1}\frac{\dot{Y}_a + a\dot{Y}_b + a^2\dot{Y}_c}{3} + \dot{E}_{AO2}\frac{\dot{Y}_a + \dot{Y}_b + \dot{Y}_c}{3} \text{ amp,} \qquad (758)$$

or

$$\dot{I}_{AO0} = \dot{E}_{AO0}\dot{Y}_{a0} + \dot{E}_{AO1}\dot{Y}_{a1} + \dot{E}_{AO2}\dot{Y}_{a2} \text{ amp,} \qquad (759)$$

$$\dot{I}_{AO1} = \dot{E}_{AO0}\dot{Y}_{a1} + \dot{E}_{AO1}\dot{Y}_{a0} + \dot{E}_{AO2}\dot{Y}_{a2} \text{ amp,} \qquad (760)$$

$$\dot{I}_{AO2} = \dot{E}_{AO0}\dot{Y}_{a2} + \dot{E}_{AO1}\dot{Y}_{a1} + \dot{E}_{AO2}\dot{Y}_{a0} \text{ amp.} \qquad (761)$$

Since, in a three-wire three-phase system, the vector sum of the line currents must be zero,

$$\dot{I}_{AO0} = 0 \text{ amp,}$$

and

$$\dot{E}_{AO0} = -\dot{E}_{AO1}\frac{\dot{Y}_{a2}}{\dot{Y}_{a0}} - \dot{E}_{AO2}\frac{\dot{Y}_{a1}}{\dot{Y}_{a0}} \text{ volts.} \qquad (762)$$

Substituting the values of E_{AO1} and E_{AO2} from Eqs. (751) and (752) into Eq. (762),

$$\dot{E}_{AO0} = -\dot{E}_{AN1}\frac{\dot{Y}_{a2}}{\dot{Y}_{a0}} - \dot{E}_{AN2}\frac{\dot{Y}_{a1}}{\dot{Y}_{a0}} \text{ volts.} \qquad (763)$$

Substituting Eqs. (751), (752), and (763) into Eqs. (760) and (761), the sequence components of the currents are

$$\dot{I}_{AO1} = -\dot{E}_{AN1}\frac{\dot{Y}_{a2}\dot{Y}_{a1}}{\dot{Y}_{a0}} - \dot{E}_{AN2}\frac{\dot{Y}_{a1}^2}{\dot{Y}_{a0}} + \dot{E}_{AN1}\dot{Y}_{a0} + \dot{E}_{AN2}\dot{Y}_{a2}$$

$$= \dot{E}_{AN1}\left[\dot{Y}_{a0} - \frac{\dot{Y}_{a2}\dot{Y}_{a1}}{\dot{Y}_{a0}}\right] + \dot{E}_{AN2}\left[\dot{Y}_{a2} - \frac{\dot{Y}_{a1}^2}{\dot{Y}_{a0}}\right] \text{ amp} \qquad (764)$$

and

$$\dot{I}_{AO2} = \dot{E}_{AN1}\left[\dot{Y}_{a1} - \frac{\dot{Y}_{a2}^2}{\dot{Y}_{a0}}\right] + \dot{E}_{AN2}\left[\dot{Y}_{a0} - \frac{\dot{Y}_{a1}\dot{Y}_{a2}}{\dot{Y}_{a0}}\right] \text{ amp.} \qquad (765)$$

If the line voltages are balanced and the star-connected loads are unbalanced,

$$\dot{E}_{AB0} = 0,\ \dot{E}_{AB2} = 0,\ \dot{E}_{AB1} = \dot{E}_{AB} \text{ volts},$$
$$\dot{E}_{AN0} = 0,\ \dot{E}_{AN2} = 0,\ \dot{E}_{AN1} = \dot{E}_{AN} \text{ volts},$$

and from Eqs. (751), (752), and (763)

$$\begin{aligned} \dot{E}_{AO1} &= \dot{E}_{AN1} = \dot{E}_{AN} \text{ volts}, \\ \dot{E}_{AO2} &= 0 \text{ volts}, \\ \dot{E}_{AO0} &= -\dot{E}_{AN}\frac{\dot{Y}_{a2}}{\dot{Y}_{A0}} \text{ volts}, \qquad (766) \end{aligned}$$

and

$$\begin{aligned} \dot{E}_{AO} &= \dot{E}_{AO0} + \dot{E}_{AO1} \text{ volts} \qquad (767) \\ &= \dot{E}_{AO0} + \dot{E}_{AN} \text{ volts}, \qquad (768) \end{aligned}$$

or the voltage E_{AO0} is the displacement of the point O from the neutral of the system. The symmetrical components of the currents become

$$\begin{aligned} \dot{I}_{AO0} &= 0 \text{ amp}, \\ \dot{I}_{AO1} &= \dot{E}_{AN}\left[\dot{Y}_{a0} - \frac{\dot{Y}_{a2}\dot{Y}_{a1}}{\dot{Y}_{a0}}\right] \text{ amp}, \qquad (769) \\ \dot{I}_{AO2} &= \dot{E}_{AN}\left[\dot{Y}_{a1} - \frac{\dot{Y}_{a2}^2}{\dot{Y}_{a0}}\right] \text{ amp}, \qquad (770) \\ \dot{I}_{AO} &= \dot{I}_{AO1} + \dot{I}_{AO2} \text{ amp}, \qquad (771) \\ \dot{I}_{BO} &= a^2\dot{I}_{AO1} + a\dot{I}_{AO2} \text{ amp}, \qquad (772) \\ \dot{I}_{CO} &= a\dot{I}_{AO1} + a^2\dot{I}_{AO2} \text{ amp}. \qquad (773) \end{aligned}$$

Example 77

Three impedances $\dot{Z}_a = 10\underline{/90^\circ}$, $\dot{Z}_b = 10\underline{/90^\circ}$, and $\dot{Z}_c = 5\underline{/-90^\circ}$ are connected in star to a balanced 220-volt three-wire three-phase line, as shown in Fig. 216. Find the voltage drop across each impedance.

The admittances corresponding to the above impedances are $\dot{Y}_a = 0.1\underline{/-90^\circ}$, $\dot{Y}_b = 0.1\underline{/-90^\circ}$, and $\dot{Y}_c = 0.2\underline{/90^\circ}$, and the value of $\dot{Y}_{a0}$ from Eq. (705) is

$$\begin{aligned} \dot{Y}_{a0} &= \tfrac{1}{3}(0.1\underline{/-90^\circ} + 0.1\underline{/-90^\circ} + 0.2\underline{/90^\circ}) \text{ mho} \\ &= 0 \text{ mho}. \end{aligned}$$

Therefore, by Eq. (766),

$$\dot{E}_{a0} = -\dot{E}_{AN}\frac{\dot{Y}_{a2}}{\dot{Y}_{a0}} = \infty \text{ volts.}$$

The voltage across each impedance must be infinitely large. This is analogous to the case of resonance in a single-phase series circuit with zero resistance.

Example 78

Three impedances $\dot{Z}_a = 10\underline{/-90^\circ}$, $\dot{Z}_b = 10\underline{/60^\circ}$, and $\dot{Z}_c = 10\underline{/60^\circ}$ are connected in star to a 220-volt three-wire three-phase line having balanced voltages, as shown in Fig. 217. Find the voltage across each impedance and the three line currents in the symmetrical-component form.

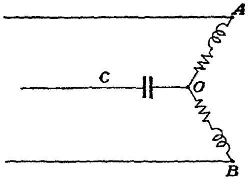

FIG. 216.—Circuit diagram for Example 77.

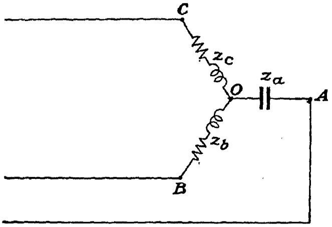

FIG. 217.—Circuit diagram for Example 78.

The admittances corresponding to the above impedances, are $\dot{Y}_a = 0.1\underline{/90^\circ}$, $\dot{Y}_b = 0.1\underline{/-60^\circ}$, and $\dot{Y}_c = 0.1\underline{/-60^\circ}$, which may be combined to give

$$\begin{aligned}\dot{Y}_{a0} &= \tfrac{1}{3}(0.1\underline{/90^\circ} + 0.1\underline{/-60^\circ} + 0.1\underline{/-60^\circ}) \text{ mho}\\ &= 0.0413\underline{/-36.2^\circ} \text{ mho,}\\ \dot{Y}_{a2} &= \tfrac{1}{3}(0.1\underline{/90^\circ} + 0.1\underline{/180^\circ} + 0.1\underline{/60^\circ}) \text{ mho}\\ &= 0.0643\underline{/105^\circ} \text{ mho.}\end{aligned}$$

By the use of Eq. (766),

$$\begin{aligned}\dot{E}_{a0} &= \dot{E}_{NA}\frac{0.0643}{0.0413}\underline{/141.2^\circ} \text{ volts}\\ &= 1.558\dot{E}_{NA}\underline{/141.2^\circ} \text{ volts.}\end{aligned}$$

Taking E_{NA} as the reference vector $\dot{E}_{NA} = 127\underline{/0^\circ}$ volts,

$$\begin{aligned}\dot{E}_{a0} &= 197.8\underline{/141.2^\circ} = -154.5 + j124 \text{ volts,}\\ \dot{E}_{AO} &= \dot{E}_{a0} + \dot{E}_{NA} = -281.5 + j124 \text{ volts,}\\ \dot{E}_{BO} &= \dot{E}_{a0} + \dot{E}_{BN} = -91 + j234 \text{ volts,}\\ \dot{E}_{CO} &= \dot{E}_{a0} + \dot{E}_{CN} = -91 + j14 \text{ volts.}\end{aligned}$$

The currents are

$$\begin{aligned}\dot{I}_{AO} &= \dot{E}_{AO}\dot{Y}_a = (-281.5 + j124)0.1\underline{/90^\circ} \text{ amp} \\ &= -12.4 - j28.15 \text{ amp}, \\ \dot{I}_{BO} &= (-91 + j234)0.1\underline{/-60^\circ} \text{ amp} \\ &= 15.73 + j19.58 \text{ amp}, \\ \dot{I}_{CO} &= (-91 + j14)0.1\underline{/-60^\circ} \text{ amp} \\ &= -3.34 + j8.58 \text{ amp}, \\ \dot{I}_{NO} &= \dot{I}_{AO} + \dot{I}_{BO} + \dot{I}_{CO} = 0 \text{ amp}.\end{aligned}$$

These relations are shown in the vector diagram (Fig. 218), where it is seen that the voltages E_{AO} and E_{BO} are greater than the impressed line voltage.

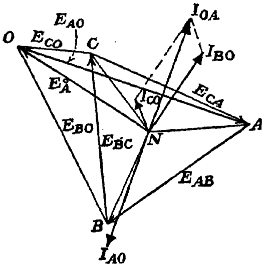

FIG. 218.—Vector diagram for Example 78.

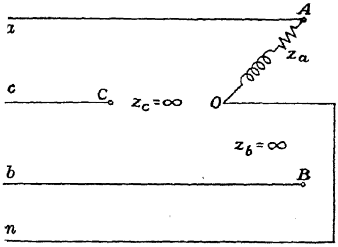

FIG. 219.—Single-phase load from line to neutral in a four-wire three-phase system.

120. Single-phase Loads on Three-phase Systems.—The most extreme case of unbalanced loads on a three-phase system is a single load connected between two of the wires of a three-wire system or between one line wire and the neutral of a four-wire system. In Fig. 219, the impedance Z_a is connected between one wire and neutral of a four-wire three-phase system. The impedances $Z_b = \infty$ and $Z_c = \infty$ are understood to complete the connection. In this system, the load currents are

$$\begin{aligned}\dot{I}_{AO} &= \dot{E}_{AN}\dot{Y}_a = \dot{I} \text{ amp}, \\ \dot{I}_{BO} &= 0, \ \dot{I}_{CO} = 0 \text{ amp}.\end{aligned}$$

Resolving these currents into symmetrical components,

$$\dot{I}_{AO0} = \frac{1}{3}(\dot{I}_{AO} + \dot{I}_{BO} + \dot{I}_{CO}) = \frac{\dot{I}}{3} \text{ amp}, \qquad (774)$$

$$\dot{I}_{AO1} = \frac{1}{3}(\dot{I}_{AO} + a\dot{I}_{BO} + a^2\dot{I}_{CO}) = \frac{\dot{I}}{3} \text{ amp,} \tag{775}$$

$$\dot{I}_{AO2} = \frac{1}{3}(\dot{I}_{AO} + a^2\dot{I}_{BO} + a\dot{I}_{CO}) = \frac{\dot{I}}{3} \text{ amp,} \tag{776}$$

$$\dot{I}_{BO0} = \dot{I}_{AO0} = \frac{\dot{I}}{3} \text{ amp,} \tag{777}$$

$$\dot{I}_{BO1} = a^2\dot{I}_{AO1} = \tfrac{1}{3}\dot{I}\underline{/-120^\circ} \text{ amp,} \tag{778}$$

$$\dot{I}_{BO2} = a\dot{I}_{AO2} = \tfrac{1}{3}\dot{I}\underline{/120^\circ} \text{ amp,} \tag{779}$$

$$\dot{I}_{CO0} = \dot{I}_{AO0} = \frac{\dot{I}}{3} \text{ amp,} \tag{780}$$

$$\dot{I}_{CO1} = a\dot{I}_{AO1} = \tfrac{1}{3}\dot{I}\underline{/120^\circ} \text{ amp,} \tag{781}$$

$$\dot{I}_{CO2} = a^2\dot{I}_{AO2} = \tfrac{1}{3}\dot{I}\underline{/-120^\circ} \text{ amp.} \tag{782}$$

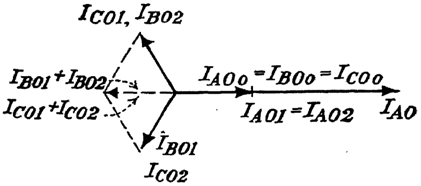

FIG. 220.—Vector diagram for a single-phase load to neutral in a four-wire three-phase system.

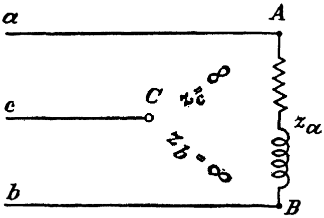

FIG. 221.—Single-phase load from line to line in a three-wire three-phase system.

These components of current are shown in Fig. 220. It is seen that

$$\dot{I}_{AO} = \dot{I}_{AO0} + \dot{I}_{AO1} + \dot{I}_{AO2} = \dot{I} \text{ amp,} \tag{783}$$

$$\dot{I}_{BO} = \dot{I}_{BO0} + \dot{I}_{BO1} + \dot{I}_{BO2} = 0 \text{ amp,} \tag{784}$$

$$\dot{I}_{CO} = \dot{I}_{CO0} + \dot{I}_{CO1} + \dot{I}_{CO2} = 0 \text{ amp,} \tag{785}$$

$$\dot{I}_{NO} = \dot{I}_{AO} + \dot{I}_{BO} + \dot{I}_{CO} = \dot{I} \text{ amp.} \tag{786}$$

In Fig. 221, the impedance Z_a is connected between lines A and B of a three-wire three-phase system, the impedances Z_b and Z_c being infinite. Then

$$\dot{I}_{AB} = \dot{E}_A \quad = \dot{I} \text{ amp,} \tag{787}$$

$$\dot{I}_{BC} = 0 \text{ amp,} \tag{788}$$

$$\dot{I}_{CA} = 0 \text{ amp.} \tag{789}$$

These currents resolve into symmetrical components in the same manner as Eqs. (774) to (776). That is,

$$\dot{I}_{AB0} = \frac{\dot{I}}{3} \text{ amp,} \tag{774}$$

$$\dot{I}_{AB1} = \frac{\dot{I}}{3} \text{ amp,} \tag{775}$$

$$\dot{I}_{AB2} = \frac{\dot{I}}{3} \text{ amp.} \tag{776}$$

The line currents are

$$\dot{I}_{aA} = \dot{I} \text{ amp,} \tag{787}$$

$$\dot{I}_{bB} = -\dot{I} \text{ amp,} \tag{788}$$

$$\dot{I}_{cC} = 0 \text{ amp.} \tag{789}$$

The line currents resolve into the following symmetrical components:

$$\dot{I}_{aA0} = \tfrac{1}{3}(\dot{I}_{aA} + \dot{I}_{bB} + \dot{I}_{cC}) = 0 \text{ amp,} \tag{790}$$

$$\dot{I}_{aA1} = \tfrac{1}{3}(\dot{I}_{aA} + a\dot{I}_{bB} + a^2\dot{I}_{cC}) \text{ amp} \tag{686}$$

$$= \frac{1}{3}(1 - a)\dot{I} = \frac{1}{\sqrt{3}}\dot{I}\underline{/-30^\circ} \text{ amp,} \tag{791}$$

$$\dot{I}_{aA2} = \tfrac{1}{3}(\dot{I}_{aA} + a^2\dot{I}_{bB} + a\dot{I}_{cC}) \text{ amp} \tag{689}$$

$$= \frac{1}{3}(1 - a^2)\dot{I} = \frac{1}{\sqrt{3}}\dot{I}\underline{/30^\circ} \text{ amp.} \tag{792}$$

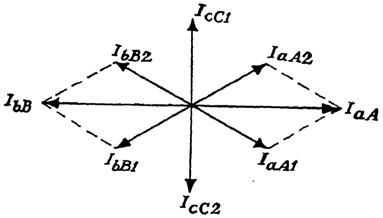

FIG. 222.—Vector diagram for a single-phase line-to-line load on a three-phase system.

These components of current are shown in Fig. 222, where it is seen that

$$\dot{I}_{aA} = \dot{I}_{aA0} + \dot{I}_{aA1} + \dot{I}_{aA2} = \dot{I} \text{ amp,} \tag{793}$$

$$\dot{I}_{bB} = \dot{I}_{bB0} + \dot{I}_{bB1} + \dot{I}_{bB2} = -\dot{I} \text{ amp,} \tag{794}$$

$$\dot{I}_{cC} = \dot{I}_{cC0} + \dot{I}_{cC1} + \dot{I}_{cC2} = 0 \text{ amp.} \tag{795}$$

It is thus demonstrated that single-phase loads on a three-phase system may be expressed in terms of three balanced three-phase systems and treated in exactly the same manner as three-phase loads.

121. Three-phase Power in Terms of Symmetrical Components.—It has been shown in Chap. V that the active power in a single-phase circuit is not expressed as the product of the effective values of E and I but that, if the voltage and current are simple sinusoidal quantities,

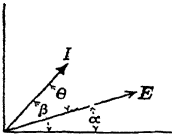

FIG. 223.—Vectors for E and I for one phase, neither in phase with the reference axis.

$$P = EI \cos \theta \text{ watts,} \tag{51}$$

and the reactive power is

$$Q = EI \sin \theta \text{ vars,}$$

where θ is the angle between the vectors E and I. If the expressed in the polar form and neither is in phase with the reference axis, then, since

$$\dot{E} = E\underline{/\alpha} \text{ volts, and } \dot{I} = I\underline{/\beta} \text{ amp,}$$
$$P = EI \cos (\beta - \alpha) \text{ watts.} \tag{335}$$

In an unbalanced three-phase system, the symmetrical components may be expressed in the polar form as

$$\dot{E}_{a0} = E_{a0}\underline{/\alpha_0},\ \dot{E}_{b0} = \dot{E}_{a0} = E_{a0}\underline{/\alpha_0},\ \dot{E}_{c0} = \dot{E}_{a0} = E_{a0}\underline{/\alpha_0} \text{ volts,}$$
$$\dot{E}_{a1} = E_{a1}\underline{/\alpha_1},\ \dot{E}_{b1} = a^2\dot{E}_{a1} = E_{a1}\underline{/\alpha_1 - 120°},$$
$$\dot{E}_{c1} = a\dot{E}_{a1} = E_{a1}\underline{/\alpha_1 + 120°} \text{ volts,}$$
$$\dot{E}_{a2} = E_{a2}\underline{/\alpha_2},\ \dot{E}_{b2} = a\dot{E}_{a2} = E_{a2}\underline{/\alpha_2 + 120°},$$
$$\dot{E}_{c2} = a^2\dot{E}_{a2} = E_{a2}\underline{/\alpha_2 - 120°} \text{ volts,}$$

and

$$\dot{I}_{a0} = I_{a0}\underline{/\beta_0},\ \dot{I}_{b0} = \dot{I}_{a0} = I_{a0}\underline{/\beta_0},\ \dot{I}_{c0} = \dot{I}_{a0} = I_{a0}\underline{/\beta_0} \text{ amp,}$$
$$\dot{I}_{a1} = I_{a1}\underline{/\beta_1},\ \dot{I}_{b1} = a^2\dot{I}_{a1} = I_{a1}\underline{/\beta_1 - 120°},$$
$$\dot{I}_{c1} = a\dot{I}_{a1} = I_{a1}\underline{/\beta_1 + 120°} \text{ amp,}$$
$$\dot{I}_{a2} = I_{a2}\underline{/\beta_2},\ \dot{I}_{b2} = a\dot{I}_{a2} = I_{a2}\underline{/\beta_2 + 120°},$$
$$\dot{I}_{c2} = a^2\dot{I}_{a2} = I_{a2}\underline{/\beta_2 - 120°} \text{ amp.}$$

The active power for each phase is

$$\begin{aligned} P_a = {} & E_{a0}I_{a0} \cos (\beta_0 - \alpha_0) + E_{a1}I_{a1} \cos (\beta_1 - \alpha_1) \\ & + E_{a2}I_{a2} \cos (\beta_2 - \alpha_2) + E_{a0}I_{a1} \cos (\beta_1 - \alpha_0) \\ & + E_{a1}I_{a2} \cos (\beta_2 - \alpha_1) + E_{a2}I_{a0} \cos (\beta_0 - \alpha_2) \\ & + E_{a0}I_{a2} \cos (\beta_2 - \alpha_0) + E_{a1}I_{a0} \cos (\beta_0 - \alpha_1) \\ & + E_{a2}I_{a1} \cos (\beta_1 - \alpha_2) \text{ watts,} \end{aligned} \tag{796}$$

$$\begin{aligned} P_b = {} & E_a I_{a0} \cos(\beta_0 - \alpha_0) + E_{a1}I_{a1} \cos(\beta_1 - \alpha_1) \\ & + E_{a2}I_{a2} \cos(\beta_2 - \alpha_2) + E_{a0}I_{a1} \cos(\beta_1 - \alpha_0 - 120°) \\ & + E_{a1}I_{a2} \cos(\beta_2 - \alpha_1 - 120°) + E_{a2}I_{a0} \cos(\beta_0 - \alpha_2 - 120°) \\ & + E_{a0}I_{a2} \cos(\beta_2 - \alpha_0 + 120°) + E_{a1}I_{a0} \cos(\beta_0 - \alpha_1 + 120°) \\ & + E_{a2}I_{a1} \cos(\beta_1 - \alpha_2 + 120°) \text{ watts,} \end{aligned} \tag{797}$$

$$\begin{aligned} P_c = {} & E_{a0}I_{a0} \cos(\beta_0 - \alpha_0) + E_{a1}I_{a1} \cos(\beta_1 - \alpha_1) \\ & + E_{a2}I_{a2} \cos(\beta_2 - \alpha_2) + E_{a0}I_{a1} \cos(\beta_1 - \alpha_0 + 120°) \\ & + E_{a1}I_{a2} \cos(\beta_2 - \alpha_1 + 120°) + E_{a2}I_{a0} \cos(\beta_0 - \alpha_2 + 120°) \\ & + E_{a0}I_{a2} \cos(\beta_2 - \alpha_0 - 120°) + E_{a1}I_{a0} \cos(\beta_0 - \alpha_1 - 120°) \\ & + E_{a2}I_{a1} \cos(\beta_1 - \alpha_2 - 120°) \text{ watts.} \end{aligned} \tag{798}$$

The total active power is

$$P_t = P_a + P_b + P_c \text{ watts,} \tag{799}$$

but, since

$$\cos(\beta_1 - \alpha_0) + \cos(\beta_1 - \alpha_0 - 120°) + \cos(\beta_1 - \alpha_0 + 120°) = 0,$$

all terms in Eqs. (796) to (798) add to zero except the first three in each equation, and

$$\begin{aligned} P_t &= 3E_{a0}I_{a0} \cos(\beta_0 - \alpha_0) + 3E_{a1}I_{a1} \cos(\beta_1 - \alpha_1) + 3E_{a2}I_{a2} \cos(\beta_2 - \alpha_2) \text{ watts} \\ &= 3E_{a0}I_{a0} \cos\theta_0 + 3E_{a1}I_{a1} \cos\theta_1 + 3E_{a2}I_{a2} \cos\theta_2 \text{ watts,} \end{aligned} \tag{800}$$

where θ_0, θ_1, and θ_2 are the phase angles between the corresponding symmetrical components of current and voltage. Then

$$P_t = 3P_0 + 3P_1 + 3P_2 \text{ watts} \tag{801}$$

$$= P_{00} + P_{01} + P_{02} \text{ watts.} \tag{802}$$

It is seen that each phase sequence produces its own active power and that the total active power is the sum of the active powers due to the separate sequence components.

In a similar manner, the total reactive power in terms of the symmetrical components is defined to be

$$Q_t = 3E_{a0}I_{a0} \sin(\beta_0 - \alpha_0) + 3E_{a1}I_{a1} \sin(\beta_1 - \alpha_1) + 3E_{a2}I_{a2} \sin(\beta_2 - \alpha_2) \text{ vars,} \tag{803}$$

which may be written

$$Q_t = 3Q_0 + 3Q_1 + 3Q_2 \text{ vars.} \tag{804}$$

If the preceding voltages E_a, E_b, and E_c are taken as the impedance drop IZ in the lines of a three-phase system and the currents are I_a, I_b, and I_c in these lines, then

$$\begin{aligned} E_{a0}I_{a0} \cos \theta_0 &= \quad {}^2R_0 \text{ watts,} \\ E_{a1}I_{a1} \cos \theta_1 &= \quad {}^2R_1 \text{ watts,} \\ E_{a2}I_{a2} \cos \theta_2 &= \quad \text{watts,} \end{aligned}$$

and the total heating loss in the three lines is

$$\text{Copper loss} = 3I_{a0}{}^2R_0 + 3I_{a1}{}^2R_1 + 3I_{a2}{}^2R_2 \text{ watts} \tag{805}$$

or each phase sequence of current produces its own line loss. In most cases, these resistances offered to the different sequences of current are alike, or $R_0 = R_1 = R_2$ ohms, but this is not true, in general, for circuits containing rotating machines.

Fundamental Positive-sequence Active Power.—The fundamental positive-sequence active power at the four points of entry (one of which is the neutral) of a three-phase circuit is three times the sum of the products obtained by multiplying the positive-sequence symmetrical component of the potential difference to neutral, the positive-sequence component of current, and the cosine of the angular phase difference between them.

Fundamental Negative-sequence Active Power and Fundamental Zero-sequence Active Power.—The fundamental negative-sequence (and zero-sequence) active power of a three-phase circuit is defined from the definition for fundamental positive-sequence active power by replacing the word positive by the word negative (or zero).

Fundamental Positive-sequence Reactive Power.—The fundamental positive-sequence reactive power at the four points of entry (one of which is the neutral) of a three-phase circuit is three times the sum of the products obtained by multiplying the positive-sequence symmetrical component of the potential difference to neutral, the positive-sequence component of current, and the sine of the angular phase difference between them.

Fundamental Negative-sequence Reactive Power and Zero-sequence Reactive Power.—The fundamental negative-sequence (and zero-sequence) reactive power of a three-phase circuit is defined from the definition for fundamental positive-sequence reactive power by replacing the word positive by the word negative (or zero).

Problems

1-10. In a three-phase system of vector currents the first-order component is $\dot{I}_{A1} = 10 + j0$, the second-order component is $\dot{I}_{A2} = 10 + j6$, and the zero-order component is $\dot{I}_{A0} = 0 - j4$. Find the other components in each system, and draw the complete vector diagrams. Find the original vectors I_A, I_B, and I_C, and draw the vector diagram.

2-10. In a three-phase system, the positive-sequence component is $\dot{I}_{A1} = 60\underline{/30^\circ}$; the negative-sequence component, $\dot{I}_{A2} = 40\underline{/0^\circ}$; and the zero-sequence component, $\dot{I}_{A0} = 35\underline{/-30^\circ}$; find the original system of vectors I_A, I_B, and I_C, and draw the complete vector diagram.

3-10. The line currents in a three-wire three-phase system are $\dot{I}_A = 125 + j50$, $\dot{I}_B = 30 - j68.3$, and $\dot{I}_C = -230 + j18.3$. What are the symmetrical phase components of these line currents? Draw the complete vector diagram showing these components and the original currents.

4-10. The line voltages on a three-phase system are $\dot{E}_{AB} = 63 + j10$, $\dot{E}_{BC} = -35 - j56$, and $\dot{E}_{CA} = -28 + j46$. Find the symmetrical phase components of these line voltages, and draw the complete vector diagram.

5-10. The line currents in a four-wire three-phase system are $\dot{I}_A = 291.3 - j13.4$, $\dot{I}_B = -291.3 - j100$, and $\dot{I}_C = 75 + j188.4$. What are the symmetrical components of these currents? What current flows in the neutral wire? Draw a complete vector diagram.

6-10. The load voltages on a four-wire three-phase system with star-connected loads are $\dot{E}_{NA} = 127\underline{/0^\circ}$, $\dot{E}_{NB} = 114\underline{/250^\circ}$, and $\dot{E}_{NC} = 113\underline{/110^\circ}$; find the symmetrical phase components, and draw the vector diagram.

7-10. Given that $\dot{I}_{A1} = 12\underline{/-60^\circ}$, $\dot{I}_{A2} = 10\underline{/30^\circ}$ and $\dot{I}_{A0} = 5\underline{/0^\circ}$, find I_C graphically.

8-10. Given that $\dot{E}_A = 100\underline{/30^\circ}$, $\dot{E}_B = 80\underline{/270^\circ}$, and $\dot{E}_C = 150\underline{/180^\circ}$, find E_{A2} graphically.

9-10. Three loads are connected in delta to a three-wire three-phase system, the load currents being $\dot{I}_{AB} = 20 + j50$, $\dot{I}_{BC} = 33.3 - j7.7$, and $\dot{I}_{CA} = -53.3 - j42.3$; find the symmetrical phase components of the load and line currents.

10-10. In a four-wire three-phase system, the line currents are $\dot{I}_A = 100 + j0$, $\dot{I}_B = 0 + j100$, and $\dot{I}_C = 0 - j100$, the impedances to neutral being expressed as $\dot{Z}_a = 3 + j10$, $\dot{Z}_b = 10 + j5$, and $\dot{Z}_C = 8 + j6$. Express the line currents in terms of symmetrical phase components. Express the neutral and line voltages in terms of symmetrical phase components. What is the neutral current?

11-10. Three impedances $\dot{Z}_a = 0 - j10$, $\dot{Z}_b = 5 + j8.66$, and $\dot{Z}_c = 5 + j8.66$ are connected in star to a three-wire three-phase system whose line voltages are given by the expressions $\dot{E}_{AB} = -500 - j1{,}500$, $\dot{E}_{BC} = -1{,}049 + j317$, and $\dot{E}_{CA} = 1{,}549 + j1{,}183$. Find the symmetrical phase components of the voltages across the impedances and the line currents.

12-10. An impedance $\dot{Z}_{NA} = 8 + j6$ is connected between line A and the neutral N of a symmetrical four-wire three-phase system having a line voltage of 230 volts. Find the symmetrical phase components of this single-phase current in the three-phase system, using E_{NA} as the reference. Draw the vector diagram.

13-10. An impedance $\dot{Z}_{AB} = 4 + j2$ is connected between the lines A and B of a symmetrical three-wire three-phase 110-volt system. Find the symmetrical phase components of the load and line currents, using E_{AB} as the reference. Draw the vector diagram.

14-10. Three impedances $\dot{Z}_a = 25 + j25$, $\dot{Z}_b = 17.32 + j10$, and $\dot{Z}_c = 17.32 + j10$ are connected in star to a balanced four-wire three-phase system having 2,300 volts to neutral. Find the symmetrical phase components of the line currents. What current flows in the neutral wire? Draw the complete vector diagram.

15-10. The load voltages of a four-wire three-phase system with loads connected between the lines and neutral are $\dot{E}_{NA} = 165\underline{/0°}$, $\dot{E}_{NB} = 106\underline{/310°}$, and $\dot{E}_{NC} = 130\underline{/100°}$. Find the symmetrical phase components of the load and line voltages.

16-10. Given in a three-phase system that $\dot{I}_A = 10\underline{/0°}$, $\dot{I}_B = 8\underline{/270°}$, and $\dot{I}_C = 12\underline{/135°}$; find graphically the symmetrical phase components.

17-10. An impedance $\dot{Z}_{NA} = 10 + j0$ is connected between line A and neutral N of a symmetrical four-wire three-phase 240-volt system, and an impedance $\dot{Z}_{NB} = 8 + j6$ is connected between line B and the neutral; resolve the load currents into the symmetrical phase components.

18-10. An impedance $\dot{Z}_a = 15 + j0$ is connected between line A and the neutral N of a balanced 2,200-volt four-wire three-phase system, and a second impedance $\dot{Z}_{CA} = 25 - j25$ is connected between lines A and C of the same system; resolve each single-phase load into its symmetrical phase components, and obtain the symmetrical phase components of the line currents for the phase sequence ABC.

19-10. (*a*) Three wattmeters are connected with their current coils in lines A, B, and C of Prob. 10-10 and with their potential coils from their respective lines to the common point N. What is the reading of each wattmeter? (*b*) If these wattmeters have potential coils of equal resistance, what will be their indications when the potential coils are connected in star and disconnected from the neutral?

CHAPTER XI

COMPARISON OF THE AMOUNT OF CONDUCTOR MATERIAL REQUIRED FOR DIFFERENT SYSTEMS OF TRANSMISSION AND DISTRIBUTION

A comparison of the relative weights of conductor material required for different systems of transmission and distribution can, in some cases, be made the basis of the selection of the system to be used. It should be remembered, however, that there are other controlling factors of design which influence the total cost of the installation and which also must be weighed, such as the mechanical strength of the conductor, number and cost of the insulators, space required and available for the conductors, equipment available to supply power to the system, and motors already on hand requiring some definite system. The following analysis will assume the same controlling factor of voltage in each case, the analysis on the basis of difference of voltage being reserved for more advanced treatment.

The following cases will be assumed as the basis of the comparison:

Case A. The same emf to neutral in all cases.

Case B. The same emf between line wires.

Case C. The same emf to neutral polyphase as between lines single phase.

The first of these comparisons is on the basis of the same insulation strain and applies particularly to high-voltage lines, the last of the comparisons is on the basis of the use of standard apparatus on medium- and low-potential systems. Each of the systems assumes the use of standardized voltages for generators, transformers, motors, etc.

122. Comparison of Conductor Weight on the Basis of the Same Emf to Neutral.—Assume the same power delivered P, the same per cent power loss due to the resistance of the conductors, and balanced conditions of load.

1. *Single-phase System* (Fig. 224).—In a single-phase system, the current flowing in each line wire, assuming nonreactive load, is

$$I = \frac{P}{2E} \text{ amp} \qquad (806)$$

and the total power loss in the two wires becomes

$$P.L. = I^2 \times 2R_1 \text{ watts},$$

where R_1 is the resistance of each conductor of the single-phase line.

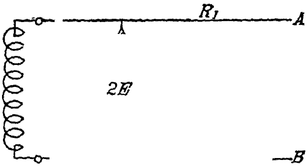

FIG. gle-
phase E to
neutral.

Substituting the value of I from Eq. (806),

$$P.L. = \frac{P^2}{4E^2} \times 2R_1 \text{ watts}$$

$$= \frac{P^2R_1}{2E^2} \text{ watts.} \qquad (807)$$

Hence, the resistance of each conductor is

$$R_1 = \frac{P.L.}{P} \times \frac{2E^2}{P} = \frac{2E^2}{P} \times \text{per cent } P.L. \text{ ohms}, \qquad (808)$$

since the per cent power loss is

$$\text{per cent } P.L. = \frac{P.L.}{P}.$$

2. *Four-wire Two-phase System* (Fig. 225).—In the four-wire two-phase system, the current in each wire becomes

$$I = \frac{P}{4E} \text{ amp}, \qquad (809)$$

the power loss being

$$P.L. = I^2 \times 4R_2 \text{ watts},$$

since there are four equal wires, where R_2 is the resistance of each wire of the two-phase line. Substituting the value of I from Eq. (809),

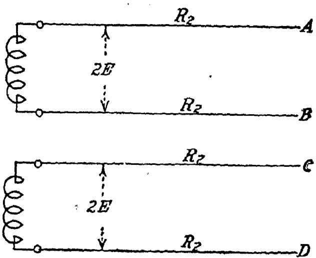

FIG. 225.—A four-wire two-phase circuit, with a voltage E to neutral.

$$P.L. = \frac{P^2}{16E^2} \times 4R_2 \text{ watts}$$
$$= \frac{P^2R_2}{4E^2} \text{ watts.} \tag{810}$$

The resistance of each conductor is

$$R_2 = \frac{P.L.}{P^2} \times 4E^2 \text{ ohms}$$
$$= \frac{4E^2}{P} \times \text{per cent } P.L. \text{ ohms.} \tag{811}$$

Using the resistance of each conductor of the single-phase line as a standard, the relative resistance of each conductor of the four-wire two-phase line becomes

$$\frac{R_2}{R_1} = \frac{\frac{P.L.}{P^2} \times 4E^2}{\frac{P.L.}{P^2} \times 2E^2} = 2.00, \tag{812}$$

or each wire has twice the resistance of one wire of the single-phase circuit. Since the resistivity and length of each conductor are assumed the same in each case, the cross section of each wire for the two-phase circuit will be half that for the single-phase circuit. The relative total weight of copper required in each case will be found as follows, since the weight varies inversely with the resistance, *i.e.*, directly with the cross section:

Size of each wire, single phase	100 per cent
Size of each wire, two phase	50 per cent
Number of wires, single phase	2
Number of wires, two phase	4
Total weight of copper, single phase	W_1
Total weight of copper, two phase	W_2

Then the relative total weights of copper on the basis chosen are

$$\frac{W_2}{W_1} = \frac{4 \times 0.50}{2 \times 1.00} = 1.00. \tag{813}$$

Hence, both systems require the same total weight of copper. The two-phase system will require twice as many insulators and more space on the cross arms for overhead distribution, but it has the advantage in respect to motor performance of being a polyphase system.

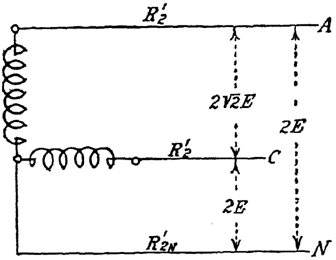

Fig. 226.—A three-wire two-phase circuit, with a voltage $2E$ to the common return.

3. *Three-wire Two-phase System* (Fig. 226).—The current required in each outside wire is the same as for the four-wire two-phase system, or

$$I = \frac{P}{4E} \text{ amp,} \tag{809}$$

and in the common-return wire

$$I_N = \sqrt{2}\, I = \frac{\sqrt{2}P}{4E} \text{ amp.} \tag{814}$$

The power loss in the three conductors will be

$$P.L. = I^2 \times 2R_2' + I_N{}^2 R_{2_N}' \text{ watts,} \tag{815}$$

where R_2' is the resistance of each outside wire and R_{2_N}' is the resistance of the common-return wire. The system may be designed (*a*) either on the basis of the same I^2R loss per unit length in all three conductors or (*b*) on the basis of the same current density in amperes per square inch of cross section for the common-return wire as for the outside wires.

a. Basis of Same I^2R Loss per Unit Length in All Three Conductors.—Equating the loss in the neutral wire to the loss in one outside conductor,

$$I^2R_2' = I_N{}^2R_{2_N}' = 2I^2R_{2_N}' \text{ watts,} \tag{816}$$

the relative resistance becomes

$$R_{2_N}' = \frac{R_2'}{2} \text{ ohms} \tag{817}$$

or the common-return wire must have twice the cross section of each of the two outside wires. Hence, from Eq. (815), the total loss will be

$$P.L. = 2I^2R_2' + \frac{2I^2R_2'}{2} = 3I^2R_2' \text{ watts.} \tag{818}$$

From Eqs. (809) and (818), the total power loss becomes

$$P.L. = \frac{3P^2R_2'}{16E^2} \text{ watts,} \tag{819}$$

the resistance of each outside conductor,

$$R_2' = \frac{16E^2}{3P} \times \text{per cent } P.L. \text{ ohms,} \tag{820}$$

and the resistance of the common-return wire,

$$R_{2_N}' = \frac{8E^2}{3P} \times \text{per cent } P.L. \text{ ohms.} \tag{821}$$

From Eqs. (820) and (821), compared with Eq. (808),

$$\frac{R_2'}{R_1} = \frac{\dfrac{16E^2}{3P} \times \text{per cent } P.L.}{\dfrac{2E^2}{P} \times \text{per cent } P.L.} = \tfrac{8}{3} = 2.67, \tag{822}$$

and

$$\frac{R_{2_N}'}{R_1} = \frac{\dfrac{8E^2}{3P} \times \text{per cent } P.L.}{\dfrac{2E^2}{P} \times \text{per cent } P.L.} = \tfrac{4}{3} = 1.33. \tag{823}$$

The relative weights and cross sections of the wires will be

$$\frac{w_2'}{w_1} = \frac{3}{8} = 0.375, \text{ or } 37.5 \text{ per cent,} \tag{824}$$

and

$$\frac{w_{2_N}'}{w_1} = \frac{3}{4} = 0.75, \text{ or } 75 \text{ per cent,} \tag{825}$$

of that for a single-phase line, where w_1 is the weight per unit length for a single-phase line, w_2' for the outer wires, and w_{2_N}' for the common-return wire of the two-phase line. Since there are two outside wires and one neutral wire, the relative total weight to the weight of the single-phase conductors will be

$$\frac{W_2'}{W_1} = \frac{2 \times \tfrac{3}{8} + \tfrac{3}{4}}{2 \times 1} = \frac{3}{4} = 0.75, \tag{826}$$

or only 75 per cent as much copper is required as for either the single-phase system or for the four-wire two-phase system.

b. Basis of the Same Current Density in Each Wire.—This condition may be expressed as a ratio in comparing the relative resistances, or

$$\frac{R_{2_N}''}{R_2''} = \frac{I_2}{I_{2_N}} = \frac{I_2}{\sqrt{2}I_2} = \frac{1}{\sqrt{2}}$$

and

$$R_{2_N}'' = \frac{R_2''}{\sqrt{2}}, \tag{827}$$

where R_2'' is the resistance of each outside wire and R_{2N}'' the resistance of the common-return wire for this condition. The total power loss becomes

$$\begin{aligned} P.L. &= I_2^2(2R_2'') + (I_N^2)R_{2_N}'' \text{ watts} \\ &= I_2^2R_2''(2 + \sqrt{2}) \text{ watts} \\ &= \frac{P^2}{16E^2}R_2''(2 + \sqrt{2}) \text{ watts.} \end{aligned} \tag{828}$$

Solving for R_2'',

$$\begin{aligned} R_2'' &= \frac{16E^2}{P^2(2 + \sqrt{2})} \times P.L. \text{ ohms} \\ &= \frac{16E^2}{3.41P} \times \text{per cent } P.L. \text{ ohms,} \end{aligned} \tag{829}$$

and

$$R_{2_N}'' = \frac{16E^2}{3.41\sqrt{2}P} \times \text{per cent } P.L. \text{ ohms.} \tag{830}$$

From Eqs. (808), (829), and (830), the relative resistances to the single-phase resistance become

$$\frac{R_2''}{R_1} = \frac{\frac{16E^2}{3.41P} \times \text{per cent } P.L.}{\frac{2E^2}{P} \times \text{per cent } P.L.} = \frac{8}{3.41} = 2.35 \tag{831}$$

and

$$\frac{R_{2_N}''}{R_1} = \frac{\frac{16E^2}{3.41\sqrt{2}P} \times \text{per cent } P.L.}{\frac{2E^2}{P} \times \text{per cent } P.L.} = \frac{8}{3.41\sqrt{2}} = 1.67. \tag{832}$$

The relative weight and cross section of each wire to the weight of the single-phase conductor are

$$\frac{w_2''}{w_1} = \frac{3.41}{8} = 0.426 \tag{833}$$

and

$$\frac{w_{2_N}''}{w_1} = \frac{3.41\sqrt{2}}{8} = 0.602. \tag{834}$$

Comparing the total weight of copper for this case to the single-phase system,

$$\frac{W_2''}{W_1} = \frac{2 \times 0.426 + 0.602}{2 \times 1} = 0.727, \tag{835}$$

or only 72.7 per cent as much total copper is required as was required for the single-phase system.

4. *Three-wire Three-phase System* (Fig. 227).—The emf between wires for this system will be

$$E_L = \sqrt{3}E \text{ volts,}$$

and the current in each wire

$$I = \frac{P}{3E} \text{ amp,}$$

where E is the emf to neutral. The total power loss is

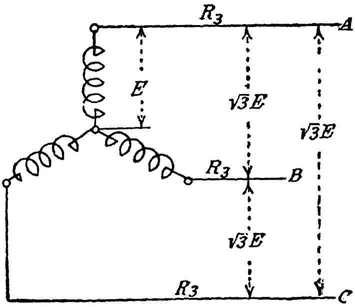

FIG. 227.—A three-wire three-phase circuit, with voltage E to neutral.

$$P.L. = I^2 \times \qquad = 3I^2R_3 \text{ watts,} \tag{836}$$

since there are three conductors of equal size. Substituting the value of I,

$$P.L. = \frac{3P^2}{9E^2}R_3 = \frac{P^2}{3E^2}R_3 \text{ watts,} \tag{837}$$

and the resistance of each wire becomes

$$R_3 = \frac{3E^2}{P} \times \text{per cent } P.L. \text{ ohms.} \tag{838}$$

From Eqs. (808) and (838), the relative resistances become

$$\frac{R_3}{R_1} = \frac{\frac{3E^2}{P} \times \text{per cent } P.L.}{\frac{2E^2}{P} \times \text{per cent } P.L.} = 1.50 \tag{839}$$

and the relative cross sections and weights become

$$\frac{w_3}{w_1} = \frac{R_1}{R_3} = \frac{2}{3} = 0.667 \text{ or } 66.7 \text{ per cent} \tag{840}$$

and

$$\frac{W_3}{W_1} = \frac{3 \times \frac{2}{3}}{2 \times 1} = 1, \tag{841}$$

or the same total weight of copper is required for three-phase as for single-phase or two-phase four-wire distribution on the basis of the same emf to neutral.

5. *Four-wire Three-phase System* (Fig. 228).—Assume that the neutral wire has the same cross section as each outside wire and that the loads are balanced. The currents and cross sections of the wires will be the same as for the three-wire three-phase circuit, or

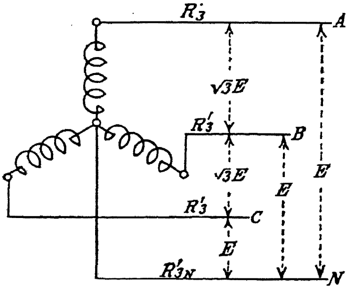

FIG. 228.—A four-wire three-phase circuit, with voltage E to neutral.

$$R_3' = R_3 = \frac{3E^2}{P} \times \text{per cent } P.L. \text{ ohms}, \tag{838}$$

and the relative resistance is

$$\frac{R_3'}{R_1} = 1.50, \tag{839}$$

where R_3' is the resistance of each conductor. The total relative weights of copper become

$$\frac{W_3'}{W_1} = \frac{4 \times \frac{2}{3}}{2 \times 1} = 1.33, \tag{842}$$

where W_3' is the total weight of conductor material, or 33 per cent more copper is required than for the three-wire three-phase system.

The results of the above comparisons are given in Table IV.

TABLE IV.—COMPARISON OF RESISTANCE AND TOTAL WEIGHT OF MATERIAL IN CONDUCTORS FOR DIFFERENT SYSTEMS OF DISTRIBUTION ON THE BASIS OF THE SAME EMF TO NEUTRAL, THE SAME POWER TRANSMITTED, AND THE SAME PER CENT POWER LOSS

	Single-phase system	Two-phase, four-wire system	Two-phase, three-wire system, same I^2R	Two-phase, three-wire system, same I per square inch	Three-phase, three-wire system	Three-phase, four-wire system
1. Voltage to neutral	E	E	E	E	E	E
2. Voltage between lines	$2E$	$2E$	$2E$	$2E$	$\sqrt{3}E$	$\sqrt{3}E$
3. Line current	$\frac{P}{2E}$	$\frac{P}{2E}$	$\frac{P}{2E}$	$\frac{P}{2E}$	$\frac{P}{3E}$	$\frac{P}{3E}$
4. $\frac{\text{Resistance per wire}}{\text{Per cent power loss}} = \frac{R}{P.L.}$	$\frac{2E^2}{P}$	$\frac{4E^2}{P}$	$\frac{16E^2}{3P}$	$\frac{16E^2}{3.41P}$	$\frac{3E^2}{P}$	$\frac{3E^2}{P}$
5. $\frac{\text{Resistance outside wire}}{\text{Resistance single-phase}}$	1.00	2.00	2.67	2.35	1.50	1.50
6. $\frac{\text{Resistance neutral wire}}{\text{Resistance single-phase}}$	.....	.....	1.33	1.67	.....	1.50
7. $\frac{\text{Weight outside wire}}{\text{Weight single-phase}}$	1.00	0.50	0.375	0.426	0.667	0.667
8. $\frac{\text{Weight neutral wire}}{\text{Weight single-phase}}$	.....	.....	0.75	0.602	.....	0.667
9. Number of outer wires	2	4	2	2	3	3
10. Number of neutral wires	.....	.....	1	1	.....	1
11. $\frac{\text{Total weight of material}}{\text{Total weight single-phase}}$	1.00	1.00	0.75	0.727	1.00	1.33

On the basis of this analysis, it is seen that the two-wire single-phase, the four-wire two-phase, and the three-wire three-phase systems require the same total weight of copper for the same distance of transmission, the same amount of power to be transmitted, a given per cent loss, and the same voltage to neutral. It is also seen that upon the same basis of comparison the three-wire two-phase system requires only 75 or 72.7 per cent as much copper per unit length as the preceding systems, the design with the same current density in the common-return wire requiring the least copper. The four-wire three-phase system, used for convenience, requires 33 per cent more copper than the single-phase or the three-wire three-phase systems, the extra copper being used in the neutral wire.

123. Comparison on the Basis of the Same Emf between Outside Wires.—The same assumptions as to length of line, power transmitted, and per cent power loss are used as in the previous case, but the new assumption is made that the emf between outside wires is the same for each system. Under these assumptions, the analysis will give the same relative results for the single-phase and the two-phase systems as in the previous case. However, different results will be found for the three-phase systems.

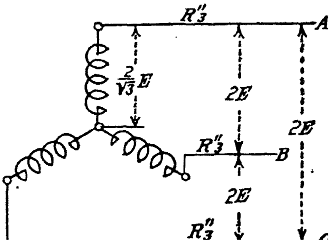

FIG. 229.—A three-wire three-phase circuit, with a voltage $2E$ to neutral.

1. *Three-wire Three-phase System* (Fig. 229).—With an emf of $2E$ between wires, the emf to neutral will be and the current in each wire

$$I = \frac{P}{3E_N} = \frac{P}{2\sqrt{3}E} \text{ amp.} \tag{843}$$

The power loss is

$$\begin{aligned} P.L. &= I^2(3R_3'') = \frac{P^2}{12E^2} \times 3R_3'' \text{ watts} \\ &= \frac{P^2R_3''}{4E^2} \text{ watts,} \end{aligned} \tag{844}$$

and the resistance of each wire becomes

$$R_3'' = \frac{4E^2}{P} \times \text{per cent } P.L. \text{ ohms.} \tag{845}$$

From Eq. (845), the relative resistance to the single-phase resistance becomes

$$\frac{R_3''}{R_1} = \frac{\frac{4E^2}{P} \times \text{per cent } P.L.}{\frac{2E^2}{P} \times \text{per cent } P.L.} = 2.00, \tag{846}$$

or each wire has twice the resistance of the corresponding wire in the single-phase circuit. The relative weights and cross sections of individual conductors become

$$\frac{w_3''}{w_1} = \frac{R_1}{R_3''} = \frac{1}{2} = 50 \text{ per cent,} \tag{847}$$

and the relative total weights of copper,

$$\frac{W_3''}{W_1} = \frac{3 \times \frac{1}{2}}{2 \times 1} = \frac{3}{4} = 0.75. \tag{848}$$

That is, the three-wire three-phase system requires only 75 per cent as much total copper as is required for the single-phase system or for the four-wire two-phase system.

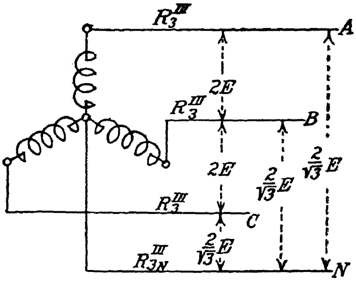

FIG. 230.—A four-wire three-phase circuit, with a voltage $2E$ between lines.

2. *Four-wire Three-phase System* (Fig. 230).—On the assumption that the wires are the same size and that there is the same emf between outside wires as before, the resistance for each wire becomes

$$R_3''' = \frac{4E^2}{P} \times \text{per cent } P.L. \text{ ohms,} \tag{845}$$

being the same value as that found for the three-wire three-phase system, each wire having twice the resistance or half the cross section and weight as for the single-phase system. The relative total weights become

$$\frac{W_3'''}{W_1} = \frac{4 \times \frac{1}{2}}{2 \times 1} = 1.00 = 100 \text{ per cent,} \tag{849}$$

or this system requires the same total amount of copper as for the single-phase or the four-wire two-phase systems.

124. Comparison on the Basis of the Same Emf to Neutral for the Four-wire Three-phase System as between Wires Single Phase.—This is the ordinary primary power distribution for lighting and power customers in the larger cities where the same standard primary voltage is to be used on the transformers as for the single-phase circuits, the primaries being connected between the neutral and each outside wire. Assume the same emf $2E$ to neutral as for the previous considerations and the connection, as shown in Fig. 231. The emf between outside lines will be

$$E_{\text{line}} = 2\sqrt{3}E \text{ volts,} \tag{850}$$

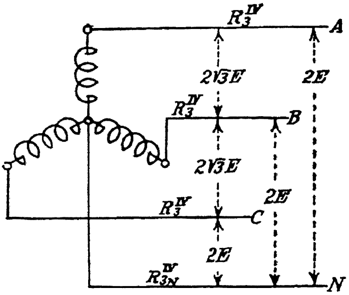

FIG. 231.—A four-wire three-phase circuit, with a voltage 2*E* to neutral.

and the line currents

$$I = \frac{P}{3E_N} = \frac{P}{6E} \text{ amp.} \tag{851}$$

The power loss will be

$$\begin{aligned} P.L. &= I^2 \times 3R_3^{\text{IV}} \text{ watts} \\ &= \frac{P^2}{36E^2} \times 3R_3^{\text{IV}} \text{ watts} \\ &= \frac{P^2}{12E^2} R_3^{\text{IV}} \text{ watts,} \end{aligned} \tag{852}$$

by substitution of the value of I from Eq. (857). Solving Eq. (852) for resistance,

$$R_3^{\text{IV}} = \frac{12E^2}{P} \times \text{per cent } P.L. \text{ ohms.} \tag{853}$$

From Eqs. (808) and (853), the relative resistances become

$$\frac{R_3^{\text{IV}}}{R_1} = \frac{\dfrac{12E^2}{P} \times \text{per cent } P.L.}{\dfrac{2E^2}{P} \times \text{per cent } P.L.} = 6.00; \tag{854}$$

and the relative weights of the wires per unit length, or the cross sections, will be

$$\frac{w_3^{\text{IV}}}{w_1} = \frac{1}{6} = 0.167 \text{ or } 16.7 \text{ per cent.} \tag{855}$$

The relative total weights of material required are

$$\frac{W_3^{\text{IV}}}{W_1} = \frac{4 \times 1/6}{2 \times 1} = \frac{1}{3} = 33.3 \text{ per cent,} \tag{856}$$

or this system requires only one-third the total weight of material in the line wires as is required for the single-phase system. Advantage may be taken of this saving for long primary lines or in heavily loaded sections of a city, but for smaller loads the mechanical strength required in each span will not allow the reduction of the size of conductor represented by the above comparison.

The results obtained by these considerations are presented in Table V.

TABLE V.—COMPARISON OF RESISTANCE AND TOTAL WEIGHT OF MATERIAL IN CONDUCTORS FOR DIFFERENT SYSTEMS OF DISTRIBUTION ON THE BASIS OF THE SAME EMF BETWEEN WIRES, THE SAME POWER TRANSMITTED, AND THE SAME PER CENT POWER LOST

	Two-wire, single-phase system	Four-wire, two-phase system	Three-wire, two-phase system, same I^2R	Three-wire, two-phase system, same current per square inch	Three-wire, three-phase system	Four-wire, three-phase system	Four-wire, three-phase system, same E to neutral as between wires single-phase
1. Voltage to neutral..........	E	E	E	E	$\frac{2E}{\sqrt{3}}$	$\frac{2E}{\sqrt{3}}$	$2E$
2. Voltage between lines.......	$2E$	$2E$	$2E$	$2E$	$2E$	$2E$	$2\sqrt{3}E$
3. Line current...............	$\frac{P}{2E}$	$\frac{P}{4E}$	$\frac{P}{4E}$	$\frac{P}{4E}$	$\frac{P}{2\sqrt{3}E}$	$\frac{P}{2\sqrt{3}E}$	$\frac{P}{6E}$
4. $\frac{\text{Resistance of each wire}}{\text{Per cent power loss}}$.....	$\frac{2E^2}{P}$	$\frac{4E^2}{P}$	$\frac{16E^2}{3P}$	$\frac{16E^2}{3.41P}$	$\frac{4E^2}{P}$	$\frac{4E^2}{P}$	$\frac{12E^2}{P}$
5. $\frac{\text{Resistance of outside wire}}{\text{Resistance single-phase}}$...	1.00	2.00	2.67	2.35	2.00	2.00	6.00
6. $\frac{\text{Resistance of neutral wire}}{\text{Resistance single-phase}}$..	....	....	1.33	1.67	.....	2.00	6.00
7. $\frac{\text{Weight of outside wire}}{\text{Weight single-phase}}$......	1.00	0.50	0.375	0.426	0.50	0.50	0.167
8. $\frac{\text{Weight of neutral wire}}{\text{Weight single-phase}}$......	....	....	0.75	0.602	.....	0.50	0.167
9. Number of line wires.......	2	4	2	2	3	3	3
10. Number of neutral wires....	....	....	1	1	.....	1	1
11. $\frac{\text{Total weight of material}}{\text{Total weight single-phase}}$...	1.00	1.00	0.750	0.727	0.75	1.00	0.333

CHAPTER XII

CONSTANTS OF TRANSMISSION LINES

In the transmission and distribution of electrical energy over metallic conductors by either direct or alternating current, varying amounts of current flow along the conductors and a voltage must also be maintained between the conductors. The current produces power losses in each conductor proportional to the resistance of the conductor and to the square of the current flowing; it also produces a voltage drop along each conductor proportional to the resistance of the conductor and to the current flowing; and it also sets up a magnetic field, with lines in circular paths nearly concentric with each conductor, the effect of which is measured by the units of inductance and reactance. The emf between conductors sets up an electrostatic field with lines in circular paths at right angles with the paths of the electromagnetic lines and terminating at the surface of the conductors at right angles with the conductor surface. This latter effect is measured by the capacitance or the capacitive susceptance of the conductor as one wire of the line.

With high-voltage transmission, there are also direct losses produced by leakage currents over, or through, the insulation of the lines and losses from wire to wire caused by ionization or by electrical stress in the dielectric separating the wires. This is known as "corona loss" when occurring in the air and as "dielectric hysteresis" when occurring in solid dielectrics, as in cables. These losses will be replaced by parallel conductance paths as will be seen when transmission-line operating characteristics are considered in later chapters.

The resistance, inductive reactance, capacitive susceptance, and the conductance replacing the wire-to-wire losses are known as the "constants" of the transmission line. It is the purpose of this chapter to discuss these constants and to derive the equations for their calculation in so far as they lie within the scope of this text. In all cases, a discussion will be given of the dimensions and other physical conditions upon which these constants depend.

125. Resistance of a Conductor for Direct Current.—In a d-c circuit, it has been found that the resistance of a conductor depends upon the specific resistance of the material and the length and cross section of the conductor. This may be expressed

$$R = \rho\frac{l}{A}\text{ohms,} \tag{857}$$

where R is the resistance of the conductor in ohms;

ρ is the specific resistance of the material;

l is the length of the conductor; and

A is its cross section.

The value of ρ, for a given temperature, depends upon the units in which R, l, and A are expressed and upon the quality of the material. When R is in ohms, l is in feet, and A is in circular mils, the value of ρ is about 10.4 ohms per mil foot at 20°C. for copper of 100 per cent conductivity. When l is expressed in centimeters, A in square centimeters, and R in abohms, the value of ρ in abohms per centimeter cube is about 1,724 for copper, 2,800 for aluminum, and 10,000 for iron and steel.

The value of ρ, and in consequence the value of R, changes with the temperature of the conductor such that

$$R_t = R_0(1 + \alpha t) \text{ ohms,} \tag{858}$$

where R_t is the resistance at any temperature;

R_o is the resistance at 0°C.;

α is the temperature coefficient of resistance; and

t is the temperature of the wire, °C.

From Eq. (858) it is seen that

$$R_o = \frac{R_{t_1}}{1 + \alpha t_1} = \frac{R_{t_2}}{1 + \alpha t_2} = \ . \ . \ . \text{ ohms,} \tag{859}$$

where R_{t_1} is the resistance at any temperature t_1. Dividing both sides of Eq. (859) by $1/\alpha$,

$$\frac{R_o}{\frac{1}{\alpha}} = \frac{R_{t_1}}{\frac{1}{\alpha} + t_1} = \frac{R_{t_2}}{\frac{1}{\alpha} + t_2} = \ . \ . \ . \ . \tag{860}$$

In the "Standardization Rules" of the American Institute of Electrical Engineers, the value of $1/\alpha$ for copper is defined as

$$t + \frac{1}{\alpha} = 274.5 \tag{861}$$

at 40°C., or

$$\frac{1}{\alpha} = 234.5^{1} \tag{862}$$

at 0°C. The value of α is not a constant for copper, and the above equations yield results with sufficient accuracy only within ordinary operating temperatures of the material. Equation (860) offers a convenient slide-rule calculation method for computing the resistance at any temperature, or the rise in temperature corresponding to any change in resistance.

126. Resistance of a Conductor Carrying an Alternating Current.—The resistance of a conductor carrying an alternating current is larger than for the same conductor at the same temperature when carrying direct current. This increased resistance is due to the decrease in effective cross section of the conductor

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[1] **Standard Annealed Copper.**—*a. General.*—The following shall be taken as normal values of standard annealed copper.

b. Resistance.—At a temperature of 20°C., the resistance of a wire of standard annealed copper 1 m in length and of a uniform section of 1 sq mm is $\frac{1}{58}$ ohm = 0.17241 . . . ohm.

c. Density.—At a temperature of 20°C., the density of standard annealed copper is 8.89 g per cubic centimeter.

d. Temperature Coefficient of Resistance.—At a temperature of 20°C., the "constant mass" temperature coefficient of resistance of standard annealed copper, measured between two potential points rigidly fixed in the wire, is 0.00393 = 1/254.45 per degree centigrade.

e. Resistance of Standard Annealed Copper at 20°C.—As a consequence it follows from (*a*) and (*b*) that, at a temperature of 20°C., the resistance of a wire of standard annealed copper of uniform section, 1 m in length and weighing 1 g, is $\frac{1}{58} \times 8.89 = 0.15328$ . . . ohm.

See International Electrical Commission Publication 28, "International Standard Resistance for Copper," March, 1914.

Paragraphs (*b*) and (*c*) define what are sometimes called "volume resistivity" and "mass resistivity," respectively. This may be expressed in other units as follows:

Volume Resistivity.—1.7241 microhms-centimeter (microhms in a centimeter cube) at 20°C.

Mass Resistivity.—875.20 ohms (mile-pound) at 20°C.

For detailed specifications of commercial copper, see the Standard Specifications of the American Society for Testing Materials.

by the nonuniformity of the current density over the cross section of the wire, defined by the terms "skin effect" and "proximity effect." The skin effect is the increase in resistance of the conductor because of the fact that current density in the outer elements is higher than in the center of the conductor owing to the alternating flux produced within the conductor by the current flowing in the conductor; and the proximity effect is the increase in resistance due to the crowding of the current into the elements on one side of a conductor because of the alternating flux passing through the conductor and produced by the current flowing in the adjacent conductors. At ordinary frequencies, the skin effect may be neglected for copper or aluminum conductors less than 250,000 circular mils. For ordinary frequencies and spacing of conductors in transmission lines, the proximity effect may be entirely neglected. The latter effect is more noticeable in cables and in the windings of a-c machines.

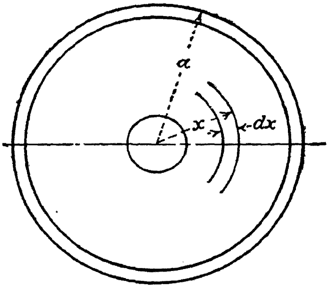

FIG. 232.—Cross section of a wire.

127. Skin Effect.—Let the outer circle in Fig. 232 represent the cross section of a circular conductor of radius *a* carrying a current in a direction away from the observer. With direct current, each unit element, or concentric tube, of the conductor will carry a current $i = I/A$. The central current element will set up a magnetic flux in the conductor outside this element, decreasing in amount toward the surface. Each succeeding element will add to this mmf and flux in the space outside until, at the surface, the entire amount of current in the conductor will be useful in producing a uniform mmf outside the conductor (developed later under inductance of the conductor). With alternating current flowing, the flux within the conductor produced by the current in the central element interlinks the short-circuited turn formed by this element and an equal element of current at the surface of the conductor. By Faraday's law, this alternating flux will induce an emf that will tend to send a current toward the observer at the center and away from the observer at the surface of the conductor, assuming that the original instantaneous current was away from the observer over the whole cross section. This element of current will superimpose on the current already flowing in the conductor in such a manner as to subtract from the current already flowing at the center and to add to that flowing at the surface and to each element of current from the surface toward the center. The next element of current in a unit area surrounding the current at the center will increase the current at the surface and decrease that in its own element, etc., as shown in Fig. 233 by the line *HK*. It is seen that the effect of the flux set up by the alternating current is to crowd the current toward the outer elements of the conductor and that this effect will be greater for large conductors than for small, for higher frequencies, for higher specific conductivities of material, and for conductors made of magnetic materials. In the case of conductors made of magnetic materials, the permeability varies with the flux density, and other losses are introduced owing to hysteresis of the material.

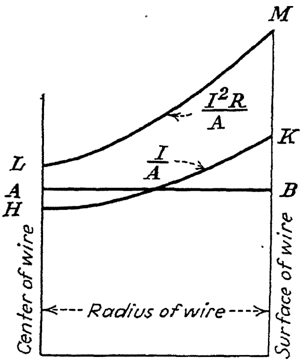

FIG. 233.—Current distribution and distribution of the unit losses across a wire carrying an alternating current.

128. Some Common Equations for Skin Effect.—The losses in each concentric element of the area due to resistance are proportional to the square of the current in that element and to the resistance of that element. Hence, the equivalent resistance of a conductor carrying alternating current will be greater than when carrying a direct current of the same amount, since the mean of the square of the alternating currents is greater than the square of their mean values, *i.e.*, the equivalent resistance of the conductor carrying alternating current is

$$R = \frac{\sum i^2 \frac{R_d}{A}}{I^2} \text{ ohms,} \qquad (863)$$

where i is the effective current in each element;

R_d/A is the resistance of each element due to direct current; and

I is the total current in the wire, *i.e.*, the algebraic mean of the various elements of current.

For ordinary frequencies of transmission, the resistance of the conductor carrying an alternating current may be found as follows:

Let R = the resistance of the wire in abohms for alternating-currents,
R_d = the resistance of the wire in abohms for direct currents,
$\omega = 2\pi f$,
l = the length of the wire, cm,
μ = the permeability of the material (assumed constant),
ρ = the specific resistance in c.g.s. units,
d = the diameter of the wire, cm.

Then, for ordinary frequencies of power transmission and ordinary sizes of conductor,

$$R = R_d\left[1 + \frac{\omega^2 l^2 \mu^2}{12 R_d^2} - \frac{\omega^4 l^4 \mu^4}{180 R_d^4} + \cdots\right] \text{abohms,} \tag{864}$$

or

$$R = R_d\left[1 + \frac{f^2 \pi^4 d^4 \mu^4}{48 \rho^2} - \frac{f^4 \pi^8 d^8 \mu^8}{2{,}880 \rho^4} + \cdots\right] \text{abohms.} \tag{865}$$

For nonmagnetic materials,

$$R = R_d\left[1 + \frac{\omega^2 l^2}{12 R_d^2} - \frac{\omega^4 l^4}{180 R_d^4} + \cdots\right] \text{abohms,} \tag{866}$$

or

$$R = R_d\left[1 + \frac{f^2 \pi^4 d^4}{48 \rho^2} - \frac{f^4 \pi^8 d^8}{2{,}880 \rho^4} + \cdots\right] \text{abohms.} \tag{867}$$

The solution given by Lord Kelvin[1] and applied to high-frequency circuits used the Bessel functions in the solution of its equations as follows:

$$\frac{R}{R_d} = \frac{x}{2}\,\frac{(ber\,x\; bei'x - ber'x\; bei\,x)}{(ber'x)^2 + (bei'x)^2}, \tag{868}$$

where

$$ber\,x = 1 - \frac{x^4}{2^2 \cdot 4^2} + \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \cdots, \tag{869}$$

$$bei\,x = \frac{x^2}{2^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{x^{10}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - \cdots, \tag{870}$$

$$x = \pi d\sqrt{\frac{2\mu f}{\rho}} \tag{871}$$

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[1] "Mathematical and Physical Papers," Vol. III, p. 291.

and where the new terms are Bessel functions and *ber'* and *bei'* are the first derivatives of *ber* and *bei*, respectively, in respect to x. The very complicated expression of Eq. (868) reduces, approximately, to the equation of a straight line[1] for values of x exceeding 3.0 and can then be expressed as

$$\frac{R}{R_d} = 0.25 + 0.355\pi d\sqrt{\frac{2\mu f}{\rho}}. \tag{872}$$

At 60 cycles and ordinary separation of the conductors, the proximity effect and the skin effect may be neglected for conductors of 250,000 circular mils or smaller. For copper or aluminum conductor with steel core, owing to the skin effect, the alternating current is usually assumed as concentrated entirely in the outer elements, *i.e.*, in the copper or the aluminum. For iron and steel wire, the values of R and the loss vary with the current density as well as with the frequency and the size of wire, and these values are usually derived from tests rather than from calculations. For more exact calculations of skin effect, for proximity effect, and for values at intermediate frequencies, the solution and the calculation of values involve the use of Bessel functions.

Example 79

Find the resistance of 1,000 ft. of 1,000,000-circular-mil copper conductor at 20°C. and 60 cycles.

From Eq. (864),

$$R = R_d\left[1 + \frac{\omega^2 l^2}{12R_d^2} - \frac{\omega^4 l^4}{180R_d^4} + \cdot\cdot\cdot\right] \text{ abohms.} \tag{864}$$

For this problem, $R_d = 0.0108$ ohm $= 108 \times 10^5$ abohms, $l = 1{,}000$ ft. $= 30.48 \times 10^3$ cm, $\omega = 2\pi f = 377$, and

$$R = \frac{108 \times 10^5}{10^9}\left[1 + \frac{\overline{377}^2 \times \overline{30.48}^2 \times 10^6}{12 \times \overline{108}^2 \times 10^{10}} - \frac{\overline{377}^4 \times \overline{30.48}^4 \times 10^{12}}{180 \times \overline{108}^4 \times 10^{20}} + \cdot\cdot\cdot\right] \text{ ohms}$$

$$= 0.0108[1 + 0.09433 - 0.007120 + \cdot\cdot\cdot] \text{ ohms}$$
$$= 0.0108 \times 1.08721 \text{ ohms}$$
$$= 0.01176 \text{ ohms.}$$

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[1] WEBB, J. S., The Variation in the High-frequency Resistance and Permeability of Ferromagnetic Materials Due to a Superimposed Magnetic Field, *Proc. I.R.E.*, Vol. 26, No. 4, April, 1938.

The a-c resistance at this frequency is seen to be 8.7 per cent greater than the d-c resistance for this size of wire at the same temperature. The higher the temperature, the smaller will be this correction for a given size of wire. The higher the specific resistance or the smaller the size of wire, the smaller will be this correction.

129. Inductance of a Single-phase Line.—The inductance of the two wires of a single-phase line may be determined as follows: In Fig. 234, let A and B represent the two conductors under consideration, spaced D cm apart, the spacing of the conductors being considered large in regard to the diameter of the wires. The electromagnetic lines will enclose the conductors in circular paths having their centers nearly concentric with each conductor.

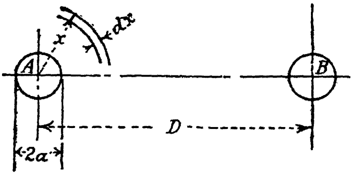

FIG. 234.—Two wires of any transmission line.

Let D = distance between conductors, cm,
a = radius of each conductor, cm,
d = diameter of each conductor, cm, and
i = current in each wire at any instant, abamp.

Consider, first, the space outside the surface of conductor A. In the space dx cm wide and 1 cm along the wire and at a distance x cm from the center of conductor A, the magnetic intensity will be

$$h = \frac{4\pi ni}{l} \text{ oersteds,}$$

but, since $n = 1.00$ and $l = 2\pi x$,

$$h = \frac{4\pi i}{2\pi x} = \frac{2i}{x} \text{ oersteds.} \tag{873}$$

The flux per centimeter along the conductor in this space is

$$\phi = \mu \times h \times (\text{area}) \text{ maxwells,}$$

but, since the area of the path per centimeter length of conductor is dx ($\mu = 1.00$ for air), then, for a given value of current,

$$d\phi_A = \frac{2i_A}{x} dx \text{ maxwells.} \tag{874}$$

Equation (874) being integrated from the surface of the wire A to S, where S is a very large finite distance, the entire flux produced outside the conductor by the current i_A becomes

$$\phi_A = \int_a^S d\phi_A = \int_a^S 2i_A \frac{dx}{x} \text{ maxwells}$$
$$= 2i_A \log_\epsilon \frac{S}{a} \text{ maxwells.} \tag{875}$$

Consider, second, the flux produced by the current i flowing in conductor B and which surrounds the current in conductor A, *i.e.*, that flux between the axis of A and S; then

$$\phi_A' = \int_D^S d\phi = \int_D^S 2i_B \frac{dx}{x} \text{ maxwells,} \tag{876}$$

where x is now the distance of the ring dx from the axis of conductor B and concentric with B. Integrating Eq. (876) as indicated,

$$\phi_A' = 2i_B \log_\epsilon \frac{S}{D} \text{ maxwells.} \tag{877}$$

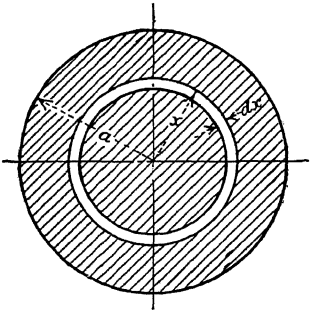

FIG. 235.—Cross section of a wire.

Consider, lastly, the flux within the radius a of conductor A, *i.e.*, the flux within the conductor. In this derivation, assume the material of the conductor as nonmagnetic and the current as distributed uniformly over the cross section of the conductor. This assumption will be close enough for ordinary spacing and diameter of conductors and for ordinary frequencies of power transmission. The more exact value may be derived at the same time with the skin effect of the conductor by the use of Bessel functions. In Fig. 235, the current in the portion of conductor A within the radius x from its center will be

$$i_x = \frac{x^2}{a^2} i_A \text{ abamp,} \tag{878}$$

and the magnetic intensity at x will be

$$h_x = \frac{4\pi n i_x}{2\pi x} = \frac{2i_x}{x} \text{ oersteds,} \tag{879}$$

or, substituting the value of i_x from Eq. (878),

$$\begin{aligned} h_x &= \frac{x^2}{a^2}\left(\frac{2}{x}\right)i_A \text{ oersteds} \\ &= \frac{2x}{a^2}i_A \text{ oersteds.} \end{aligned} \tag{880}$$

The portion of the flux in the ring 1 cm along the wire of radius x and width dx is

$$d\phi_x = \mu h_x dx \text{ maxwells,} \tag{881}$$

which, for nonmagnetic materials, becomes

$$d\phi_x = \frac{2i_A}{a^2}x dx \text{ maxwells} \tag{882}$$

per centimeter length of wire. This flux interlinks with the portion of current

$$i_x = \frac{x^2}{a^2}i_A \text{ abamp} \tag{878}$$

within this ring, or the linkages of flux and current are

$$i_x d\phi_x = \frac{x^2}{a^2}i_A d\phi_x. \tag{883}$$

The equivalent flux at the surface of the wire and interlinked with the entire current i_A will be

$$i_A d\phi_A'' = i_x d\phi_x, \tag{884}$$

or

$$d\phi_A'' = \frac{x^2}{a^2}d\phi_x \text{ maxwells} \tag{885}$$

since this flux, for the same linkages, must be decreased by the ratio of increase in current enclosed, or

$$\begin{aligned} d\phi_A'' &= \frac{x^2}{a^2}\left(\frac{2i_A}{a^2}\right)x dx \text{ maxwells} \\ &= \frac{2i_A}{a^4}x^3 dx \text{ maxwells.} \end{aligned} \tag{886}$$

The equivalent total value of this flux at the surface of the wire for the entire current i_A is then

$$\begin{aligned}\phi_A'' &= \int_0^a \frac{2i_A}{a^4}x^3dx \text{ maxwells}\\ &= \frac{i_A}{2} \text{ maxwells,} \qquad (887)\end{aligned}$$

and for magnetic materials of constant permeability this becomes

$$\phi_A'' = \frac{\mu i_A}{2} \text{ maxwells.} \qquad (888)$$

This flux is seen to be independent of the size of the wire; therefore for conductors of all sizes the flux produced by a current within the conductor is equivalent to a flux ϕ_A'' linking the entire conductor and depending only on the amount of current flowing. For each instantaneous value of alternating current, this flux will be the same as for an identical value of direct current except as modified by the skin effect as explained in a previous paragraph.

Since, in a single-phase line at any instant,

$$i_A = -i_B = i \text{ abamp,} \qquad (889)$$

the total flux interlinked with the current in conductor A will be the sum of the amounts given in Eqs. (875), (877), and (887), or

$$\begin{aligned}\phi &= 2i \log_\epsilon \frac{S}{a} - 2i \log_\epsilon \frac{S}{D} + \frac{i}{2} \text{ maxwells}\\ &= 2i \log_\epsilon \frac{D}{a} + \frac{i}{2} \text{ maxwells} \qquad (890)\end{aligned}$$

per centimeter length of wire A, for nonmagnetic materials. For magnetic materials, Eq. (890) becomes

$$\phi = 2i \log_\epsilon \frac{D}{a} + \frac{\mu i}{2} \text{ maxwells.} \qquad (891)$$

The emf induced along A due to a change in this flux is a function of the time and not of the dimensions already considered. Applying Faraday's law,

$$e_i = -N\frac{d\phi}{dt} \text{ abvolts,} \qquad (2)$$

to Eq. (890),

$$e_x = \left[2 \log_\epsilon \frac{D}{a} + \frac{1}{2}\right]\frac{di}{dt} \text{ abvolts} \tag{892}$$

$$= L\frac{di}{dt} \text{ abvolts.} \tag{47}$$

From Eqs. (892) and (47) or from the definition of inductance as the coefficient of di/dt,

$$L = 2 \log_\epsilon \frac{D}{a} + \frac{1}{2} \text{ abhenrys} \tag{893}$$

per centimeter length of wire for nonmagnetic materials, or

$$L = 2 \log_\epsilon \frac{D}{a} + \frac{\mu}{2} \text{ abhenrys} \tag{894}$$

for magnetic materials.

Reducing from c.g.s. to practical units,

$$L = \left[2 \log_\epsilon \frac{D}{a} + \frac{1}{2}\right]10^{-9} \text{ henrys} \tag{895}$$

per centimeter length of wire. Reducing to common logarithms

$$L = \left[2 \times 2.31 \log_{10} \frac{D}{a} + \frac{1}{2}\right]10^{-9}$$

$$= \left[4.62 \log_{10} \frac{D}{a} + \frac{1}{2}\right]10^{-9} \text{ henrys} \tag{896}$$

per centimeter length of wire. Since there are 1.61 × 10^5 cm in 1 mile,

$$L = 1.61 \times 10^5 \left[4.62 \qquad \frac{D}{a} + \frac{1}{2}\right.$$

$$0.00074 \log_{10} \frac{D}{a} + 0.00008 \text{ henrys} \tag{897}$$

per mile of wire for nonmagnetic materials. For approximate calculations, it is sufficiently accurate to use

$$L = 0.00074 \log_{10} \frac{2D}{d} \text{ henrys} \tag{898}$$

per mile of wire, particularly where the wires are small and spaced at a considerable distance from each other. Considering the complete line of two wires,

$$L = 0.00148 \log_{10} \frac{2D}{d} + 0.00016, \tag{899}$$

or, approximately,

$$L = 0.00148 \log_{10} \frac{2D}{d} \text{ henrys} \tag{900}$$

per mile of line, for use with single-phase distribution systems or single-phase transmission lines.

130. Reactance of Lines Due to Inductance.—The reactance of the line is found from the preceding equations for the inductance and the value of ω, as

$$x = \omega L = 2\pi f L \text{ ohms}, \tag{71}$$

where L is in henrys from Eqs. (895) to (900), above. It should be noted in Eqs. (895) to (900) that D/a and $2D/d$ are ratios so that the dimensions of the wires and the distances between the wires may both be measured in inches or other units as well as in centimeters, since multiplying both the numerator and the denominator of these ratios by the same unit of conversion does not change the numerical value of the fraction.

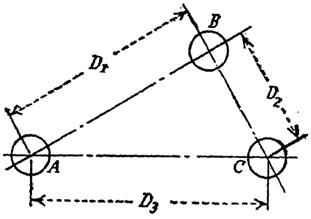

FIG. 236.—Low triangular arrangement of wires in a three-phase transmission line.

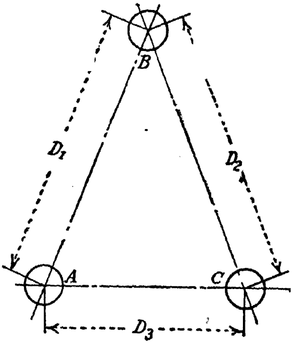

FIG. 237.—High triangular arrangement of wires in a three-phase transmission line.

131. Inductance of Three-phase Transmission Lines.—The early three-phase transmission lines were constructed with the conductors equally spaced at the corners of an equilateral triangle, called "delta configuration." For single lines, one conductor was mounted on an insulator supported on top of the poles and the other two conductors on insulators on a single crossarm at the proper distance below the first conductor to give equal spacing between the conductors, *i.e.*, an equilateral triangle. When duplicate lines were necessary, it was customary to use either two separate lines, or two crossarms with the wires properly arranged to secure uniform spacing for each line or from line to line. With the advent of the overhead ground wire for lightning protection and the use of steel poles and towers with longer spans, unequal spacings became common. Figures 236 to 241 illustrate a few of the common unequal spacings. Figure 236 is a horizontal low triangular spacing; Fig. 237, a high triangular

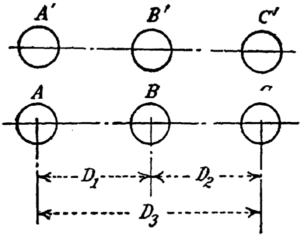

FIG. 238.—Uniform horizontal spacing of duplicate lines in a three-phase transmission system.

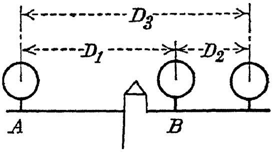

FIG. 239.—Nonuniform horizontal spacing of wires in a three-phase transmission line.

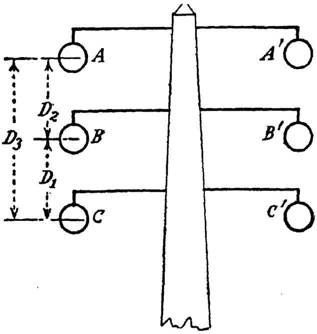

FIG. 240.—Uniform vertical spacing of duplicate lines in a three-phase transmission system.

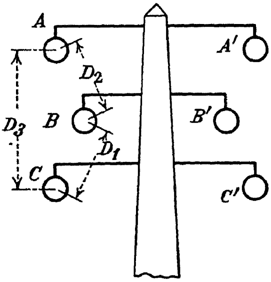

FIG. 241.—Vertical low triangular spacing of wires in a three-phase transmission line.

spacing; Fig. 238, a uniform horizontal spacing, with duplicate lines; Fig. 239, a nonuniform horizontal spacing; Fig. 240, a uniform vertical spacing with duplicate lines; and Fig. 241, a vertical low triangular spacing used to give greater vertical clearance between wires. For each of these nonuniform spacings, except Fig. 237, it is seen that D_3 is greater than D_1 or D_2.

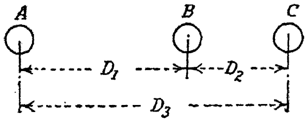

Fig. 242.—Arrangement of conductors before transposition.

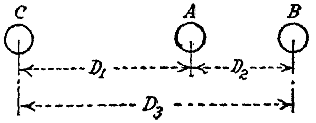

Fig. 243.—Arrangement of conductors after first transposition.

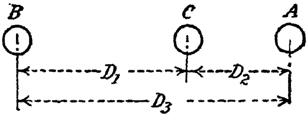

Fig. 244.—Arrangement of conductors after second transposition.

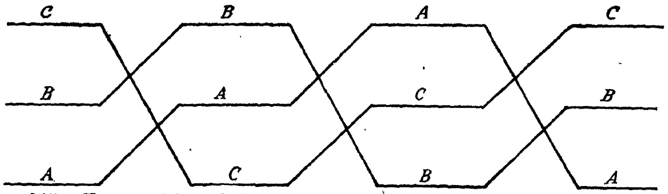

Fig. 245.—Transposition of the three conductors of a three-phase transmission line.

In order to equalize the spacings of the conductors over the entire line as regards each of the other conductors as well as to reduce the danger of interference by induction with other parallel lines, either power or communication, the transmission line is transposed between stations in multiples of three transpositions of equal length, as shown in Figs. 242 to 245. Three transpositions are sufficient for equalizing the spacings of the power line, but, to eliminate disturbances with adjacent telephone lines it is necessary to transpose three times, or in multiples of three, in each section of the line paralleled by a telephone line or major change in distance between transmission and telephone line. The telephone line must also be transposed as well as protected by special devices if the lines are relatively close together. In consequence of the unequal spacing of the conductors and the transpositions, the equation for the inductance of a three-phase line must contain a development for an equivalent uniform spacing.

Let Fig. 242 represent any spacing of conductors in a three-phase system. The flux interlinked with the current i_A in A will be, from Eqs. (875), (877), and (887),

$$\phi_A' = \frac{i_A}{2} + 2i_A \log_\epsilon \frac{S}{a} + 2i_B \log_\epsilon \frac{S}{D_1} + 2i_C \log_\epsilon \frac{S}{D_3} \text{ maxwells.} \tag{901}$$

After the first transposition of the line (Fig. 243),

$$\phi_A'' = \frac{i_A}{2} + 2i_A \log_\epsilon \frac{S}{a} + 2i_B \log_\epsilon \frac{S}{D_2} + 2i_C \log_\epsilon \frac{S}{D_1} \text{ maxwells} \tag{902}$$

and, after the next transposition (Fig. 244),

$$\phi_A''' = \frac{i_A}{2} + 2i_A \log_\epsilon \frac{S}{a} + 2i_B \log_\epsilon \frac{S}{D_3} + 2i_c \log_\epsilon \frac{S}{D_2} \text{ maxwells.} \tag{903}$$

For equal transposition spacings, the average flux interlinked with i_A will be

$$\phi_A = \tfrac{1}{3}[\phi_A' + \phi_A'' + \phi_A'''] \text{ maxwells,} \tag{904}$$

or

$$= \frac{i_A}{2} + 2i_A \log_\epsilon \frac{1}{a} + \frac{2}{3}(i_B + i_C) \log_\epsilon \left(\frac{1}{D_1} \times \frac{1}{D_2} \times \frac{1}{D_3}\right) + 2(i_A + i_B + i_C) \log_\epsilon S \text{ maxwells.} \tag{905}$$

In Eq. (905),

$$i_A + i_B + i_C = 0 \text{ abamp,} \tag{906}$$

for an ungrounded three-phase system, *i.e.*, a three-wire three-phase system without grounded neutral, and Eq. (905) becomes

$$\phi_A = \frac{i_A}{2} + 2i_A \log_\epsilon \frac{1}{a} - \frac{2}{3}i_A \log_\epsilon \left(\frac{1}{D_1 \times D_2 \times D_3}\right) \text{ maxwells} \tag{907}$$

$$= \frac{i_A}{2} + 2i_A \log_\epsilon \frac{1}{a} + \frac{2}{3}i_A \log_\epsilon (D_1 \times D_2 \times D_3) \text{ maxwells} \tag{908}$$

$$= \frac{i_A}{2} + 2i_A \log_\epsilon \frac{\sqrt[3]{D_1 \times D_2 \times D_3}}{a} \text{ maxwells.} \tag{909}$$

Hence, since

$$e_{L_A} = N\frac{d\phi}{dt} = L\frac{di}{dt} \text{ abvolts,}$$

then

$$L_A = \frac{1}{2} + 2\log_e \frac{\sqrt[3]{D_1 \times D_2 \times D_3}}{a} \text{ abhenrys} \tag{910}$$

per centimeter length of wire and of line, and

$$L_A \quad 0.00074 \log_{10} \frac{2\sqrt[3]{D_1 \times D_2 \times D_3}}{d} + 0.00008 \tag{911}$$

henrys per mile of line. Comparing Eq. (910) with Eq. (893) and Eq. (911) with Eq. (897), it is seen that

$$D = \sqrt[3]{D_1 \times D_2 \times D_3} \tag{912}$$

is a mean spacing of conductors. If the conductors are placed at the corners of an equilateral triangle, this mean spacing becomes the actual spacing, since, in that case

$$D_1 = D_2 = D_3 = D.$$

It is also apparent that the inductance per wire of a three-phase line with balanced loads is the same as the inductance per wire of a single-phase line for the same spacing. The reactance will be found as

$$\begin{aligned} x_L &= 2\pi f L \text{ ohms} \\ &= 2\pi f\left[0.00074 \log_{10} \frac{2\sqrt[3]{D_1 \times D_2 \times D_3}}{d} + 0.00008\right] \end{aligned} \tag{913}$$

ohms per mile of wire or line.

Example 80

Find the inductance, resistance, and reactance at 75°F. of a 33,000-volt three-phase 60-cycle transmission line 40 miles long constructed of No. 2 B. & S. copper wire spaced at the corners of an equilateral triangle 30 in. on a side.

From the wire table,

$$\begin{aligned} \text{resistance of 1,000 ft. of No. 2 wire} &= 0.159 \text{ ohm at } 20°\text{C.} \\ &= 0.1615 \text{ ohm at } 75°\text{F.,} \end{aligned}$$

or

$$R = 5.280 \times 40 \times 0.1615 = 34.1 \text{ ohms.}$$

From Eq. (912), the inductance in henrys per mile is

$$L_{\text{mi.}} = 0.00074 \log_{10} \frac{2D}{d} + 0.00008,$$

where

$$\begin{aligned} d &= 0.258 \text{ in.}, \qquad D = 30 \text{ in.}, \\ \log_{10} \frac{2D}{d} &= \log_{10} \frac{2 \times 30}{0.258} = 2.377, \\ L_{\text{mi.}} &= 0.00074 \times 2.377 + 0.00008 \\ &= 0.00184 \text{ henry}. \end{aligned}$$

The total inductance becomes

$$L = 40 \times 0.00184 = 0.0736 \text{ henry}$$

and the reactance,

$$\begin{aligned} x_L &= 2\pi f L = 2\pi \times 60 \times 0.0736 \\ &= 27.8 \text{ ohms}. \end{aligned}$$

Example 81

What are the inductance, resistance, and reactance of two wires of Example 80 used as a single-phase line?

$$\begin{aligned} R &= 2 \times R \text{ (per wire)} \\ &= 2 \times 34.1 = 68.2 \text{ ohms}, \\ L &= 2 \times 0.0736 = 0.1472 \text{ henry}, \end{aligned}$$

and

$$x_L = 2 \times 27.8 = 55.6 \text{ ohms}.$$

Example 82

Find the inductance and reactance of the line of Example 80 if the wires are placed on the same cross arm 30 in. from the middle wire to each of the outside wires?

From the problem,

$$D_1 = 30 \text{ in.}; \qquad D_2 = 30 \text{ in.}, \qquad D_3 = 60 \text{ in.},$$

and from Eq. (912) the mean distance is

$$D = \sqrt[3]{30 \times 30 \times 60} = 37.8 \text{ in.},$$

and, since

$$\begin{aligned} \log_{10} \frac{2D}{d} &= \log_{10} \frac{2 \times 37.8}{0.251} = 2.480, \\ L &= 40(0.00074 \times 2.480 + 0.00008) \\ &= 0.0767 \text{ henry}, \end{aligned}$$

and

$$x_L = 377 \times 0.0767 = 28.9 \text{ ohms}.$$

132. Design Conditions Affecting the Inductance Equation.—The inductance equations for single- and three-phase lines, as derived in this chapter, assume a solid conductor of unit permeability and with uniform distribution of current over the cross section of the conductor. These equations are sufficiently accurate for even small spacings if proximity and skin effects may be neglected. As was explained before, the proximity effect is small except in cables at very narrow spacings. The skin effect is also small in its effect on inductance except for large conductors at narrow spacings, for high frequencies such as switching conditions and lightning surges, or for wires in which the permeability is considerably greater than unity. If the entire current of the conductor is in the outer layer, it will reduce the constant term in the inductance equation to zero; hence, the skin effect tends to reduce the inductance only slightly for conductors of unit permeability. Both the proximity effect and the skin effect have a greater influence on the resistance than on the inductance of the line.

For stranded conductors, the diameter of the wire to be used in the inductance equations is some value between that of the actual conductor and that of a solid conductor of the same cross section. When the diameter of the equivalent solid conductor is used, the inductance will be too large by perhaps 2 to 3 per cent. For copper-clad wire and for steel-reinforced aluminum cables, the magnetic effect of the steel core is usually neglected and the current is considered as flowing in the outer conducting layers. This effect reduces slightly the value of the constant term and reduces the inductance below the value for the same conductivity of aluminum in a solid conductor. The same is true for a steel-reinforced copper conductor or for stranded conductors with hemp cores.

Any percentage change in the spacing of the conductors of a transmission line to accommodate the design to different voltages or altitudes will change the inductance of the line by adding to the original inductance 0.00074 times the logarithm of the ratio of the spacings. Any change in the size of wire for the same spacing will effect the inductance in the opposite manner to the change in spacing.

133. Capacitance of a Transmission Line.—Every pair of wires separated by any insulating medium has a capacitance depending upon the area of the metallic surface of the conductor, the separation of the conductors, and the character of the dielectric. For short low-voltage transmission lines, the charging current is neglected; however, this does not mean that these lines have no capacitance, but that this effect is not of sufficient importance for consideration in the steady operation of these lines. For long transmission lines, the voltage must be high for efficient transmission of power, and the charging current of the line is greater owing to the increased area of the conductor as well as to the higher voltage of the line. At the same time, the load current for a given amount of power is less for the higher voltage of transmission, and the value of the charging current and its effects may not be neglected as is done for the short low-voltage lines. In order to calculate the charging current, the capacitance of the line must be known.

For the first consideration, take two wires of a transmission line A and B, Fig. 246, spaced S cm apart. These wires are supposed to have no dimensions other than length, and each wire is charged with q units of charge per unit length. From the definition of the potential of a point, as the work in ergs required to bring a unit charge of positive electricity from the boundary of the field to that point without disturbing the distribution of the electrostatic lines, the potential of the point P is found from

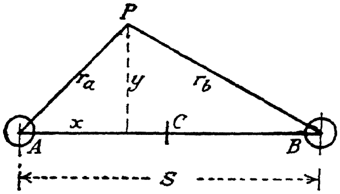

FIG. 246.—A point P in relation to two conductors A and B.

$$Fds = -dv \text{ ergs},$$

or

$$F = -\frac{dv}{ds} \text{ dynes} \tag{914}$$

where F is the average force, dynes,

ds is the differential distance, cm, and

dv is the change in potential in distance ds,

the negative sign being used in Eq. (914) since the potential increases as s decreases, or as the charge causing the difference in potential is approached. With two parallel wires, the forces may be considered in a plane perpendicular to the axes of the wires rather than in all space, as about a point, which eliminates the use of space coordinates.

The force exerted on a unit charge at any point is numerically equal to the intensity of the electric field at that point, and since the intensity of the field at a distance r is

$$F = \frac{4\pi q}{2\pi r} = \frac{2q}{r} \text{ dynes,}$$

then

$$F = -\frac{dv}{ds} = \frac{2q}{r} \text{ dynes} \tag{915}$$

per unit charge, air being assumed as the dielectric. Take C, a point midway between A and B on line AB, and consider the charge per centimeter length to be $+q$ on A and $-q$ on B. The potential difference between C and P due to A is $V_A - V_A'$, where V_A is the potential of P and V_A' is the potential of C due to the charge on A. Since the difference in potential is measured by the amount of work required to move a unit positive charge from one point to the other,

$$\frac{dv_A}{dr} = \frac{2q}{r}, \tag{915}$$

the difference in potential between P and C becomes

$$\begin{aligned} V_A - V_A' &= \int_{r_a}^{S/2} 2q\frac{dr}{r} \\ &= 2q \log_\epsilon \frac{S}{2r_a}. \end{aligned} \tag{916}$$

Due to conductor B this difference in potential is

$$V_B - V_B' = -2q \log_\epsilon \frac{S}{2r_b}. \tag{917}$$

The potential difference between P and C due to both wires is the algebraic sum of the two potentials, represented by Eqs. (916) and (917), or

$$V = V_A + V_B - (V_A' + V_B') = V_A + V_B, \tag{918}$$

si[redacted] $V_A' + V_B' = 0$, C being a point of zero potential by sym-[redacted]. Hence,

$$V = 2q \log_\epsilon \frac{r_b}{r_a}. \tag{919}$$

For a given spacing of the wire, the locus of all points having the same potential as P will be determined by making the ratio r_b/r_a a constant. Let the coordinates of the point P with the origin at A (Fig. 246) be x and y, then

$$r_b^2 = (S - x)^2 + y^2,$$
$$r_a^2 = x^2 + y^2.$$

The square of the ratio of the distance will be

$$\frac{r_b^2}{r_a^2} = \frac{S^2 - 2Sx + x^2 + y^2}{x^2 + y^2} = k_A^2. \qquad (920)$$

Hence,

$$\frac{S^2 - 2Sx}{x^2 + y^2} = k_A^2 - 1,$$

$$x^2 + \frac{2S}{k_A^2 - 1}x + \frac{S^2}{(k_A^2 - 1)^2} + y^2 = \frac{S^2}{k_A^2 - 1} + \frac{S^2}{(k_A^2 - 1)^2},$$

or

$$\left(x + \frac{S}{k_A^2 - 1}\right)^2 + y^2 = \frac{S^2k_A^2}{(k_A^2 - 1)^2}. \qquad (921)$$

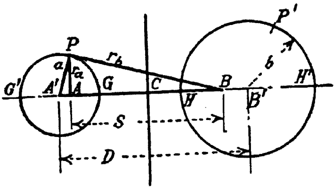

FIG. 247.—Two conductors of a transmission system.

This is the equation of a circle (Fig. 247) of radius $a = \frac{Sk_A}{k_A^2 - 1}$ and with its center displaced from A by the distance

$$AA' = \frac{S}{k_A^2 - 1}, \qquad (922)$$

but

$$\frac{a}{AA'} = k_A, \qquad (923)$$

$$\frac{BA'}{a} = \frac{S + AA'}{a} = \frac{\frac{Sk_A^2}{k_A^2 - 1}}{\frac{Sk_A}{k_A^2 - 1}} = k_A. \qquad (924)$$

Then

$$\frac{a}{AA'} = \frac{BA'}{a}, \qquad (925)$$

$$(AA')(A'B) = (A'B)(A'B - S) = a^2 \qquad (926)$$
$$= (A'A)(A'A + S) = a^2. \qquad (927)$$

In a similar manner, it may be shown that the equipotential line about B passing through the point P' is a circle of radius,

$$b = \frac{Sk_B}{k_B^2 - 1},$$

with its center displaced a distance,

$$B'B = \frac{S}{k_B^2 - 1},$$

from B, where

$$k_B = \frac{r_a}{r_b}.$$

Hence,

$$\frac{b}{BB'} = k_B, \tag{928}$$

$$\frac{B'A}{b} = \frac{S + B'B}{b} = k_B, \tag{929}$$

and

$$(B'B)(B'A) = (B'A)(B'A - S) = b^2 \tag{930}$$

$$= (B'B)(B'B + S) = b^2. \tag{931}$$

Lines of electrostatic force show the direction of the force acting on a unit positive charge if placed at any point within the field. Since this force can at no place have a component along an equipotential line or surface, the lines of electrostatic force must at all points be perpendicular to the equipotential lines. It can be shown that the electrostatic lines are circles passing through A and B with their centers receding along the zero potential line ab (Fig. 248). In Fig. 247, GPG' and $HP'H'$ represent two equipotential surfaces with centers at A' and B', D cm apart. Since all points within a conductor must be at the same potential, if solid conducting cylinders of radii a and b are placed with their centers at A' and B', the charges q and $-q$ will be distributed over their surfaces in such a way that the potential of the surfaces GPG' and $HP'H'$ will not be altered. The electrostatic field external to the conductor will, therefore, remain in intensity and distribution as though the charges were concentrated along A and B.

The potential of the conductors with centers at A' and B' will be that of any point on their surfaces such as G and H. From Eq. (919),

$$V_A = 2q \log_\epsilon \frac{r_b}{r_a} = 2q \log_\epsilon \frac{BG}{AG}, \tag{932}$$

and

$$V_B = 2q \log_\epsilon \frac{r_b'}{r_a'} = 2q \log_\epsilon \frac{BH}{AH}, \tag{933}$$

and the difference of potential between A and B is

$$V_A - V_B = 2q \log_\epsilon \left[\left(\frac{BG}{AG}\right)\left(\frac{AH}{BH}\right)\right]. \tag{934}$$

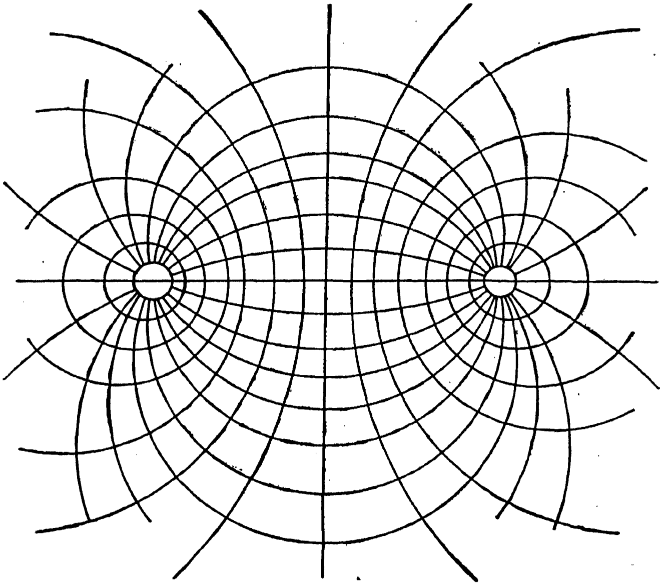

FIG. 248.—Electrostatic and electromagnetic lines about two conductors.

By the definition of capacitance as $VC = q$,

$$V_A - V_B = \frac{q}{C}, \tag{935}$$

or

$$C = \frac{1}{2 \log_\epsilon \left(\frac{BG}{AG}\right)\left(\frac{AH}{BH}\right)} \tag{936}$$

is the capacitance per unit length of line. From Eqs. (924) and (929),

$$\frac{BG}{AG} = \frac{r_b}{r_a} = k_A = \frac{BA'}{a}, \tag{937}$$

and

$$\frac{AH}{BH} = \frac{r_a'}{r_b'} = k_B = \frac{AB'}{b}; \tag{938}$$

then

$$\frac{(BG)(AH)}{(AG)(BH)} = \frac{(BA')(AB')}{ab}, \tag{939}$$

or from Eqs. (936) and (939),

$$C = \frac{}{2 \log_\epsilon \frac{(BA')(AB')}{ab}}. \tag{940}$$

From Eqs. (926), (927), and (930),

$$(A'B)^2 - (A'B)S = a^2, \tag{941}$$

$$(A'A)^2 + (A'A)S = a^2, \tag{942}$$

and

$$(B'A)^2 - (B'A)S = b^2. \tag{943}$$

Solving these quadratics,

$$A'B = \frac{S}{2} + \sqrt{a^2 + \frac{S^2}{4}}, \tag{944}$$

$$A'A = -\frac{S}{2} + \sqrt{a^2 + \frac{S^2}{4}}, \tag{945}$$

and

$$B'A = \frac{S}{2} + \sqrt{b^2 + \frac{S^2}{4}}. \tag{946}$$

From Fig. 247 and Eqs. (945) and (946), it is seen that

$$D = B'A + A'A = \sqrt{a^2 + \frac{S^2}{4}} + \sqrt{b^2 + \frac{S^2}{4}}. \tag{947}$$

From Eqs. (944) and (946),

$$(A'B)(B'A) = \frac{S^2}{4} + \frac{S}{2}\left[\sqrt{a^2 + \frac{S^2}{4}} + \sqrt{b^2 + \frac{S^2}{4}}\right] + \sqrt{a^2 + \frac{S^2}{4}}\sqrt{b^2 + \frac{S^2}{4}}, \tag{948}$$

and

$$(A'B)(B'A) = \frac{S^2}{4} + \frac{S}{2}D + \sqrt{a^2 + \frac{S^2}{4}}\sqrt{b^2 + \frac{S^2}{4}}. \qquad (949)$$

The value of $(BA')(AB')$ must be expressed in terms of the known dimensions of the line, *i.e.*, the radii a and b and the spacing of the line wires D, before it can be substituted in Eq. (940). Squaring both sides of Eq. (947),

$$\begin{aligned} D^2 &= a^2 + \frac{S^2}{4} + 2\sqrt{a^2 + \frac{S^2}{4}}\sqrt{b^2 + \frac{S^2}{4}} + b^2 + \frac{S^2}{4} \\ &= 2\frac{S^2}{4} + 2\sqrt{a^2 + \frac{S^2}{4}}\sqrt{b^2 + \frac{S^2}{4}} + a^2 + b^2, \end{aligned}$$

or

$$\frac{D^2 - a^2 - b^2}{2} = \frac{S^2}{4} + \sqrt{a^2 + \frac{S^2}{4}}\sqrt{b^2 + \frac{S^2}{4}}. \qquad (950)$$

Squaring both sides of Eq. (950) and collecting terms not containing S,

$$\begin{aligned} \left(\frac{D^2 - a^2 - b^2}{2}\right)^2 - a^2b^2 &= \frac{S^2}{4}\left[2\frac{S^2}{4} + 2\sqrt{a^2 + \frac{S^2}{4}}\sqrt{b^2 + \frac{S^2}{4}} + a^2 + b^2\right] \\ &= \frac{S^2}{4}D^2, \end{aligned}$$

or

$$\frac{S}{2}D = \sqrt{\left(\frac{D^2 - a^2 - b^2}{2}\right)^2 - a^2b^2}. \qquad (951)$$

Substituting Eqs. (950) and (951) in Eq. (949),

$$(B'A)(A'B) = \frac{D^2 - a^2 - b^2}{2} + \sqrt{\left(\frac{D^2 - a^2 - b^2}{2}\right)^2 - a^2b^2}. \qquad (952)$$

Equation (940) then becomes

$$C = \frac{1}{2\log_\epsilon\left[\frac{D^2 - a^2 - b^2}{2ab} + \sqrt{\left(\frac{D^2 - a^2 - b^2}{2ab}\right)^2 - 1}\right]} \qquad (953)$$

$$= \frac{1}{2\log_\epsilon(\alpha + \sqrt{\alpha^2 - 1})}, \qquad (954)$$

where

$$\alpha = \frac{D^2 - a^2 - b^2}{2ab} \tag{955}$$

or

$$C = \frac{1}{2 \cosh^{-1} \alpha}. \tag{956}$$

For transmission lines, each conductor is an equipotential surface and the conductors have the same radius a, D becoming the distance between the centers of these conductors. Putting $a = b$ in Eq. (955),

$$\alpha = \frac{D^2 - 2a^2}{2a^2}, \tag{957}$$

and

$$\begin{aligned} \sqrt{\alpha^2 - 1} &= \sqrt{\frac{D^4 - 4D^2a^2 + 4a^4}{4a^4} - 1} \\ &= \frac{D}{2a^2}\sqrt{D^2 - 4a^2}, \end{aligned}$$

or

$$\begin{aligned} \alpha + \sqrt{\alpha^2 - 1} &= \frac{2[D^2 - 2a^2 + D\sqrt{D^2 - 4a^2}]}{4a^2} \\ &= \frac{D^2 + 2D\sqrt{D^2 - 4a^2} + D^2 - 4a^2}{4a^2} \\ &= \left(\frac{D + \sqrt{D^2 - 4a^2}}{2a}\right)^2, \end{aligned} \tag{958}$$

and Eq. (954) becomes

$$\begin{aligned} C &= \frac{1}{2 \log_\epsilon \left[\frac{D}{2a} + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]^2} \\ &= \frac{1}{4 \log_\epsilon \left[\frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1}\right]} \qquad (959) \\ &= \frac{1}{4 \cosh^{-1} \frac{D}{d}}, \qquad (960) \end{aligned}$$

where d is the diameter of the conductor. But, since D is large compared with the diameter of the wire,

$$C = \frac{1}{4 \log_\epsilon \frac{D}{a}} = \frac{1}{4 \log_\epsilon \frac{2D}{d}} \tag{961}$$

may be taken as the capacitance of 1 cm length of line in electrostatic units, for approximate results.

In order to make the above equations useful in calculations, they must be reduced to either c.g.s. or practical units in the electromagnetic system. The ratio between the electrostatic unit of emf and the c.g.s. unit of the electromagnetic system is the square of the velocity of light in centimeters per second, or $(3 \times 10^{10})^2$. Hence, C in c.g.s. electromagnetic units is

$$C_{\text{c.g.s.}} = \frac{C \text{ (in electrostatic units)}}{(3 \times 10^{10})^2},$$

also

$$C_{\text{farads}} = C_{\text{c.g.s.}} \times 10^9, \tag{963}$$

and

$$C_{\text{microfarads}} = C_{\text{farads}} \times 10^6. \tag{964}$$

Hence,

$$C_{\text{microfarads}} = \frac{C_{\text{(in electrostatic units)}} \times 10^{15}}{(3 \times 10^{10})^2}$$

$$= 1.11 \times 10^{-6}\, C_{\text{electrostatic}}. \tag{966}$$

Applying the conversion factor of Eq. (966) to Eq. (960),

$$C = \frac{1.11 \times 10^{-6}}{4 \cosh^{-1} \frac{D}{d}}\ \mu\text{f} \tag{967}$$

per centimeter length of line, or

$$C = \frac{(1.11 \times 10^{-6})(1.61 \times 10^5)}{4 \cosh^{-1} \frac{D}{d}}$$

$$= \frac{0.0447}{\cosh^{-1} \frac{D}{d}}\ \mu\text{f} \tag{968}$$

per mile of single-phase line, or, from Eq. (959),

$$C = \frac{0.0447}{\log_\epsilon \left[\frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1}\right]}\ \mu\text{f} \tag{969}$$

per mile of single-phase line, or, for approximate results,

$$C = \frac{0.0447}{\log_\epsilon \frac{2D}{d}}. \tag{970}$$

Reducing Eqs. (969) and (970) to common logarithms,

$$C = \frac{0.0194}{\log_{10}\left[\frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1}\right]}, \tag{971}$$

or

$$C = \frac{0.0194}{\log_{10} \frac{2D}{d}}\ \mu\text{f} \tag{972}$$

per mile of single-phase line. Equations (971) and (972) may be written,

$$C = \frac{0.0388}{\log_{10}\left[\frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1}\right]} \tag{973}$$

$$= \frac{0.0388}{\log_{10} \frac{2D}{d}}\ \mu\text{f} \tag{974}$$

per mile of wire to neutral since two condensers each of capacitance $2C$ in series between lines may be considered as replacing the single capacitance of Eqs. (971) and (972). Each of these condensers of value $2C$ may then be considered as being between one line and neutral.

The capacitance of any conductor to ground may be found by the method of images. In Fig. 247, it is seen that the line through C is a straight line perpendicular to the line AB. Hence, the assumption may be made that the ground, being a conductor, is this zero equipotential line and that conductor A is D cm above conductor B, its image. If h is the average distance of conductor A above ground, then $D = 2h$ and the value of the capacitance of A to ground is

$$C = \frac{0.0388}{\log_{10} \frac{4h}{d}}. \tag{975}$$

The value of the capacitance of each conductor to ground is usually small and is neglected in the calculation of transmission lines within ordinary accuracy. The value of the capacitance in Eq. (975) is in microfarads per mile of wire where h is the distance above ground in the same units as d.

134. Capacitance of a Three-phase Line.—For a three-phase line with conductors A, B, C placed at the corners of an equilateral triangle, the capacitance of conductor A to conductor B cannot be affected by any charge on conductor C. Hence, the capacitance of A to any point on the neutral plane will be

$$C = \frac{0.0388}{\log_{10} \frac{2D}{d}} \mu\text{f} \tag{976}$$

per mile of wire. The capacitance of C is likewise the same in regard to either A or B. In this case. the three equipotential

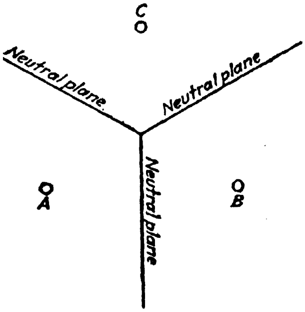

FIG. 249.—Arrangement of conductors and neutral planes of a three-phase transmission line.

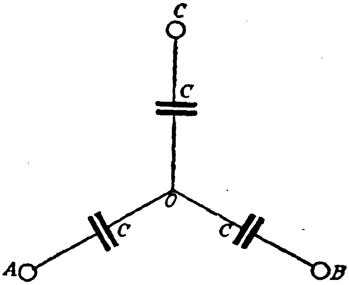

FIG. 250.—Capacitance of a three-phase transmission line.

planes of zero potential are perpendicular bisectors of each side of the triangle formed by the conductors, and they meet at the neutral point of the triangle (Fig. 249). This system of condensers may be replaced by three condensers of capacitance C to neutral as shown in Fig. 250. Hence, the capacitance of each wire to neutral for a delta spacing is given by Eq. (976). For unequal spacing of a transposed line, the same mean spacing,

$$D = \sqrt[3]{D_1 \times D_2 \times D_3}, \tag{912}$$

may be used in the equations for capacitance as was used for the inductance of the same three-phase line, when considering emfs of the primary frequency. For triple frequencies, however, and for capacitance to ground, residual voltages of a different character result for the different configurations of the line, resulting in danger of interference by capacitive inductance with neighboring communication circuits.

It should be noted that since

$$L = 0.00148 \log_{10} \frac{2D}{d} \text{ henrys (approximately)} \tag{900}$$

and

$$C = \frac{0.0194}{\log_{10} \frac{2D}{d}} \mu\text{f (approximately)} \tag{972}$$

the product of the capacitance in microfarads and the inductance in henrys is a constant, as

$$LC = \frac{0.00148 \log_{10} \frac{2D}{d} \times 0.0194}{1 \quad 2D}$$

$$= 28.7 \times 10^{-6}, \tag{977}$$

which is independent of the size of the wires and their spacing within the accuracy of Eqs. (900) and (972). If L is expressed in henrys and C in farads per mile of conductor, then the reciprocal of the square root of their product is the velocity of light in miles per second.

Example 83

Calculate the capacitance of 150 miles of a 66,000-volt three-phase transmission line constructed of No. 2/0 B. & S. copper conductors spaced 6 ft. apart at the corners of an equilateral triangle.

Diameter of wire from table, $d = 0.3648$ in.,

Distance between centers, $D = 72$ in.

From Eq. (973),

$$C = \frac{0.0388}{\log_{10}\left[\frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1}\right]} \mu\text{f} \tag{973}$$

per mile of conductor, or, substituting values of D and d,

$$C = \frac{0.0388}{\log_{10}\left[\frac{72}{0.3648} + \sqrt{\left(\frac{72}{0.3648}\right)^2 - 1}\right]}$$

$$= \frac{0.0388}{\log_{10}[197.4 + \sqrt{197.4^2 - 1}]}$$

$$= \frac{0.0388}{\log_{10} 394.7} = \frac{0.0388}{2.596}$$

$$= 0.0149\ \mu\text{f per mile.}$$

For the total length of line,

$$C = 150 \times 0.0149 = 2.24\ \mu\text{f.}$$

Using the approximate equations,

$$C = \frac{0.0388}{\log_{10}\frac{2D}{d}}$$

$$= \frac{0.0388}{\log_{10}\frac{144}{0.3648}}$$

$$= \frac{0.0388}{2.596} = 0.0149\ \mu\text{f per mile,}$$

or the two results agree with sufficient accuracy.

Example 84

Suppose that the wires of Example 83 are vertically above each other, with 6 ft. between the middle wire and each of the other two wires. Since

$$D = \sqrt[3]{D_1 \times D_2 \times D_3}, \tag{912}$$

$$D = \sqrt[3]{72 \times 72 \times 144}$$

$$= 72\sqrt[3]{2} = 90.8\text{ in.,}$$

and

$$C = \frac{0.0388}{\log_{10}\frac{2 \times 90.8}{0.3648}} = \frac{0.0388}{2.698}$$

$$= 0.0143\ \mu\text{f per mile,}$$

or

$$C = 150 \times 0.0143 = 2.15\ \mu\text{f.}$$

135. Charging Current, Reactance, Susceptance.—A transmission line of capacitance C farads to neutral and of length such that the emf is approximately constant over the whole line has a charging current

$$I_c = 2\pi f C E_N\text{ amp,} \tag{978}$$

where I is the current for each wire,
C is the capacitance in farads to neutral, and
E is the voltage to neutral.

The capacitance of a single-phase line to neutral is twice the capacitance from wire to wire or per mile of line. For a polyphase line, the capacitance to neutral should always be used in the calculation of the line characteristics. The capacitive reactance is then

$$x_c = \frac{E_N}{I_c} = \frac{-1}{2\pi f C} \text{ ohms,} \tag{118}$$

and the capacitive susceptance is

$$b_c = \frac{-1}{x_c} = \frac{I_c}{E_N} = 2\pi f C \text{ mhos,} \tag{979}$$

since the resistance in series with the charging current path may be neglected.

136. Corona Phenomena on Transmission Lines.—Under ordinary atmospheric conditions, air is very nearly a nonconductor of electricity and it is so considered for low voltages. It is well known, however, that an electric current may be sent through a vacuum at low voltages and also through air at ordinary pressure if the voltage applied is sufficient; *e.g.*, when the voltage applied between two needle points is gradually increased, a voltage is finally reached when a faint purplish glow appears and extends outward from the points and a faint hissing sound is heard. If the voltage is increased much above this value, a disruptive spark passes between the points, followed by a continuous rush of current along the ionized path until the arc is broken by lowering the voltage or by the lengthening of the path by the heated arc stream rising and becoming longer. These same effects may be observed in the case of the smooth cylindrical wires of a transmission line. In this case, the first evidence of a breakdown in the air appears as streamers or brush discharge from dust particles or irregularities on the surface of the conductors, similar to the discharge from the needle points. At a slightly higher voltage, corona appears as a glow spreading nearly uniformly along the entire length of the wire, the hissing sound becomes pronounced, and a faint odor is noticeable due to compounds of oxygen and nitrogen being formed in the air as well as to the production of ozone from the oxygen of the air. Since these phenomena are an evidence of a loss of energy, a wattmeter placed in the circuit will indicate power input to the line although there is no load at the receiving end.

The phenomena of brush discharge, or corona, and the subject of the conductivity of gases in general have been studied by many scientists, but a detailed discussion of the many theories advanced is not within the scope of this text. The subject remained of scientific interest only and of no practical importance to the engineer until the voltages used for transmission reached such a value that corona appeared at the spacings used for the higher voltage systems; and it became necessary to determine the minimum spacing for different sizes of wires at the different voltages then in use or contemplated for new designs. The earlier research being limited to laboratory conditions, it became necessary to apply the results of these investigations to practical operating conditions of transmission lines. This involved experimental research on a large scale at great expense and as nearly as possible following actual conditions of climate and other determining factors. This was followed by laboratory work under controlled conditions to determine the variables of the problem and the effect of a variation of each of these determinants. The brief summary presented here is taken from the extensive research and published results of F. W. Peek, J. B. Whitehead, and others, the empirical equations presented being taken from these sources. The use of such equations is essential in the calculation of the operating characteristics of a transmission line, if corona is present, and to obtain the proper dimensions of the line before it is constructed.

According to the electron theory, air, under ordinary conditions, contains very few free electrons and, hence, its conductivity is very low. When a low voltage is applied between the two conductors, any free electrons in the air near the conductors are set in motion. As the potential is increased, the velocity, and hence the kinetic energy, of the electrons is increased. The moving electrons collide with the atoms of the air, but, as long as their kinetic energy is less than a certain minimum, the impact is elastic and the electrons are reflected without loss of energy. If the kinetic energy is greater than this amount at the time of collision, free electrons are detached from the atoms of the air, leaving positively charged atoms or ions, and the air is said to become ionized by the collision. The kinetic energy of the electrons is determined by the voltage gradient along the line of motion. When the voltage gradient exceeds the dielectric strength of air (30 kv per centimeter) for a definite radial distance beyond the surface of the conductor, the kinetic energy, or the $\Sigma mv^2/2$ of the moving electrons, is sufficient to produce ionization, the conductivity of the air is greatly increased, and corona appears for the radial distance represented by the limits of the ionization.

137. Disruptive Critical Voltage.—The voltage gradient at a distance x cm from the center of a conductor is given by the expression

$$\frac{de}{dx} = v_x = \frac{E_m}{x \log_\epsilon \frac{2D}{d}} \text{ kv per centimeter,} \tag{980}$$

where v_x is the voltage gradient, kv per centimeter;
E_m is the maximum kv to neutral;
D is the spacing of the conductors, cm;
d is the diameter of conductors, cm.

The gradient at the surface of the conductor is

$$v = \frac{E_m}{\frac{d}{2} \log_\epsilon \frac{2D}{d}} \text{ kv per centimeter.} \tag{981}$$

The voltage that will produce the critical gradient of 30 kv per centimeter at the surface of the conductor is, therefore,

$$E_{m_o} = 30 \frac{d}{2} \log_\epsilon \frac{2D}{d} \text{ kv} \tag{982}$$

to neutral. This is called the "disruptive critical voltage," but corona cannot start at this voltage because the gradient is less than 30 kv at any finite distance beyond the conductor surface.

138. Visual Critical Voltage.—Since for a given spacing and voltage, the voltage gradient at the conductor surface increases with a decrease in the diameter of the conductor, it would appear that the radial distance over which the critical gradient of 30 kv per centimeter must be exceeded in order to produce corona would be less for small wires than for those of larger diameter. This has been found to be true and it has been experimentally determined that, for corona to form, the voltage gradient must exceed the critical value for a radial distance of $0.301\sqrt{d/2}$ cm beyond the surface of the conductor. Thus, when visual corona appears first, the gradient is known to be equal to 30 kv per centimeter at a distance of

$$x = \frac{d}{2} + 0.301\sqrt{\frac{d}{2}} \text{ cm} \tag{983}$$

from the center of the conductor. The voltage to produce this condition is found by substituting these values in Eq. (980),

$$\begin{aligned} e_{m_v} &= 30\left(\frac{d}{2} + 0.301\sqrt{\frac{d}{2}}\right) \log_\epsilon \frac{2D}{d} \text{ kv} \\ &= 30\,\frac{d}{2}\left(1 + \frac{0.301}{\sqrt{\frac{d}{2}}}\right) \log_\epsilon \frac{2D}{d} \text{ kv} \end{aligned} \tag{984}$$

to neutral. This value of voltage is called the "visual critical voltage."

139. Spark-over Voltage.—If the impressed voltage is increased above the value for the visual critical voltage at which corona first appears, the region of the corona spreads rapidly in a radial direction from the conductors and at a certain voltage a disruptive spark jumps from one conductor to the other. The voltage at which this occurs is called the "spark-over voltage." This is given by the expression

$$E_{m_s} = 30\,\frac{d}{2}\left(1 + \frac{0.01\frac{2D}{d}}{\sqrt{\frac{d}{2}}}\right) \log_\epsilon \frac{2L}{d} \text{ kv} \tag{985}$$

to neutral. It is seen that, when

$$\frac{2D}{d} = 30.1,\ E_{m_s} = E_{m_v} \text{ kv},$$

spark-over occurs at the same time that corona forms. At smaller ratios of spacing to diameter, spark-over will occur before corona forms.

140. Corona Loss in Transmission.—When the voltage between the conductors is below the value E_{m_v}, the energy loss in the air dielectric is negligible. Above this voltage, however, the loss becomes appreciable, increasing very rapidly with an increase in voltage, and it can be measured by the use of a wattmeter connected into the line. It has been found by experiment that the corona loss varies as the square of the excess voltage above the effective value of the disruptive critical voltage E_0. This loss is not independent of the frequency, but consists of a constant component and a component varying directly with the frequency, the loss being given by the expression

$$P_c = 241\sqrt{\frac{d}{2D}}(f + 25)(E - E_0)^2 \text{ kw/km.} \qquad (986)$$

Some of the factors affecting the corona loss at a given voltage are the diameter and spacing of the conductors, air density, surface condition of the conductor (such as rough or smooth, solid or stranded), wave form of the voltage, and some weather conditions, particularly snowstorms. These factors are discussed below.

Diameter and Spacing of the Conductors.—An inspection of the above equations will show that, for a given spacing, D, E_{m_0}, and E_{m_v} increase rapidly with an increase in the diameter d. With a given diameter of conductor, an increase in the spacing produces a similar effect but to a smaller degree, since D enters only in the logarithmic term. The corona loss is decreased by an increase in D and also by an increase in d, since the effect of an increase in d on the term $\sqrt{d/2D}$ is less than on the last term $(E - E_o)^2$.

Air Density.—The density of the air affects the process of ionization, lower densities permitting the corona to spread to a greater distance for the same voltage gradient, and hence it has an effect on all the quantities considered. Since the density of the air varies directly as the barometric pressure and inversely as the absolute temperature, the density correcting factor is

$$\delta = \frac{3.92b}{273 + t}, \qquad (987)$$

where b is the barometric pressure, cm of mercury, and
t is the temperature, °C.
Under standard, or sea-level, conditions (76 cm and 25°C.), the value of the air-density factor is 1.00.

Conductor Surface.—The experimental data from which the above expressions were derived were obtained from smooth, polished wires. For weathered wires or stranded cable, the voltage must be reduced by a constant m depending on the condition of the surface. These correction factors will be discussed later in their application to the different voltages considered.

Wave Form.—The expressions for the voltages are given above in maximum values which can be easily converted to effective values for sine waves by dividing by $\sqrt{2}$. For peaked voltage waves, the effective voltages are lower, and for flat-topped waves they are higher than for sine waves of the same maximum values. For the determination of corona voltages and corona losses, the crest factor of the wave should be determined.

141. Corona Calculations.—Introducing the various correction factors, expressions may be obtained for use in calculation of the characteristics of transmission lines, all quantities being expressed in metric units and sine waves of voltage being assumed. Disruptive critical voltage,

$$E_o = 21.1 m_0 \delta \frac{d}{2} \log_\epsilon \frac{2D}{d} \text{ kv}, \tag{988}$$

effective to neutral.

Visual critical voltage,

$$E_v = 21.1 m_v \delta \frac{d}{2} \left(1 + \frac{0.301}{\sqrt{\delta \frac{d}{2}}} \right) \log_\epsilon \frac{2D}{d} \text{ kv}, \tag{989}$$

effective to neutral.

Spark-over voltage,

$$E_s = 21.1 \delta \frac{d}{2} \left(1 + \frac{0.01 \frac{2D}{d}}{\sqrt{\delta \frac{d}{2}}} \right) \log_\epsilon \frac{2D}{d} \text{ kv}, \tag{990}$$

effective to neutral.

Power loss,

$$P_c = \frac{241}{\delta}\sqrt{\frac{d}{2D}}(f + 25)(E - E_o)^2 \tag{991}$$

kilowatts per kilometer of single conductor. In these equations the values of the correction factors are

$m_o = m_v = 1.00$ for polished wires,
$m_o = 0.93$ to 0.98 for weathered wires
$= 0.83$ to 0.87 for seven-strand cables,
$m_v = 0.72$ for local corona all along the line
$= 0.82$ for decided corona along the line.

142. Exciting Conductance.—The exciting conductance g, due to corona loss, if present, must be used in connection with the capacitive susceptance b as previously derived to obtain the value of the admittance y per unit length of line. This is usually calculated at the normal voltage of the line and assumed to be a constant. This assumption is not strictly correct since the value of the loss varies directly with the square of the excess voltage above the disruptive critical voltage. In this calculation, the value of g is found as

$$g = \frac{\text{corona current}}{E_{RN}} \text{ mhos,} \tag{992}$$

or

$$g = \frac{P_C}{E_{RN}^2} \text{ mhos.} \tag{993}$$

Since the loss is greater for low values of air density, it is greater for higher altitudes near the generating plant if the energy is generated by hydroelectric stations situated in mountainous regions. The loss will vary along the line with the air-density factor and also with the variation of the voltage distribution along the line. For this reason, long transmission lines over country of variable altitude must be broken up into sections for calculation of the voltage distribution, regulation, and efficiency.

Problems

1-12. A single-phase distribution circuit 3.5 miles long is constructed of No. 00 B. & S. stranded copper conductor of 0.414-in. diameter spaced 24 in. apart. Find the total resistance, inductance, and inductive reactance of the line for 60 cycles.

2-12. A single-phase distribution circuit 4,000 ft. long is constructed of 500,000-circular-mil stranded copper conductor spaced 15 in. on centers. Find the total resistance, inductance, and inductive reactance of this line for both conductors for 60 cycles.

3-12. A single-phase transmission line 150 miles long is constructed of No. 2 B. & S. copper conductor spaced 4 ft. apart. Find the resistance, inductance, and inductive reactance per conductor of this line at 60 cycles.

4-12. A three-phase line 400 miles long is composed of three 250,000-circular-mil aluminum conductors spaced delta 6 ft. apart. Find the resistance and inductive reactance of this line at 50 cycles.

5-12. A three-phase transmission line 200 miles long is constructed of 500,000-circular-mil aluminum steel-cored conductors, the conductors being spaced 8 ft. apart horizontally. Calculate the resistance and inductive reactance of this line at 60 cycles.

6-12. Find the line constants of a 60-cycle three-phase transmission line 100 miles long constructed of No. 00 B. & S. copper conductor spaced 6 ft. apart in a vertical plane.

7-12. A three-phase transmission line is constructed of stranded copper conductors of 250,000-circular-mil cross section spaced 4, 5, and 6 ft. apart in a triangle. What is the inductance and resistance of 150 miles of this line?

8-12. Find the resistance per wire and the inductance and capacitance to neutral of a three-phase transmission line 250 miles long, constructed of 500,000-circular-mil stranded copper conductors spaced horizontally 10 ft. apart.

9-12. Find the inductive reactance, capacitive susceptance, and resistance of a 220,000-volt three-phase 50-cycle transmission line 120 miles long, constructed of 300,000-circular-mil stranded aluminum steel-cored conductors spaced at the corners of an equilateral triangle 20 ft. on each side.

10-12. Find the same quantities as required in Prob. 9-12 if the wires are spaced 20 ft. apart in horizontal spacing.

11-12. A three-phase 60-cycle transmission line is constructed of No. 00 B. & S. copper conductors and has a resistance of 36 ohms and an inductive reactance of 70 ohms. The spacing of these conductors is now doubled; what are the new resistance and reactance?

12-12. The following data apply to the line of the Southern California Edison Co. from Big Creek to Eagle Rock: voltage, 220,000; frequency, 50 cycles; length of line, 240 miles; conductors, 605,000-circular-mil steel-core aluminum; arrangement, 17 ft. 3 in. horizontal spacing. (*a*) Calculate the inductance and the inductive reactance of the line, the conductors having an outside diameter of 0.953 in., there being 54 aluminum conductors, each 0.1059 in. in diameter; neglect the conductance and inductance of the steel core. (*b*) Calculate the capacitance, the capacitive reactance and susceptance, and the charging current and kva per wire.

13-12. A three-phase 60-cycle transmission line has a resistance of 19 ohms per wire, an inductive reactance of 38 ohms per wire, and a capacitive susceptance to neutral of 0.000136 mho per wire. The line is to be operated at a frequency of 30 cycles; what are the new values of the preceding constants?

CHAPTER XIII

CALCULATION OF SHORT TRANSMISSION LINES AND DISTRIBUTION CIRCUITS

The early d-c stations had a very limited radius for the distribution of energy and a very limited station capacity in consequence. It was found impossible, however, to commutate large amounts of direct current at high voltages to make d-c systems of advantage for the longer distances of transmission in this country, and since, as a consequence, alternating current has become universally used for transmission, it is necessary to study the relation of the various phenomena common to such systems. Alternating current is used almost exclusively for the transmission of energy on account of the relative economy in copper and other materials made possible by the use of high voltages, the weight of the conductor material varying inversely with the square of the voltage of transmission. Also, owing to the ease and efficiency of transforming large or small amounts of power from one voltage to another at the same frequency by means of transformers, the adoption of alternating current has made possible great advances in distance of transmission of energy during the past 35 years.

143. Standard Voltages of Transmission.—The standardization of 115 volts for incandescent lamps fixes a unit, the multiples of which give the common transmission voltages in use in this country. The following partly standardized voltages are in use at present:

115 volts for incandescent lights and fractional-horsepower motors.

230 volts for motors of medium size and for the outside wires of the three-wire single-phase lighting circuit.

2,300 volts for primary distribution in cities of medium size and for larger motors.

4,000 volts between the outside wires of 2,300-volt four-wire three-phase distribution systems with 2,300 volts to neutral.

6,600 volts for distances of from 3 to 10 miles and in longer primary distribution lines in larger cities.

22,000 volts for distances of from 20 to 30 miles (the first long-distance polyphase voltage from Niagara Falls to Buffalo, New York).

33,000 volts for distances of from 30 to 50 miles. Standard for many interurban railway transmission lines and medium length of power transmission.

66,000 volts for distances of from 50 to 100 miles. About the present dividing point between the pin- and suspension-type insulators.

110,000 volts for distances of from 100 to 200 miles where large amounts of power are available.

285,000 volts is the present upper limit of transmission-line voltage (1938).

The progress from the first 22,000-volt transmission line to the present 285,000-volt lines has been a matter of about 35 years of development. The whole field of investigation of the proper material for, and the design of the shape of, the insulators for the lines and the bushings of transformers, as well as the internal design of the apparatus to permit these advances in transmission voltage, has been governed by a large amount of research, both mathematical and experimental, into the phenomena of the voltage stresses in the insulation itself and in the media surrounding the wires. The results of these researches have been employed in determining the spacing of the conductors and the diameter of the conductor necessary to limit corona loss from wire to wire at the higher line potentials.

144. Constants of Short Overhead Transmission Lines.—A single-phase overhead transmission line or distribution circuit may be considered as a coil of wire of one turn. The area of this coil is determined by the spacing of the wires and by the length of the line. The resistance may be taken either as the resistance of the two wires together, *i.e.*, twice the resistance of one wire, or the resistance of one wire may be used together with the voltage to neutral, as is done in the calculation of polyphase lines. The inductance to be considered is that of one wire for polyphase systems and also for single-phase systems when neutral voltage is employed in the calculation. The reactance is calculated from the inductance and the frequency of transmission as discussed in Chap. XII.

In polyphase lines, the current is assumed to return through a neutral of zero resistance and reactance. Hence, the current in one wire and the constants of one wire for the required length of lines are used, as determined in Chap. XII. It is seen that a zero current, for balanced loads, flowing in any neutral resistance and reactance is equivalent to any value of current flowing in a return having zero constants. Obviously, the drop in one line wire for a single-phase circuit will be one-half the total drop in the circuit; hence, the neutral voltage, one-half the line voltage, may be used with the constants of one wire for the calculation of single-phase circuits.

For low voltages, transmission is limited, by economic reasons, to relatively small distances, and in consequence, the amount of charging current due to line capacitance is small. The line current, for a given amount of power transmitted, being large for low voltages, the small charging current is negligible in affecting the operating characteristics of short lines. Transmission lines of 66,000 volts or less, and less than 100 miles in length, may be considered in this class. The short line may be defined as a line in which the effect of capacitance is so small that it may be omitted in making the calculations for the operating characteristics of the line.

145. Characteristics of Loads.—Transmission lines must be built to serve all classes of loads or combinations of loads. These loads may be for residence or commercial lighting with unity power factor, but requiring a close voltage regulation on account of the characteristics of the incandescent lamp filament as to candle power and life. They may be arc lamps with a relatively high power-factor, constant, inductive load during the night. They may be induction motors with current lagging behind the impressed voltage and with power factors varying from 0.20 at light loads to 0.85 at full load. They may be synchronous motors with power-factor adjustment for unity or for either leading or lagging currents of any power factor. They may be electric-heating loads of unity power factor; or electric-furnace loads with high or low power factor, depending upon the type of furnace; or other loads with characteristics peculiar to themselves, or combinations of these loads. These loads are all combined at the receiving end of the line or at points along the line and will vary from hour to hour and from day to day in amount and power factor according to the particular combination of the several demands. They may be balanced, *i.e.*, have equal currents in all phases and line wires, or they may be unbalanced; however, they are usually fairly well balanced automatically by the polyphase-motor loads. It is the duty of the engineer so to design and construct the lines that they will give acceptable service to the customers at all points along the line at all times.

The following study of the transmission line will be made with all these factors of operation in view. In the consideration of such lines, the customer's voltage, or receiving-end voltage, is usually considered to be kept constant, the station voltage being varied according to load characteristics to maintain this condition. With the load conditions known, the operating characteristics usually desired are the sending-end voltage, current, and power factor, as well as the efficiency and regulation of the line. The load, or receiving-end, voltage usually is used as the reference in making the calculation for these quantities, although any reference may be taken, the load current sometimes being chosen.

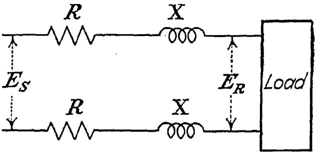

FIG. 251.—Equivalent circuit for a short transmission line.

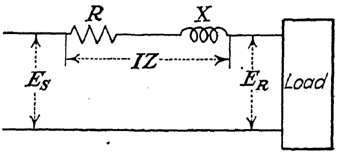

FIG. 252.—Equivalent circuit for a short transmission line.

146. Calculation of the Operating Characteristics of a Single-phase Transmission Line.—Since the charging current for a short transmission line or a distribution circuit may be neglected when compared with the load current, the resistance and the inductance of the wires may be lumped, *i.e.*, taken as if the line were a coil, and the line may be represented by an equivalent circuit, as shown in Fig. 251. The resistance and reactance of the two wires may be further combined, giving the equivalent circuit shown in Fig. 252, where R is the resistance and X is the reactance of both wires, calculated from equations in Chap. XII. This combination of resistances and reactances is only possible when the same current flows in each of the wires, as is the case for the type of circuits here considered.

The line current and its phase relation to E_R must be known in order to calculate the effect of impedance drop in the lines. If the load is given in terms of power and power factor, the current may be obtained from the power relation, previously derived,

$$P_R = E_R I_R \cos \theta_R \text{ watts,} \tag{51}$$

where P_R is the load or power delivered, watts;
I_R is the load or line current, amp;
E_R is the voltage between lines at the receiving end; and
$\cos \theta_R$ is the power factor of the load, θ_R being positive for leading currents and negative for lagging currents.

Solving Eq. (51) for the current,

$$I_R = \frac{P_R}{E_R \cos \theta_R} \text{ amp.} \tag{994}$$

This current expressed in vector form, with the receiving-end voltage as the reference, is given as

$$= i_R + j i_R' \text{ amp} \tag{995}$$

where

$$i_R = I_R \cos \theta_R \text{ amp} \tag{996}$$

is the active component of the load current and

$$i_R' = I_R \sin \theta_R \text{ amp} \tag{997}$$

is the reactive component of the load current. The sign of the numerical value of i_R will be positive for power output of the line or power input to the load; and for i_R' it will be positive for capacitive loads and negative for inductive loads, carrying the same sign as $\sin \theta_R$, the reactive factor of the load.

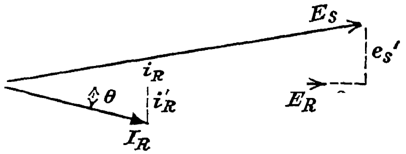

FIG. 253.—Vector diagram for a short transmission line.

Using the equivalent circuit diagram, as shown in Fig. 252, and the line voltage at the load as the reference,

$$E_R = E_R + j0 \text{ volts.} \tag{998}$$

The line current is the same as the load current and is expressed as

$$\dot{I}_R = i_R + j i_R' \text{ amp} \tag{999}$$

as shown in the vector diagram, Fig. 253.

The voltage at the sending end of the line, as derived by applying Kirchhoff's laws to the circuit shown in Fig. 252, becomes

$$\dot{E}_S = \dot{E}_R + \dot{I}\dot{Z} \text{ volts.} \tag{369}$$

This equation is independent of the reference chosen. Since, for this solution, the receiver or load voltage has been chosen as the reference, Eq. (998), the sending-end voltage becomes

$$\begin{aligned} \dot{E}_S &= E_R + (i_R + ji_R')(R + jX) \text{ volts} && (370)\\ &= E_R + i_RR - i_R'X + j(i_RX + i_R'R) \text{ volts} &&\\ &= e_S + je_S' \text{ volts,} && (371) \end{aligned}$$

where

$$e_S = E_R + i_RR - i_R'X \text{ volts} \tag{372}$$

and

$$e_S' = i_RX + i_R'R \text{ volts.} \tag{373}$$

The magnitude of E_S is

$$E_S = \sqrt{e_S^2 + (e_S')^2} \text{ volts,} \tag{1000}$$

and the power input to the sending-end of the line is

$$P_S = e_Si_S + e_S'i_S' \text{ watts.} \tag{336}$$

The vector and apparent power at the sending end of the line is

$$\text{U}_S = E_SI_S \text{ volt-amp.} \tag{1001}$$

The impedance drop in the line is

$$\text{Impedance drop} = IZ \text{ volts,} \tag{1002}$$

and the line drop is

$$\text{Line drop} = E_S - E_R \text{ volts,} \tag{1003}$$

giving the regulation of the line as

$$\begin{aligned} \text{Regulation} &= \frac{E_{R0} - E_R}{E_R} 100 \text{ per cent} &&\\ &= \frac{E_S - E_R}{E_R} 100 \text{ per cent,} && (1004) \end{aligned}$$

where E_{R0} is the receiving-end voltage at no load with E_S held constant at the sending end at its value given in Eq. (1000).

Since there is no current flowing in the line at no load, then E_{R0} is equal to E_S for the short transmission line.

The efficiency of the line is

$$\text{Efficiency} = \frac{P_R}{P_S} 100 \text{ per cent,} \tag{1005}$$

and the sending-end power factor is

$$P.F._S = \frac{P_S}{E_S I_S} = \frac{P_S}{E_S I_R}. \tag{969}$$

The following definitions are used in the comparison of transmission lines as well as in their specification:

Impedance Drop.—The impedance drop in a transmission line is the product of the line current and the line impedance.

Line Drop.—The line drop in a transmission line is the numerical difference between the voltages at the sending and receiving ends of the line for any load.

Regulation.—The regulation of a transmission line is the percentage change in voltage at the receiving end (the customer) in terms of the receiving-end voltage when the load is reduced to zero, the voltage at the sending end being held constant at such a value as to give rated voltage at the receiving end for the particular load chosen.

Efficiency.—The efficiency of a transmission line is the active-power output of the line (the load) divided by the active-power input.

Power Factor.—The power factor at the receiving end of a transmission line is the power factor of the combined loads. The power factor at the sending end of the line is the active-power input divided by the apparent-power input.

147. Vector Diagrams for Single-phase Lines Using Emf as the Reference.—The vector diagram for a short single-phase transmission line with a nonreactive load is shown in Fig. 254, where E_R is taken as the reference, I_R is drawn in phase with E_R, and the impedance triangle is erected on this current vector. The resistance drop $I_R R$ is in phase with the current vector and is added directly to E_R. The reactance drop $I_R X$ is 90 deg. ahead of the current and is added to the vector $E_R + I_R R$, giving the vector E_S, as shown. The triangle formed by $I_R R$, $I_R X$, and $I_R Z$ is the impedance triangle for the line, $I_R Z$ being the impedance drop in the line, which is added vectorially to E_R to obtain E_S.

The impedance triangle may be formed either directly on the current line at the origin, as shown in the capacitive load diagram, Fig. 255, or at the end of the E_R vector, as shown in Fig. 254. The vector E_S is the sending-end voltage in its proper relation to the line current I_R and has the components e_S and e_S' in phase with and 90 deg. out of phase with the reference axis. The inductive-load diagram is shown in Fig. 256. It is seen from these diagrams that, for a given current, the impedance drop in the line is constant but that the line drop may be positive, zero, or negative, resulting in positive, zero, or negative line regulation.

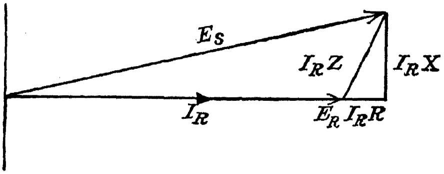

FIG. 254.—Vector diagram for a short line, nonreactive load.

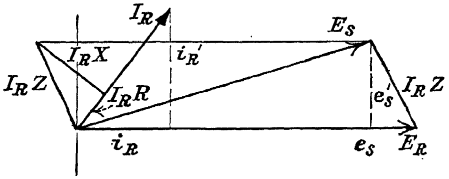

FIG. 255.—Vector diagram for a short line, capacitive load.

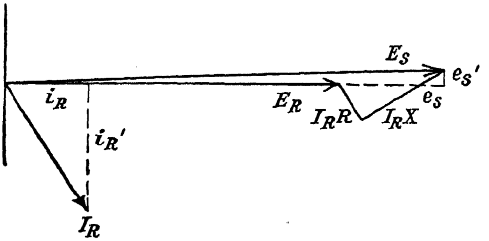

FIG. 256.—Vector diagram for a short line, inductive load.

Example 85

Calculate the operating characteristics of a 2,200-volt single-phase distribution line delivering 0, 50, 100, and 200 amp to loads of unity power factor and to inductive and capacitive loads of 0.80 power factor. The line has a total resistance of 1 ohm and a total reactance of 3 ohms for both wires.

The equivalent circuit diagram is shown in Fig. 252. Taking the receiver voltage as the reference,

$$\dot{E}_R = 2{,}200 + j0 \text{ volts},$$

and making the calculation for 200-amp inductive load,

$$\begin{aligned} I_R &= 200\underline{/-36°52'} \text{ amp} \\ &= 160 - j120 \text{ amp.} \end{aligned}$$

Then

$$\begin{aligned} \dot{E}_S &= E_R + I_R Z \text{ volts} \\ &= 2{,}200 + j0 + (160 - j120)(1 + j3) \text{ volts} \\ &= 2{,}200 + 520 + j360 \text{ volts} \\ &= 2{,}720 + j360 \text{ volts,} \end{aligned}$$

having a magnitude of

$$E_S = \sqrt{2720^2 + 360^2} = 2{,}744 \text{ volts.}$$

The active-power input to the line is

$$\begin{aligned} P_S &= ei + e'i' \text{ watts} \\ &= \frac{2{,}720 \times 160 + 360 \times (-120)}{1{,}000} \text{ kw} \\ &= 392 \text{ kw.} \end{aligned}$$

The active-power output of the line is

$$P_R = \frac{2{,}200 \times 160}{1{,}000} = 352 \text{ kw.}$$

The efficiency of the line at this load is

$$\text{Efficiency} = \frac{352}{392}100 = 89.8 \text{ per cent.}$$

When the load is removed, the sending-end voltage being maintained constant at 2,744 volts, the receiving-end voltage rises to this same value, so that E_{R0} becomes 2,744 volts, and the regulation is

$$\begin{aligned} \text{Regulation} &= \frac{2{,}744 - 2{,}200}{2{,}200}100 \text{ per cent} \\ &= 24.7 \text{ per cent.} \end{aligned}$$

The power factor at the sending end is

$$P.F._S = \frac{392}{2.744 \times 200}100 = 71.3 \text{ per cent.}$$

In making a study of the characteristics of a transmission line, it is necessary to calculate the preceding quantities for various loads and power factors. The tabular method of presenting these calculations is very convenient and easily visualized. An illustration of this tabular method of calculation is given in Table VI for this example.

TABLE VI.—CALCULATION OF THE OPERATING CHARACTERISTICS OF A 2,200-VOLT SINGLE-PHASE DISTRIBUTION LINE USING EMF BETWEEN WIRES

Example 85

Kind of load	Zero	Non-reactive			Inductive			Capacitive		
Power factor of load	.....	1.00	1.00	1.00	0.80	0.80	0.80	0.80	0.80	0.80
Line current, I_R	0	50	100	200	50	100	200	50	100	200
Active component, i_R	0	50	100	200	40	80	160	40	80	160
Reactive component, i_R'	0	0	0	0	−30	−60	−120	30	60	120
Line voltage, E_R	2,200	2,200	2,200	2,200	2,200	2,200	2,200	2,200	2,200	2,200
i_RR	0	50	100	200	40	80	160	40	80	160
$i_R'X$	0	0	0	0	−90	−180	−360	90	180	360
$e_S = E_R + i_RR - i'_RX$	2,200	2,250	2,300	2,400	2,330	2,460	2,720	2,150	2,100	2,000
$i_R'R$	0	0	0	0	−30	−60	−120	30	60	120
i_RX	0	150	300	600	120	240	480	120	240	480
$e_S' = i_R'R + i_RX$	0	150	300	600	90	180	360	150	300	600
$E_S = \sqrt{e^2_S + e'^2_S}$	2,200	2,255	2,320	2,475	2,332	2,467	2,744	2,155	2,121	2,090
Apparent power input $= \frac{E_SI_S}{1,000}$	0	112.75	232.0	495.0	116.6	246.7	548.8	107.75	212.1	418.0
Power output, $P_R = \frac{E_Ri_R}{1,000}$	0	110	220	440	88	176	352	88	176	352
Apparent power output $= \frac{E_RI_R}{1,000}$	0	110	220	440	110	220	440	110	220	440
$\frac{e_Si_R}{1,000}$	0	112.5	230	480	93.2	196.8	435.2	86	168	320
$\frac{e_S'i_R'}{1,000}$	0	0	0	0	−2.7	−10.8	−43.2	4.5	18	72
Power input $= \frac{e_Si_R + e_S'i_R'}{1,000} = P_S$	0	112.5	230	480	90.5	186	392	90.5	186	392
Power factor at the sending end $= \frac{P_S}{E_SI_R}$	.....	99.7	99.1	97	77.5	75.4	71.3	84.0	87.7	93.7
Efficiency $= \frac{P_R}{P_S}$	.....	97.8	95.7	91.8	97.4	94.6	89.8	97.4	94.6	89.8
Impedance drop $= IZ$	0	158	316	632	158	316	632	158	316	632
Line drop $= E_S - E_R$	0	55	120	275	132	267	544	−45	−79	−110
Regulation $= \frac{E_S - E_R}{E_R}$	.....	2.5	5.45	12.5	6.05	12.2	24.7	−2.02	−3.58	−5.0

148. Operating Characteristics of Three-phase Transmission Lines Using Voltage at Load as the Reference.—For a three-phase transmission line, the calculation should be made on the basis of the emf to neutral and line current, the system being assumed balanced. In Chap. VIII, it was shown that, for balanced polyphase systems, the line current bears the same phase relation to the neutral voltage that the load current does to the load voltage. In Fig. 257, for nonreactive load, E_{OA} is the neutral voltage and I_A is the current in line wire A, in phase with E_{OA}, where

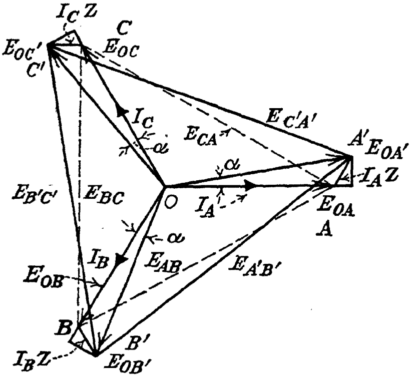

FIG. 257.—Complete vector diagram for a short three-phase transmission line with nonreactive load.

$$E_{OA} = \frac{E_{AB}}{\sqrt{3}} \text{ volts}$$

and

$$I_A = \frac{P_R}{\sqrt{3}E_{AB}} = \frac{P_R}{3E_{OA}} \text{ amp.}$$

The emf. E_{OA}' is found in its proper length and phase relation to E_{OA} by adding to E_{OA} the impedance drop due to the current I_A flowing through the resistance and reactance of one wire. In a similar manner, E_{OB}' is found by the addition of E_{OB} and the impedance drop caused by I_B, and E_{OC}' by the addition of E_{OC} and the impedance drop caused by I_C. Then, since $I_A = I_B = I_C$ and the impedances are equal, the emfs E_{OA}', E_{OB}', and E_{OC}' are equal and are revolved in a counterclockwise direction by the same angle α from the positions of E_{OA}, E_{OB}, and E_{OC}. But E_{OA}, E_{OB}, and E_{OC} are equal and at 120 deg. with each other, their ends forming the corners of the equilateral triangle ABC of the line voltages, E_{AB}, E_{BC}, and E_{CA}. Hence, E_{OA}', E_{OB}', and E_{OC}' are also at 120 deg. with each other, and their ends form the corners of the equilateral triangle $A'B'C'$ of sending-end voltages $E_{A'B'}$, $E_{B'C'}$, and $E_{C'A'}$. The triangle of line voltages at the sending end has its neutral coincident with the neutral of the voltage triangle of the receiving end, and the triangle is revolved α deg. about this neutral. Figure 258 shows a similar construction for an inductive load.

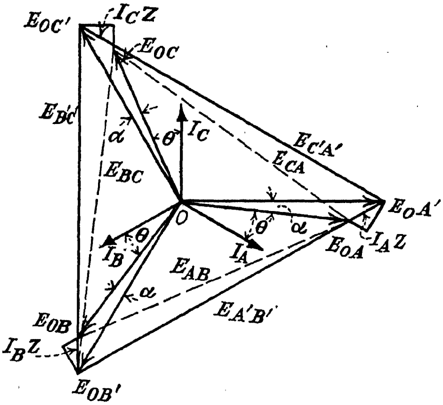

FIG. 258.—Complete vector diagram for a short three-phase transmission line with inductive load.

The line voltages of the three-phase system form a closed triangle at any given point along the line. The triangle of voltages ABC in Figs. 257 and 258 for the receiving end must be converted to the triangle of voltages $A'B'C'$ at the sending end by adding the impedance drops AA', BB', and CC', respectively, to obtain the neutral voltages E_{OA}', E_{OB}', and E_{OC}' for the balanced system, such as is assumed in the calculation of three-phase transmission lines. Since the triangles are equiangular and coaxial, the calculation for the three-phase line may be

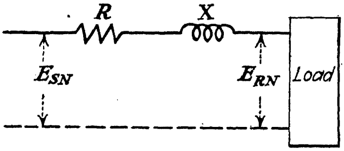

FIG. 259.—Equivalent circuit diagram for calculation of three-phase transmission lines.

carried out by using one of the vector triangles, such as OAA'. This reduces the calculation to a single-phase basis by using the emfs to neutral and the constants of one line wire. The equivalent circuit diagram for this calculation is shown in Fig. 259.

The emf at the sending end between line wires is found by multiplying E_{SN} by $\sqrt{3}$. The regulation, efficiency, and power factor at the sending end may be found, as in the single-phase system, by using neutral voltages and line currents, as illustrated in Example 86.

Example 86

A 6,600-volt three-phase 60-cycle transmission line is designed to transmit power to a 500-kva load. The line has a resistance of 4 ohms and a reactance of 12 ohms per wire. Find the regulation, the efficiency, the apparent-power input, and the power factor at the sending end for full kva output and for half load and no load at unity power factor and at power factors of 0.80 lead and lag.

Making the calculation for full-load inductive at 0.8 power factor and using E_{RN} as the reference,

$$I_R = \frac{500 \times 1{,}000}{6{,}600\sqrt{3}} = 44 \text{ amp, and}$$

$$E_{RN} = \frac{6{,}600}{\sqrt{3}} = 3{,}810 \text{ volts.}$$

Then

$$\dot{E}_{RN} = 3{,}810 + j0 \text{ volts.}$$

The sending-end voltage to neutral is

$$\begin{aligned} \dot{E}_{SN} &= E_{RN} + \dot{I}\dot{Z} \text{ volts} \\ &= 3{,}810 + 44(0.8 - j0.6)(4 + j12) \text{ volts} \\ &= 3{,}810 + (35.2 - j26.4)(4 + j12) \text{ volts} \\ &= 4{,}268 + j317 \text{ volts, and} \\ E_{SN} &= \sqrt{4{,}268^2 + 317^2} = 4{,}280 \text{ volts.} \end{aligned}$$

The power input to the line at the sending end is

$$\begin{aligned} P_S &= 3(e_{SN}i_s + e_{SN}'i_{SN}') \text{ kw} \\ &= \frac{3(4{,}268 \times 35.2 + 317 \times -26.4)}{1{,}000} \text{ kw} \\ &= 426 \text{ kw.} \end{aligned}$$

The apparent-power input to the sending end is

$$\begin{aligned} U_S = V_S &= 3E_{SN}I_S \text{ vector kva} \\ &= \frac{3 \times 4{,}280 \times 44}{1{,}000} = 565 \text{ vector kva.} \end{aligned}$$

The regulation of the line becomes

$$\text{Regulation} \quad \frac{-E_{RN}}{E_{RN}} = \frac{4{,}280 - 3{,}810}{3{,}810}$$

$$= 0.123 = 12.3 \text{ per cent.}$$

$$\text{Efficiency} = \frac{400}{426}100 = 94.5 \text{ per cent.}$$

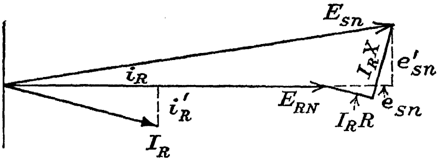

FIG. 260.—Vector diagram for the three-phase line of Example 86.

The sending-end power factor is

$$P.F._s = \frac{P_S}{V_S} = \frac{426}{565}100 \qquad 75.5 \text{ per cent.}$$

The vector diagram for this calculation is similar to that shown in Fig. 260. The calculation for the other loads is given in Table VII.

149. Equations of a Transmission Line Using Load Current as the Reference.

—The equivalent circuit diagram, Fig. 259, the corresponding equation for the voltage at the sending end in terms of the voltage at the receiving end, and the line constants are the same for any reference axis. However, when using the load current as the reference, the *IR* drop is always in phase with the reference axis and the *IX* drop is 90 deg. ahead of this axis. In the case of inductive loads, Fig. 261, the impressed emf E_{SN} becomes

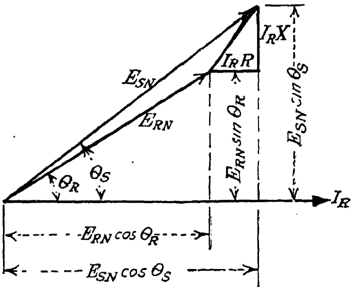

FIG. 261.—Vector diagram for a short transmission line with inductive load, using current as the reference vector.

$$\dot{E}_{SN} = (E_{RN}\cos\theta_R + I_R \quad + j(I_RX + E_{RN}\sin\theta_R) \text{ volts,} \qquad (1006)$$

having the magnitude,

$$E_{SN} = \sqrt{(E_{RN}\cos\theta_R + I_RR)^2 + (I_RX + E_{RN}\sin\theta_R)^2} \text{ volts,} \qquad (1007)$$

TABLE VII.—CALCULATION OF THE OPERATING CHARACTERISTICS OF A THREE-PHASE TRANSMISSION LINE USING THE VOLTAGE AT THE LOAD AS A REFERENCE

Example 86

Kind of load	Zero	Non-reactive		Inductive		Capacitive	
Power factor of load	.....	1.00	1.00	0.80	0.80	0.80	0.80
Reactive factor of load	.....	0	0	−0.60	−0.60	0.60	0.60
Load current $= \frac{P_R}{\sqrt{3}E_R} = I_R$	0	22	44	22	44	22	44
Active component of current, i_R	0	22	44	17.6	35.2	17.6	35.2
Reactive component of current, i'_R	0	0	0	−13.2	−26.4	13.2	26.4
Voltage to neutral, $E_{RN} = \frac{E_R}{\sqrt{3}}$	3,810	3,810	3,810	3,810	3,810	3,810	3,810
$i_R R = 4i_R$	0	88	176	70.4	140.8	70.4	140.8
$i_R' X = 12i'_R$	0	0	0	−158.5	−317	158.5	317
$e_{SN} = E_{RN} + i_R R - i'_R X$	3,810	3,898	3,986	4,039	4,268	3,722	3,634
$i_R R = 4i'_R$	0	0	0	−52.8	−105.6	52.8	105.6
$i_R X = 12i_R$	0	264	528	212	423	212	423
$e'_{SN} = i'_R R + i_R X$	0	264	528	159	317	265	529
$E_{SN} = \sqrt{e^2_{SN} + e'^2_{SN}}$	3,810	3,907	4,021	4,042	4,280	3,731	3,672
Apparent power input $= \frac{3E_{SN}I_R}{1,000}$	0	258	531	267	565	246	484
Power output $= \frac{3E_{RN}i_R}{1,000}$	0	250	500	200	400	200	400
Apparent power output $= \frac{3E_{RN}I_R}{1,000}$	0	250	500	250	500	250	500
$3\frac{e'_{SN}i'_R}{1,000}$	0	258	527	214	452	196.5	384
$3\frac{e_{SN}i'_R}{1,000}$	0	0	0	−6	−25	10.5	42
Power input $= \frac{3(e_{SN}i_R + e'_{SN}i'_R)}{1,000}$	0	258	527	208	427	207	426
Power factor, sending end $= \frac{P_S}{3E_{SN}I_R}$	.....	100	99.3	78.0	75.5	83.5	87.8
Efficiency $= \frac{P_R}{P_S}$	.....	97.0	95.0	96.2	94.5	96.2	94.5
Impedance drop $= I_R Z$	0	279	558	279	558	279	558
Voltage at sending end $= E_S = \sqrt{3}E_{SN}$	6,600	6,770	6,970	7,000	7,430	6,470	6,370
Line drop $= E_S - E_R$	0	170	370	400	830	−130	−230
Wire drop $= E_{SN} - E_{RN}$	0	97	211	232	470	−79	−138
Regulation $= \frac{E_{SN} - E_{RN}}{E_{RN}}$	0	2.5	5.5	6.1	12.3	−2.1	−3.63

where, as before, $\cos \theta_R$ is the power factor and $\sin \theta_R$ is the reactive factor of the load. For capacitive loads, Fig. 262, θ_R becomes negative and $\sin \theta_R$ is negative in Eqs. (1006) and (1007). For nonreactive loads, $\cos \theta_R = 1.00$, $\sin \theta_R = 0$, and Eq. (1006) reduces to

$$\dot{E}_{SN} = E_{RN} + I_R R + jI_R X \text{ volts,} \tag{1008}$$

the vector diagram being as shown in Fig. 254.

For a three-phase transmission line the emf to neutral and the line current are used with the constants of one line wire in the preceding equations as in the calculation using the receiving-end voltage to neutral. The emf at the sending end will be in the correct phase relation to the line-wire current, and the emf between lines will be this neutral emf multiplied by $\sqrt{3}$. This method of calculation gives simple equations for the rapid calculation of short transmission lines, and it is used as the basis for several graphical solutions.

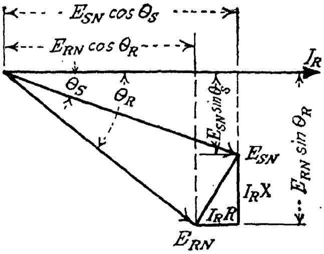

FIG. 262.—Vector diagram for a short transmission line with capacitive load, using current as the reference vector.

Example 87

Calculate the voltage to neutral at the sending end, and the active and apparent power input for the same transmission line as given in Example 86 and for the same load of 44 amp at 0.8 power factor inductive, using the load current as the reference.

From Example 86, $E_{RN} = 3{,}810$ volts, $I_R = 44$ amp, and $\dot{Z} = 4 + j12$ ohms. Then for this problem

$$\begin{aligned} \dot{I}_R &= 44 + j0 \text{ amp,} \\ \dot{E}_{RN} &= 3{,}810(0.80 + j0.60) \text{ volts} \\ &= 3{,}040 + j2{,}280 \text{ volts.} \end{aligned}$$

The sending-end voltage to neutral becomes

$$\begin{aligned} \dot{E}_{SN} &= \dot{E}_{RN} + I_R \dot{Z} \text{ volts} \\ &= 3{,}040 + 44 \times 4 + j(2{,}280 + 44 \times 12) \text{ volts} \\ &= 3{,}216 + j2{,}808 \text{ volts, and} \\ \therefore E_{SN} &= \sqrt{3{,}216^2 + 2{,}808^2} = 4{,}280 \text{ volts.} \end{aligned}$$

The power input to the sending end is

$$\begin{aligned} P_S &= 3(3{,}216 \times 44 + 2{,}808 \times 0) \text{ watts} \\ &= 423 \text{ kw.} \end{aligned}$$

The apparent power at the sending end is

$$U_S = V_S = 3E_{SN}I_S \text{ vector volt-amp}$$
$$= \frac{3 \times 4{,}280 \times 44}{1{,}000} = 565 \text{ vector kva.}$$

The vector diagram is similar to that shown in Fig. 261.

150. Per Cent and Per Unit Constants and Calculations.—Where it is necessary to compare lines of different designs and ratings as to their characteristics, it is much easier to make the comparison when the quantities involved, such as the line constants, are reduced to per cent or per unit values. A given amount of the quantity under consideration is taken as an entity of 100 units or 100 per cent of the chosen unit, all other amounts of the same quantity being given in their ratio to this amount, the ratio being expressed in per cent. A familiar example is found when the full-load rating of a machine or a transmission line is considered as such a base, it being expressed as 100 per cent of full load; any other operating load is expressed as a per cent of this full load, half load being 50 per cent of full load. For purposes of such comparisons, the per cent expressions are much more convenient than the actual rated values in kilowatts or kilovolt-amperes.

Calculations can be made using these per cent expressions, but the value 100 enters into these calculations with each quantity. This difficulty can be eliminated by using a per unit value instead of a per cent value; the base value being expressed as unity, all other ratios to this base being expressed as decimals. The per unit value is the per cent value divided by 100. Because the fractional values from one one-hundredth to unity are expressed as a whole number in the per cent system, this system is usually used in specifying the quantity, whereas the per unit system is used in the calculations. This involves no complications, since the connecting factor is always 100.

The definitions of the per unit quantities are as follows:

Per Unit Resistance Drop.—The per unit resistance drop in a transmission line is the product of rated full-load line current and the resistance of one wire divided by the rated voltage to neutral at the receiving end.

Per Unit Reactance Drop.—The per unit reactance drop in a transmission line is the product of rated full-load line current *and the reactance to neutral of one wire divided by the rated voltage to neutral at the receiving end.*

Per Unit Impedance Drop.—The per unit impedance drop in a transmission line is the product of rated full-load line current and the impedance to neutral of one wire divided by the rated voltage to neutral at the receiving end.

These definitions are reduced to equation form for convenience in calculation.

Let I_R = the rated full-load line current per wire,
E_{RN} = the rated neutral voltage at the load,
R = the resistance of one wire, and
X = the reactance of one wire due to the inductance.

Then, the full-load normal values will be

$$E_{RN} = 100 \text{ per cent voltage} = 1.00 \text{ in per unit values;}$$

$$I_R = \frac{P_R}{E_{RN}\cos\theta_R} = 100 \text{ per cent current, or } I_R = 1.00;$$

$$P_{RN} = 100 \text{ per cent power, or } P_{RN} = 1.00 \text{ at full load.}$$

Then, by definition,

$$\frac{I_R R}{E_{RN}} = \text{per unit resistance drop or per-unit resistance;} \qquad (1009)$$

$$\frac{I_R X}{E_{RN}} = \text{per unit reactance drop or per unit reactance;} \qquad (1010)$$

$$\frac{I_R Z}{E_{RN}} = \text{per unit impedance drop or per unit impedance.} \qquad (1011)$$

The number of ohms that would make the ratios expressed in Eqs. (1009), (1010), and (1011) equal to unity, *i.e.*, that would make the respective drops equal to the base voltage, would constitute a basic unit of ohms; and the unit value of R, X, and Z would equal that value in ohms. The ratio of any given value of R, X, or Z to the base unit would express the per unit value of these respective quantities. This per unit value would be identical with the per unit value of drop caused by the given number of ohms, so that Eq. (1009) expresses the per unit resistance as well as the per unit resistance drop.

Example 88

Calculate the operating characteristics of the line of Example 86, using the per unit method. The line is a 6,600-volt three-phase 60-cycle circuit designed to transmit a 500-kva load. The line has a resistance of 4 ohms and a reactance of 12 ohms per wire. The operating characteristics required are regulation, efficiency, apparent-power input, and power factor at the sending end of the line for full kva output at 0.80 power factor inductive.

$$E_{RN} = \frac{6{,}600}{\sqrt{3}} = 3{,}810 \text{ volts} = 1.00 \text{ in per unit value;}$$
$$V_{RN} = \frac{500}{3} = 166.7 \text{ vector kva} = 1.00;$$
$$I_R = \frac{500{,}000}{6{,}600\sqrt{3}} = 44 \text{ amp} = 1.00 \text{ in per unit value.}$$

$$\text{Per unit resistance} = \frac{44 \times 4}{3{,}810} = 0.046;$$
$$\text{per unit reactance} = \frac{44 \times 12}{3{,}810} = 0.138.$$

With E_{RN} as the reference,

$$\begin{aligned} \dot{E}_{RN} &= 1.00 + j0, \\ \dot{I}_R &= 0.80 - j0.60. \end{aligned}$$

Then

$$\begin{aligned} \dot{E}_{SN} &= 1.00 + (0.80 - j0.60)(0.046 + j0.138) \\ &= 1.12 + j0.083, \text{ and} \\ E_{SN} &= \sqrt{1.12^2 + 0.083^2} = 1.123 \\ &= 1.123 \times 3{,}810 = 4{,}280 \text{ volts.} \\ \text{Regulation} &= \frac{1.123 - 1.00}{1.00} = 0.123 = 12.3 \text{ per cent.} \\ P_{SN} &= [(1.12)(0.80) + (0.083)(-0.60)] = 0.849 \\ &= 0.849 \times 166.7 = 141.5 \text{ kw.} \\ \text{Efficiency} &= \frac{0.80}{0.849} = 0.944 = 94.4 \text{ per cent.} \end{aligned}$$

The apparent power to neutral is

$$\begin{aligned} V_{SN} &= E_{SN} I_R \text{ vector volt-amp} \\ &= 1.123 \times 1.00 = 1.123 \\ &= 1.123 \times 166.7 = 188 \text{ vector kva.} \end{aligned}$$

The power factor is

$$P.F._S = \frac{0.849}{1.123} = 0.755 \text{ or } 75.5 \text{ per cent.}$$

The calculation of this line for various loads and power factors is shown in Table VIII.

151. Maximum Power Transmitted over a Given Line.—It is of interest to consider the maximum amount of power that may be transmitted over a short transmission line under certain

TABLE VIII.—CALCULATION OF THE OPERATING CHARACTERISTICS OF A THREE-PHASE TRANSMISSION LINE BY THE PER UNIT METHOD USING THE EMF AT THE LOAD AS A REFERENCE

Example 88

Kind of load	Zero	Non-reactive		Inductive		Capacitive	
Power factor of load	.....	1.00	1.00	0.80	0.80	0.80	0.80
Reactive factor of load	.....	0	0	−0.60	−0.60	0.60	0.60
Load current, I_R	0	0.50	1.00	0.50	1.00	0.50	1.00
Active component, i_R	0	0.50	1.00	0.40	0.80	0.40	0.80
Reactive component, i_R'	0	0	0	−0.30	−0.60	0.30	0.60
Voltage to neutral, E_{RN}	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$i_R R = 0.046 i_R$	0	0.023	0.046	0.0184	0.0368	0.0184	0.0368
$i_R' X = 0.138 i_R'$	0	0	0	−0.0414	−0.0828	0.0414	0.0828
$e_{SN} = E_{RN} + i_R R - i_R' X$	1.00	1.023	1.046	1.0598	1.1196	0.9770	0.9540
$i_R' R = 0.046 i_R'$	0	0	0	−0.0138	−0.0276	0.0138	0.0276
$i_R X = 0.138 i_R$	0	0.069	0.138	0.0552	0.1104	0.0552	0.1104
$e'_{SN} = i_R' R + i_R X$	0	0.069	0.138	0.0414	0.0828	0.0690	0.1380
$E_{SN} = \sqrt{e^2_{SN} + e'^2_{SN}}$	1.00	1.025	1.055	1.0606	1.1200	0.9794	0.9640
Power output, $E_{RN} i_R$	0	0.50	1.00	0.40	0.80	0.40	0.80
$e_{SN} i_R$	0	0.5115	1.046	0.4239	0.8957	0.3908	0.7632
$e'_{SN} i_R'$	0	0	0	−0.0124	−0.0497	0.0207	0.0828
Power input $= P_S =$ $e_{SN} i_R + e'_{SN} i_R'$	0	0.5115	1.046	0.4115	0.8460	0.4115	0.8460
Apparent power input $= E_{SN} I_R$	0	0.5125	1.055	0.5303	1.1200	0.4897	0.9640
Efficiency $= \dfrac{P_R}{P_S}$	.....	97.8	95.5	97.5	94.6	97.5	94.6
Power factor, sending end $=$ $\dfrac{P_{SN}}{E_{SN} I_R}$	.....	99.8	99.1	77.5	75.5	83.8	87.8
Impedance drop $= I_R Z$	0	0.073	0.146	0.073	0.146	0.073	0.146
Line drop $= E_S - E_R$	0	0.025	0.055	0.0606	0.120	−0.0206	−0.0360
Regulation $= \dfrac{E_{RNO} - E_R}{E_R}$	.....	2.5	5.5	6.06	12.0	−2.06	−3.60
Line voltage sending end	6,600	6,770	6,970	7,000	7,400	6,470	6,370

conditions of operation. Consider a line with a constant impressed voltage E_{SN} in phase with the reference axis as

$$\dot{E}_{SN} = E_{SN} + j0 \text{ volts} \qquad (1012)$$

and with line impedance

$$\dot{Z} = R + jX \text{ ohms.}$$

Let the load be considered of constant power factor and represented by

$$\dot{Z}_1 = R_1 + jX_1 = R_1 + jKR_1 \text{ ohms,} \tag{1013}$$

where

$$X_1 = KR_1 = R_1 \tan \theta_R \text{ ohms.} \tag{1014}$$

The total current taken by the load and line will be

$$\dot{I}_R = \frac{\dot{E}_{SN}}{\dot{Z}_0} = \frac{\dot{E}_{SN}}{R_0 + jX_o} \text{ amp} \tag{1015}$$

of magnitude,

$$I_R = \frac{E_{SN}}{\sqrt{R_o^2 + X_o^2}} \text{ amp.} \tag{1016}$$

The power output of the line per phase becomes

$$P_{RN} = I_R^2 R_1 = \frac{E_{SN}^2 R_1}{R_0^2 + X_0^2} \text{ watts} \tag{1017}$$

$$= \frac{E_{SN}^2 R_1}{(R + R_1)^2 + (X + KR_1)^2} \text{ watts.} \tag{1018}$$

R_1 being varied, to vary the load, other conditions being kept constant, the maximum power will be found from

$$\frac{dP_{RN}}{dR_1} = 0 = \frac{d}{dR_1}\left[\frac{E_{SN}^2 R_1}{(R + R_1)^2 + (X + KR_1)^2}\right] \tag{1019}$$

$$= E_{SN}^2\left\{\frac{(R+R_1)^2+(X+KR_1)^2-2(R+R_1)R_1-2K(X+KR_1)R_1}{[(R + R_1)^2 + (X + KR_1)^2]^2}\right\}.$$

Hence,

$$(R + R_1)^2 + (X + KR_1)^2 = 2(R + R_1)R_1 + 2KR_1(X + KR_1),$$

or

$$R^2 - R_1^2 + X^2 - K^2R_1^2 = 0,$$

and

$$R^2 + X^2 = (1 + K^2)R_1^2, \tag{1020}$$

or

$$Z^2 = Z_1^2, \quad Z = Z_1 \text{ ohms,} \tag{1021}$$

i.e., for maximum power output the impedance of the load must be equal to the impedance of the line. The point of maximum power does not occur at the point of maximum efficiency.

Other power factors may produce a still greater value of power than the value at the chosen power factor. To investigate this condition consider Eq. (1020) as

$$Z^2 = (1 + K^2)R_1^2, \tag{1020}$$

or

$$R_1 = \sqrt{\frac{Z^2}{1 + K^2}} = \frac{Z}{\sqrt{1 + K^2}} \text{ ohms.} \tag{1022}$$

Substituting this value of R_1 into Eq. (1018),

$$P_{RN_{max}} = \frac{E_{SN}^2}{2Z\sqrt{1 + K^2} + 2(R + KX)} \text{ watts.} \tag{1023}$$

In view of the fact that for inductive loads K is positive, for capacitive loads K is negative, and for nonreactive loads K is zero, the following cases are interesting:
For nonreactive load $K = 0$,

$$P_{RN_{max}} = \frac{E_{SN}^2}{2(R + Z)} \text{watts.} \tag{1024}$$

For direct current $X = 0$,

$$P_{RN_{max}} = \frac{E_{SN}^2}{4R} \text{ watts.} \tag{1025}$$

For large values of K,

$$P_{RN_{max}} = \frac{E_{SN}^2}{2K(Z \pm X)} \text{ (approximately) watts.} \tag{1026}$$

The value of load resistance to produce $P_{RN_{max}}$ is

$$R_1 = \frac{Z}{\sqrt{1 + K^2}} \text{ ohms,} \tag{1022}$$

which becomes $R_1 = Z$ for nonreactive loads. For direct currents, $R_1 = R$ for maximum power, or, the load and line resistance being equal, the efficiency of transmission at maximum power direct current is 50 per cent.

The value of K to produce the very maximum value of power is found by differentiating Eq. (1023) in respect to K, as

$$\frac{dP_{RN_{max}}}{dK} = 0 = \frac{E_{SN}^2[KZ(1 + K^2)^{-\frac{1}{2}} + X]}{2[Z\sqrt{1 + K^2} + (R + KX)]^2}, \tag{1027}$$

or

$$\frac{KZ}{\sqrt{1 + K^2}} = -X, \tag{1028}$$

and

$$K = -\frac{X}{R} \tag{1029}$$

is the value of K to produce this condition. But, by assumption,

$$K = \frac{X_1}{R_1},$$

or

$$-\frac{X}{R} = \frac{X_1}{R_1}, \tag{1030}$$

and the condition is satisfied with a leading current of the same power factor as that of the line itself. But, since the condition that $Z = Z_1$ must also be satisfied for maximum power, then

$$R = R_1 \text{ ohms and } X = -X_1 \text{ ohms}, \tag{1031}$$

or the condition gives the same equation for maximum power as that for a d-c circuit [Eq. (1025)], the efficiency under this condition being 50 per cent.

Example 89

What is the maximum power that may be transmitted for unity power factor and for power factors of 0.80 inductive and capacitive with normal voltage impressed at the sending end for a 22,000-volt 60-cycle 3-phase transmission line 30 miles long constructed of No. 0000 B. & S. copper conductor spaced 4 ft. apart?

$$R = 8.30 \text{ ohms}, \qquad X = 20.4 \text{ ohms}, \qquad Z = 22.1 \text{ ohms},$$

and $E_{SN} = 12{,}700$ volts.
For unity power factor at the load, $K = 0$,

$$\begin{aligned} P_{RN_{max}} &= \frac{E_{SN}^2}{2(R + Z)} \text{ watts} \qquad (1024) \\ &= \frac{\overline{12{,}700}^2}{2(8.30 + 22.1)} = \frac{162 \times 10^6}{60.8} \text{ watts} \\ &= 2{,}660 \text{ kw}, \end{aligned}$$

and

$$P_{R_{max}} = 3 \times 2{,}660 = 7{,}980 \text{ kw}.$$

For inductive load, 0.80 power factor,

$$R_1 = 0.8Z_1 \text{ ohms}, \qquad X_1 = 0.6Z_1 \text{ ohms},$$

and

$$K = \frac{X_1}{R_1} = \frac{0.6Z_1}{0.8Z_1} = 0.75,$$

hence,

$$P_{RN_{max}} = \frac{E_{SN}^2}{2Z\sqrt{1 + K^2} + 2(R + KX)} \text{ watts} \tag{1023}$$

$$= \frac{162 \times 10^6}{2 \times 22.1\sqrt{1 + \overline{0.75}^2} + 2(8.30 + 0.75 \times 20.4)} \text{ watts}$$

$$= \frac{162 \times 10^6}{102.5} = 1{,}580 \text{ kw},$$

and

$$P_{R_{max}} = 3 \times 1{,}580 = 4{,}740 \text{ kw}.$$

For capacitive load, 0.80 power factor,

$$K = -0.75,$$

and

$$P_{RN_{max}} = \frac{162 \times 10^6}{2 \times 22.1\sqrt{1 + \overline{0.75}^2} + 2(8.30 - 15.3)}$$

$$= \frac{162 \times 10^6}{41.3} = 3{,}920 \text{ kw},$$

and

$$P_{R_{max}} = 3 \times 3{,}920 = 11{,}760 \text{ kw}.$$

Example 90

What is the very maximum power that may be transmitted over the line of Example 89 with normal voltage at the sending end of the line? Find the power factor of the load and the efficiency of transmission at this load.

$$R = 8.30 \text{ ohms}, \qquad X = 20.4 \text{ ohms}, \qquad Z = 22.1 \text{ ohms},$$

$$R_1 = R = 8.30 \text{ ohms}, \qquad X_1 = -X = -20.4 \text{ ohms},$$

$$P_{RN_{max}} = \frac{E_{SN}^2}{4R} = \frac{162 \times 10^6}{4 \times 8.30} \text{ watts}$$

$$= \frac{162 \times 10^6}{33.2} = 4{,}875 \text{ kw},$$

and

$$P_{R_{max}} = 3 \times 4{,}875 = 14{,}625 \text{ kw}.$$

$$\text{Line current} = \frac{E_{SN}}{2R} = \frac{12{,}700}{2 \times 8.30} = 765 \text{ amp}.$$

Voltage at receiving end = 765 × 22.1 = 16,900
= 29,300 volts between lines.

Efficiency = 50 per cent.

$$\text{Power factor of load} = \frac{R_1}{Z_1} = \frac{R}{Z} = \frac{8.30}{22.1} = 0.375$$

= 37.5 per cent.

Problems

1-13. A 60-cycle 2,200-volt single-phase distribution circuit is constructed of 500,000-circular-mil stranded copper conductors spaced 15 in. apart on centers and has a resistance of 0.1166 ohm per mile of conductor and an inductance of 1.246 millihenrys per mile of conductor. It delivers 1,500 kva at 0.88 power factor inductive from a substation to an industrial plant located 4,200 ft. from the substation. Find the voltage required at the substation and the efficiency and regulation of the line. Draw the vector diagram with receiving-end voltage as the reference.

2-13. A 60-cycle single-phase distribution circuit 380 ft. long delivers 40 kva at 440 volts at 0.80 power factor inductive. It is constructed of No. 6 B. & S. stranded copper conductor spaced on 12-in. centers and having a resistance of 2.17 ohms per mile of conductor and an inductance of 1.672 millihenrys per mile of conductor. Find the efficiency and regulation of the line. Draw the vector diagram, using the load emf as the reference.

3-13. A 11,000-volt single-phase distribution line 6 miles long is constructed of No. 0000 stranded copper wire on 24-in. centers and has a resistance of 0.278 ohm per mile of conductor and a reactance of 0.588 ohm per mile of conductor. A load of 3,000 kva at 0.85 power factor inductive is being delivered by the line. Calculate the voltage required at the sending end of the line, the regulation, and the efficiency. Draw the vector diagram, using voltage at the load as the reference.

4-13. A single-phase transmission line has a total resistance of 10 ohms and an inductive reactance of 30 ohms for the two wires. If the receiving-end voltage is 6,600 volts, find the regulation and efficiency for the following values of current, 0, 50, 100, 150 amp and for the following conditions of load, (*a*) direct current, (*b*) alternating current at unity power factor, (*c*) alternating current at 0.70 power factor inductive. Assemble the results in the form of a table.

5-13. A three-phase 60-cycle distribution circuit transmits power from a substation to a factory 2,600 ft. away. The line is constructed of No. 00 stranded-copper conductor on 15-in. centers and has a resistance of 0.429 ohm and an inductance of 1.482 millihenrys per mile of conductor. Calculate the regulation and efficiency of the line when it is delivering 150 kva at 0.70 power factor inductive and 550 volts. Draw the vector diagram, using load voltage to neutral as the reference.

6-13. A 2,300-volt 60-cycle three-phase feeder is 2.4 miles long. It is constructed of No. 0 B. & S. stranded-copper conductor on 18-in. centers and has a resistance of 0.54 ohm and a reactance of 0.595 ohm per mile. Calculate the regulation and efficiency of the line when it is delivering 350 kva at 0.75 power factor inductive. Draw the vector diagram, using receiver voltage to neutral as the reference.

7-13. A 25-cycle 4,000-volt three-phase distribution circuit is 3.8 miles long. It is constructed of No. 00 B. & S. stranded copper conductor having a resistance of 0.428 ohm and a reactance of 0.257 ohm per mile. Find the regulation and efficiency of the line when it is delivering 800 kva at 0.85

power factor inductive and 4,000 volts. Draw the vector diagram, using the receiver voltage to neutral as the reference.

8-13. A 60-cycle three-phase transmission line 12 miles long is constructed of No. 0 B. & S. copper conductor having a resistance of 0.54 ohm and a reactance of 0.644 ohm per mile. Find the regulation and efficiency of this line when it is delivering 3,000 kva at 16,500 volts at 0.80 power factor inductive. Use the receiver voltage to neutral as the reference for the calculation and the vector diagram.

9-13. A 66,000-volt three-phase 18-mile transmission line is constructed of 266,800-circular-mil aluminum cable, steel reinforced, with a spacing of 6 ft. horizontal. It has a resistance of 6.30 ohm and a reactance of 12 ohms per wire. Calculate the regulation and efficiency when 35,000 kva is delivered at 66,000 volts and unity power factor.

10-13. Calculate the operating characteristics of a three-phase transmission line 22 miles long constructed of No. 0000 B. & S. stranded copper conductor spaced in an equilateral triangle 3 ft. apart, for loads of 0, 800, and 1,500 kva at unity power factor and at power factors of 0.80 current leading and lagging. The delivered voltage is maintained at 11,000 volts. The line resistance is 6.11 ohms and the line reactance is 14.0 ohms. Use the delivered voltage to neutral as the reference, and express the results in tabular form.

11-13. A 44,000-volt 60-cycle three-phase transmission line is constructed of No. 0000 aluminum cable, steel reinforced, with a spacing of 5 ft. horizontal. The resistance is 15.8 ohms and the reactance is 25.1 ohms. Calculate the regulation and efficiency of this line when it is delivering 9,500 kva at 44,000 volts at 0.80 power factor inductive. Use the load current as the reference.

12-13. A 33,000-volt three-phase transmission line has a resistance of 12.5 ohms and an inductive reactance of 35 ohms. What is the regulation of the line when it is delivering a load of 8,000 kw at unity power factor? Use the load current as the reference. What is the copper loss in the line when transmitting this power?

13-13. Calculate the operating characteristics of a three-phase transmission line 40 miles long constructed of No. 0 B. & S. copper conductor spaced vertically 4 ft. apart for loads of 0, 5,000, and 10,000 kva at unity power factor and at a power factor of 0.80 current leading and lagging. The voltage is maintained constant at 50,000 volts at the receiving end. Use the load current as the reference. The line has a resistance of 22.2 ohms and a reactance of 29.6 ohms per wire. Use the tabular method of solution, and draw the vector diagram for the 10,000 kva load for all three power factors.

14-13. A 15,000-volt three-phase 60-cycle transmission line 12 miles long is constructed of No. 2 B. & S. copper conductor and has a resistance of 0.858 ohm and an inductance of 1.817 millihenrys per mile of wire. The line is rated at 2,000 kva; what are the per unit and per cent resistance and reactance of the line?

15-13. The resistance per mile for a 44,000-volt 25-cycle three-phase transmission line 25 miles long constructed of No. 0 B. & S. copper conductor with a horizontal spacing of 5 ft. is 0.54 ohm and the reactance per mile is 0.77 ohm. The line is rated at 10,000 kva; what are the per unit constants of the line?

16-13. Calculate the regulation and efficiency for all transmission lines having 8 per cent resistance and 14 per cent reactance when carrying a full load at unity power factor. Draw the vector diagram.

17-13. A three-phase load takes 10,000 kva at 0.70 power factor inductive at 66,000 volts at the receiving end of a transmission line. The line has a resistance of 4 per cent and a reactance of 10 per cent. Calculate the voltage between wires at the sending end, the regulation and efficiency of the line, using the per unit method and receiver voltage to neutral as the reference. Draw the vector diagram. What are the values of R and X in ohms per wire?

18-13. Calculate the operating characteristics of a three-phase transmission line having a resistance of 3 per cent and an inductive reactance of 8 per cent on the basis of 500 kva at 15,000 volts, for loads of 0, 400, 600, and 1,000 kva at unity power factor and 0.70 power factor inductive and capacitive loads. Use the receiving-end voltage as the reference in the calculations, and draw the vector diagrams for the 400 kva loads.

19-13. A three-phase transmission line has been operating at 25 cycles and has a resistance of 7 per cent and a reactance of 16 per cent when delivering 1,000 kva at 33,000 volts. The frequency of the line is raised to 60 cycles, and the voltage is changed to 22,000 volts. What are the new per cent constants at 1,000 kva?

20-13. What is the very maximum power that may be transmitted over a three-phase line having an impedance $\dot{Z} = 4 + j10$, with 33,000 volts impressed on the line?

21-13. Calculate the operating characteristics of a single-phase transmission line having a resistance of 10 ohms and an inductive reactance of 25 ohms per wire and operated with 13,200 volts at the customer's load for loads of 0, 300, and 600 kva at unity power factor and 0.866 power factor inductive and capacitive loads. Use the tabular method of calculation, and draw the vector diagrams for the 600 kva loads at the three power factors.

22-13. The impedance of each conductor of a three-phase transmission line is given as $\dot{Z} = 10 + j30$ at 60 cycles. The line voltage at the load is 88,000, and the load is 25,000 kva at 0.75 power factor inductive. Calculate the sending-end voltage, the regulation, and the efficiency at this load.

23-13. Calculate the regulation and efficiency of a 6,600-volt three-phase 60-cycle transmission line 8 miles long constructed of No. 0 B. & S. stranded copper conductors with a 30 in. horizontal spacing, when carrying a load of 800 kva at 90 per cent power factor inductive.

24-13. A 44,000-volt three-phase 60-cycle 38-mile transmission line has No. 000 B. & S. stranded copper conductors spaced 5 ft. on centers in the form of an equilateral triangle. Calculate the efficiency and regulation of this line when it is delivering 8,000 kva at 0.80 power factor inductive.

25-13. Consider a three-phase transmission line with impedance $\dot{Z} = 5 + j20$, a load of 15,000 kva at the receiving end, and a constant receiver voltage of 44,000 volts between lines. Vary the power factor of the load from 0.40 lagging to 0.40 leading by steps of 0.2, and calculate the values of the impressed emf between wires, regulation, efficiency, power factor, and impedance drop for these loads by tabulation.

26-13. Calculate the regulation and efficiency for all transmission lines having a resistance of 9 per cent and a reactance of 21 per cent carrying full load at 0.80 power factor inductive. Draw the vector diagram.

27-13. The impedance of one conductor of a three-phase transmission line is $\dot{Z} = 40 + j80$ at 60 cycles. The line is transmitting 6,000 kva at 66,000 volts 25 cycles and at 0.70 power factor leading current. What are the per unit values of its constants at 25 cycles?

28-13. Calculate the operating characteristics of a three-phase transmission line having a resistance of 4 per cent and a reactance of 12 per cent on a 2,000-kva base, the voltage between lines at the load being 22,000 volts. Use current as the reference vector, and calculate for loads of 0, 1,000, 1,500 and 2,000 kva at unity and 0.75 power factors lead and lag. Draw the vector diagrams for each type of load.

CHAPTER XIV

MEDIUM-LENGTH TRANSMISSION LINES

Transmission lines of considerable length with medium voltages, or branches of longer transmission lines at higher voltages, have a charging current flowing over the line sufficient appreciably to affect the operating characteristics. For such lines, the more rigorous solutions developed in the next chapter are not necessary, and a mathematical solution involving only circular functions and complex quantities is very often applied. Such solutions give results sufficiently accurate for lines 200 miles or less in length at standard frequencies for transmission of energy and for transmission voltages not over 150,000 volts between wires. They also serve as a rapid check method for the solution of longer lines and for preliminary estimates of operating characteristics of any length of line.

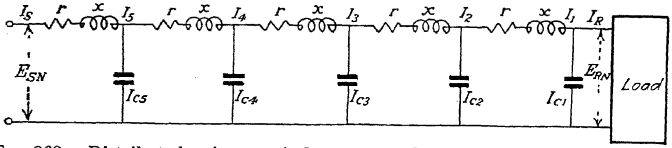

FIG. 263.—Distributed resistance, inductance, and capacitance of a transmission line.

152. Distributed Resistance, Inductance, and Capacitance.—Each unit length of line, *i.e.*, each centimeter or each mile, has a resistance, an inductive reactance, and a capacitance. From Fig. 263, representing any high-voltage transmission line, it is seen that, when the line is considered from the load end, the first r and x must carry the load current and the charging current for C_1; the next r and x must carry this same current as well as the charging current for C_2, always added vectorially; the last r and x, that at the sending end of the line, must carry the load current and the entire charging current of the line. In order to derive the proper values of E_s and I_s, the resistance, inductive reactance, and the capacitive susceptance must be considered as distributed along the entire line. Solutions in this chapter will be based upon approximate methods where equivalent values of the constants are concentrated or *bunched* at certain points along the line giving a *lumpy* instead of a *smooth* line. In the next chapter, exact solutions for long lines will be developed, the constants being considered as uniformly distributed along the line.

In order to study the effect on the sending-end voltage E_s of the distributed charging current I_c flowing through the distributed resistance and inductive reactance of the line, consider the load at the receiving end to be zero so that the only current flowing is the charging current. The line may be considered as consisting of a large number of resistances and reactances connected in series and the same number of condensers of equal susceptance connected in parallel, as represented in Fig. 264.

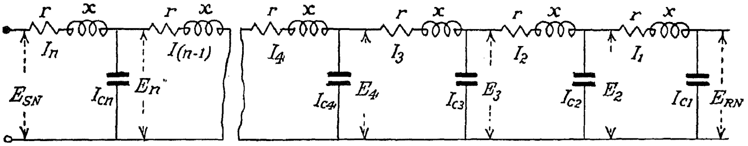

FIG. 264.—Transmission line at no load with uniformly distributed constants

Let R = the total resistance of the line,

X = the total inductive reactance of the line,

$\dot{Z}$ = the total impedance of the line = $R + jX$,

$b = 2\pi fC$ = the total charging susceptance of the line, calculated from the equations derived in Chap. XII,

I_0 = the total charging current of the line at no load,

$n = L$ = the number of mile sections, or length of the line,

$r = R/L$ = the unit resistance per mile of line,

$x = X/L$ = the inductive reactance per mile of line,

$\dot{z} = r + jx$ = the impedance per mile of line,

$\dfrac{b}{L} = 2\pi f\dfrac{C}{L}$ = the charging susceptance per mile of line.

The charging current,

$$I_{C_1} = E_{RN}\frac{b}{L} \text{ amp},$$

is proportional to, and 90 deg. ahead of, E_{RN}, as shown in Fig. 265.

The current $\dot{I}_1 = \dot{I}_{c_1}$ flows through r and x, producing a voltage drop I_1z so that

$$\dot{E}_2 = \dot{E}_{RN} + \dot{I}_1\dot{z} \text{ volts.} \tag{1032}$$

The charging current I_{C_2} is proportional to, and 90 deg. ahead of, E_2, and

$$\dot{I}_2 = \dot{I}_1 + \dot{I}_{C_2} \text{ amp.} \tag{1033}$$

The voltage drop in the second section of the line is I_2z, and

$$\dot{E}_3 = \dot{E}_2 + \dot{I}_2\dot{z} \text{ volts.} \tag{1034}$$

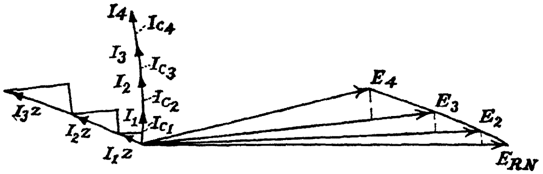

FIG. 265.—No-load charging current and voltage along a transmission line.

The charging current I_{C_3} is proportional to, and 90 deg. ahead of, E_3 so that

$$\dot{I}_3 = \dot{I}_2 + \dot{I}_{C_3} \text{ amp} \tag{1035}$$

and

$$\dot{E}_4 = \dot{E}_3 + \dot{I}_3\dot{z} \text{ amp.} \tag{1036}$$

Likewise, the charging current in the last section I_{c_n} is proportional to, and 90 deg. ahead of, E_n so that

$$\dot{I}_n = \dot{I}_{(n-1)} + \dot{I}_{c_n} \text{ amp} \tag{1037}$$

and

$$\dot{E}_{SN} = \dot{E}_n + \dot{I}_n\dot{z} \text{ amp.} \tag{1038}$$

From the vector diagram (Fig. 265), it is seen that the charging current per unit length of line is not constant, because of the change in magnitude of the voltage along the line, and that the charging currents are not in phase with each other, because of the change in phase position of the voltage vector along the line. For lines of medium length, however, where the total charging current is relatively small, E_{SN} differs from E_{RN} by only a small percentage in magnitude and by only a fraction of a degree, or at best a few degrees, in phase position, so that no great error is introduced by assuming that the charging current per unit length of line is constant in magnitude and phase and that it is equal to

$$I_C = \frac{I_o}{L} \text{ amp,} \tag{1039}$$

where I_o is the total no-load charging current of the line. Hence, the impedance drops along the line are all in phase, as shown in Fig. 266. Therefore,

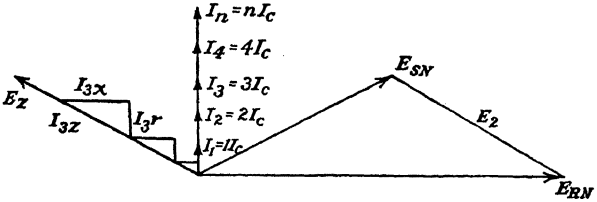

FIG. 266.—Equivalent no-load charging current and voltage along a transmission line.

$$\begin{aligned} I_1 &= I_{C_1} = I_C \text{ amp,} \\ I_2 &= I_1 + I_{C_2} = 2I_C \text{ amp,} \\ I_3 &= I_2 + I_{C_3} = 3I_C \text{ amp} \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{1040}$$

and

$$I_o = I_{n-1} + I_{C_n} = nI_C \text{ amp.} \tag{1041}$$

The total impedance drop in the line becomes

$$E_Z = I_1z + I_2z + I_3z + \ . \ . \ . \ + I_nz \text{ volts} \tag{1042}$$

$$\begin{aligned} &= I_Cz + 2I_Cz + 3I_Cz + \ . \ . \ . \ + nI_Cz \\ &= I_Cz(1 + 2 + 3 + \ . \ . \ . \ + n) \text{ volts.} \end{aligned} \tag{1043}$$

The sum of the arithmetical series of n terms having a common difference of 1 is

$$S = \frac{n(n+1)}{2}. \tag{1044}$$

When the unit of length chosen is small, as, for instance, when r and x are taken per centimeter, n becomes very large and

$$S = \frac{n^2}{2} \text{ (approximately).} \tag{1045}$$

Hence,

$$E_Z = I_C z \frac{n^2}{2} = \frac{(nI_C)(nz)}{2} = \frac{I_0 Z}{2} \text{ volts,} \tag{1046}$$

and

$$\dot{E}_{SN} = \dot{E}_{RN} + \dot{E}_Z = \dot{E}_{RN} + \frac{\dot{I}_0 \dot{Z}}{2} \text{ volts.} \tag{1047}$$

Equation (1047) may be interpreted as indicating that the effect of the charging currents on E_{SN} is equivalent to that produced by half the total charging current flowing through the entire impedance of the line or to that produced by the entire charging current flowing through half the impedance of the line. The former interpretation gives the "nominal-π" solution and the latter the "nominal-T" solution. In either case, the total load current must be transmitted over the whole line, and the effects of both load and charging current must be taken into consideration in calculating the line characteristics.

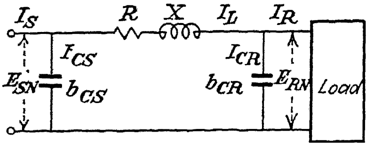

FIG. 267.—Circuit diagram for a nominal-π solution of a transmission line.

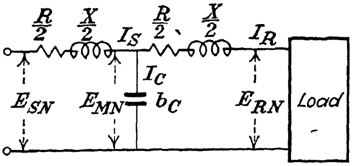

FIG. 268.—Circuit diagram for a nominal-T solution of a transmission line.

In Fig. 267, the equivalent circuit diagram for the nominal-π solution, half the capacitance of the line is concentrated at the receiving end and the charging current for this capacitance must be added to the load current to give the line current. This current is passed through the entire resistance and reactance of the line to obtain the emf E_{SN}, at the sending end of the line, and finally the charging current for the other half the line capacitance, at proper value and phase position corresponding with E_{SN}, must be added to the line current to obtain the current consumed by the line and load combined. In Fig. 268, the equivalent circuit diagram for the nominal-T solution, the entire load current is passed through one-half the line resistance and inductive reactance to obtain the emf E_{MN}, at the center of the line. The charging current of the entire line, corresponding in value and phase position with E_{MN}, is added to the load current to obtain the total line current. This current is then passed through the resistance and the inductive reactance of the remainder of the line to obtain the voltage E_{SN}, at the sending end. These two approximate solutions give results that check very closely with measured results for lines of the type for which they are derived. The nominal-T solution is a better introduction to the equations and the vector diagrams of the transformer, but the nominal-π solution is slightly shorter in application, and it compares more nearly with the solutions in Chap. XIII for the shorter lines. The nominal-π method is often convenient to use for lines forming parts of a power-system network where the capacitive kilovolt-amperes may be included with the loads at the stations in which the lines terminate, and the short-line calculation may then be applied to the line between stations.

153. Nominal-π Solution.—Consider half the capacitance of the transmission line concentrated at each end, as in Fig. 267. The capacitive susceptance agreeing with this equivalent diagram becomes

$$b_{CR} = b_{CS} = 2\pi f \frac{C}{2} \text{ mhos,} \tag{1048}$$

where C is the total capacitance of one wire to neutral from Eqs. (969) to (976). Taking E_{RN} as the reference vector,

$$\dot{E}_{RN} = E_{RN} + j0 \text{ volts,}$$

and the load current, in reference to E_{RN}, is

$$\dot{I}_R = i_R + ji_R' \text{ amp}$$

where E_{RN} is the emf to neutral. The charging current at the receiving end is

$$\dot{I}_{CR} = \dot{E}_{RN} jb_{CR} = ji_{CR}' \text{ amp,} \tag{1049}$$

where

$$i_{CR}' = E_{RN} b_{CR} \text{ amp.} \tag{1050}$$

The equivalent line current becomes

$$\begin{aligned} \dot{I}_L &= \dot{I}_R + \dot{I}_{CR} \text{ amp} \\ &= i_R + j(i_R' + i_{CR}') = i_L + ji_L' \text{ amp,} \end{aligned} \tag{1051}$$

where

$$i_L = i_R \text{ amp} \tag{1052}$$

and

$$i_L' = i_R' + i_{CR}' \text{ amp.} \tag{1053}$$

The voltage at the sending end of the line is

$$\dot{E}_{SN} = \dot{E}_{RN} + \dot{I}_L\dot{Z} \text{ volts} \tag{1054}$$

$$\begin{aligned} &= E_{RN} + (i_L + ji_L')(R + jX) \text{ volts} \\ &= E_{RN} + i_LR - i_L'X + j(i_L'R + i_LX) \text{ volts} \qquad (1055) \\ &= e_{SN} + je_{SN}' \text{ volts,} \end{aligned}$$

where

$${}_{SN} = E_{RN} + i_LR - i_L'X \text{ volts} \tag{1056}$$

and

$$e_{SN}' = i_L'R + i_LX \text{ volts.} \tag{1057}$$

The charging current for the line capacitance at the sending end becomes

$$\dot{I}_{CS} = \dot{E}_{SN}jb_{CS} = (e_{SN} + je_{SN}')jb_{CS} \text{ amp} \tag{1058}$$

$$= -\, e_{SN}'b_{CS} + je_{SN}b_{CS} = i_{CS} + ji_{CS}' \text{ amp,} \tag{1059}$$

where

$$i_{CS} = -\, e_{SN}'b_{CS} \text{ amp} \tag{1060}$$

and

$${}_{CS}' = e_{SN}b_{CS} \text{ amp.} \tag{1061}$$

The total current at the sending end of the line is the sum of the charging current and the line current of Eq. (1051), or

$$\dot{I}_S = \dot{I}_L + \dot{I}_{CS} \text{ amp} \tag{1062}$$

$$= (i_L + i_{CS}) + j(i_L' + i_{CS}') = i_S + ji_S' \text{ amp,} \tag{1063}$$

where

$$i_S = i_L + i_{CS} \text{ amp} \tag{1064}$$

and

$$i_S' = i_L' + i_{CS}' \text{ amp.} \tag{1065}$$

The magnitude of the emf at the sending end is

$$E_{SN} = \sqrt{e_{SN}^2 + e_{SN}'^2} \text{ volts} \tag{1066}$$

and the magnitude of the total current,

$$I_S = \sqrt{i_S^2 + i_S'^2} \text{ amp.} \tag{1067}$$

From these equations,

$$P_{SN} = e_{SN} i_S + e_{SN}' i_S' \text{ watts} \tag{336}$$

is the power input per phase to the line. The efficiency becomes

$$\text{Efficiency} = \frac{P_{RN}}{P_{SN}} = \frac{P_R}{P_S}, \tag{1005}$$

and, the apparent power input per phase being $E_{SN}I_S$, the power factor at the sending end of the line is

$$P.F._S = \frac{P_{SN}}{E_{SN}I_S}.$$

Figure 269 is the vector diagram for a nominal-π solution of a transmission line with an inductive load, using the receiving-end voltage to neutral as the reference. In solving the network, as shown in Fig. 267, which is the nominal-π representation of a medium-length transmission line, any reference axis may be chosen, the process of solution of the network being the same. The general equations into which the values of I_R and E_{RN}, expressed with any reference, may be inserted are

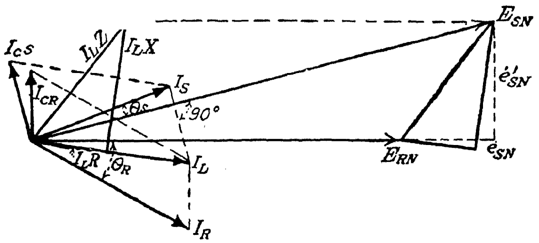

FIG. 269.—Vector diagram for a nominal-π solution of a transmission line with inductive load.

$$\dot{I}_{CR} = j b_{CR} \dot{E}_{RN} \text{ amp,} \tag{1049}$$

$$\dot{I}_L = \dot{I}_R + \dot{I}_{CR} \text{ amp,} \tag{1051}$$

$$\dot{E}_{SN} = \dot{E}_{RN} + \dot{I}_L \dot{Z} \text{ volts,} \tag{1054}$$

$$\dot{I}_{CS} = j b_{CS} \dot{E}_{SN} \text{ amp,} \tag{1058}$$

and

$$\dot{I}_S = \dot{I}_L + \dot{I}_{CS} \text{ amp.} \tag{1062}$$

In Fig. 270, the vector diagram is shown for the nominal-π solution of a transmission line with inductive load, current being used as the reference. To show the effect of the power factor of the load on the relative value of the sending- and receiving-end voltages, a vector diagram, Fig. 271, is shown for a nominal-π solution with a capacitive load.

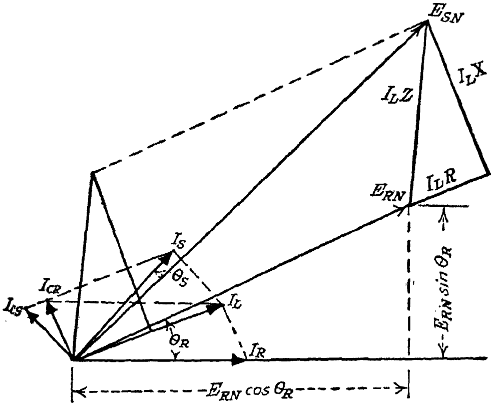

FIG. 270.—Vector diagram for the nominal-π solution of a transmission line with inductive load, using current as the reference vector.

The same definition for regulation applies to the medium-length line as to the short line, *i.e.*, *regulation is the percentage change in voltage at the receiving end of the line in terms of rated receiving-end voltage, when the load is reduced to zero, the voltage at the sending end being held constant at the value required for the particular load previously chosen.* The open-circuit voltage for the medium-length line at the receiving end differs from that at the

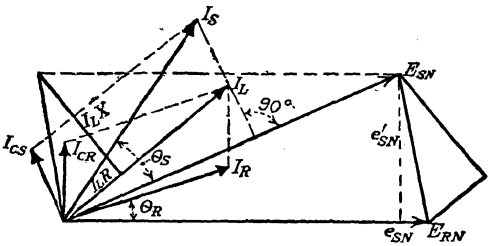

FIG. 271.—Vector diagram for a nominal-π solution of a transmission line with capacitive load.

sending end, whereas in the short line they were the same. Even when the load on the line is reduced to zero, there still remains the charging current of the line that passes through the line resistance and reactance, as shown in the circuit diagram, Fig. 267. From this diagram,

$$\dot{I}_L = \dot{I}_{CR} = \dot{E}_{RN} j b_{CR} \text{ amp,} \tag{1068}$$

and

$$\begin{aligned} \dot{E}_{SN} &= \dot{E}_{RN} + \dot{E}_{RN} j b_{CR} \dot{Z} \text{ volts} &(1069)\\ &= \dot{E}_{RN}(1 + j b_{CR} \dot{Z}) \text{ volts.} &(1070) \end{aligned}$$

The vector diagram for the open-circuit condition of the nominal-π solution is shown in Fig. 272. This vector diagram is shown for two different receiver voltages, the voltage E_{SN0} being the open-circuit sending-end voltage with rated voltage at the receiving end, whereas E_{RN}' is the open-circuit voltage at the receiving end for any given voltage E_{SN} held at the sending end. Since, for the calculation of regulation, the sending-end voltage is to remain fixed, the receiving-end voltage must be calculated for the known sending-end voltage. This may be done by solving Eq. (1070) for E_{RN}'.

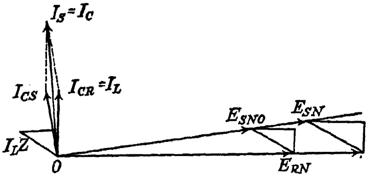

FIG. 272.—Open-circuit vector diagram for a nominal-π solution of a transmission line.

$$E_{RN}' = \frac{\dot{E}_{SN}}{1 + j b_{CR} \dot{Z}} \text{ volts,} \tag{1071}$$

where E_{RN}' is the receiving-end voltage when the load is reduced to zero, E_{SN} being held constant.

Regulation is calculated from these values and the definition in the usual manner as

$$\text{Regulation} = \frac{E_{RN}' - E_{RN}}{E_{RN}} 100 \text{ per cent.} \tag{1072}$$

In Fig. 272, the two triangles O-E_{RN}-E_{SN0} and O-E_{RN}'-E_{SN} are similar since two sides including a common angle are proportional. This gives rise to another method for calculating the receiving-end voltage when the load is reduced to zero. Since

$$\frac{E_{RN}'}{E_{SN}} = \frac{E_{RN}}{E_{SN0}},$$

$$E_{RN}' = \frac{E_{RN}}{E_{SN0}} E_{SN} \text{ volts,} \tag{1073}$$

where E_{SN0} has been calculated from Eq. (1070) for a value of E_{RN} equal to its rated value. If Eq. (1073) is substituted in Eq. (1072), there results

$$\text{Regulation} = \frac{\dfrac{E_{RN}}{E_{SN0}} E_{SN} - E_{RN}}{E_{RN}} 100 \text{ per cent} \tag{1074}$$

$$= \frac{E_{SN} - E_{SN0}}{E_{SN0}} 100 \text{ per cent.} \tag{1075}$$

Although Eq. (1075) may be found convenient to use in the calculation of regulation, it must be borne in mind that it has been derived by proportion from Fig. 272 and is not the definition of regulation.

Example 91

Determine the operating characteristics of a transmission line designed for transmitting 22,000 kva at 66,000 volts three-phase 60 cycles and having a resistance of 20 ohms per wire, an inductive reactance of 40 ohms per wire, and a capacitive susceptance of 0.00025 mho per wire, using the nominal-π solution for full kilovolt-ampere output at 0.80 power factor inductive.

The rated receiver voltage to neutral is

$$E_{RN} = \frac{66{,}000}{\sqrt{3}} = 38{,}100 \text{ volts.}$$

The load current is

$$I_R = \frac{22{,}000 \times 1{,}000}{66{,}000\sqrt{3}} = 192.5 \text{ amp.}$$

Taking E_{RN} as the reference,

$$\dot{E}_{RN} = 38{,}100 + j0 \text{ volts,}$$
$$\dot{I}_R = 154 - j115.5 \text{ amp,}$$
$$b_{CR} = b_{CS} = \frac{0.00025}{2} = 0.000125 \text{ mho.}$$

Then

$$\begin{aligned} \dot{I}_{CR} &= j0.000125(38{,}100 + j0) \text{ amp} \\ &= j4.76 \text{ amp.} \\ \dot{I}_L &= 154 - j115.5 + j4.76 \text{ amp} \\ &= 154 - j110.7 \text{ amp.} \\ \dot{E}_{SN} &= 38{,}100 + (154 - j110.7)(20 + j40) \text{ volts} \\ &= 45{,}600 + j3{,}900 \text{ volts.} \end{aligned}$$

$$
\begin{aligned}
E_{SN} &= \sqrt{45{,}600^2 + 3{,}900^2} = 45{,}750 \text{ volts.}\\
\dot{I}_{CS} &= j0.000125(45{,}600 + j3{,}900) \text{ amp}\\
&= -0.495 + j5.70 \text{ amp, and}\\
\dot{I}_S &= 154 - j110.7 - 0.495 + j5.70 \text{ amp}\\
&= 153.5 - j105 \text{ amp, and}\\
I_S &= \sqrt{153.5^2 + 105^2} = 186 \text{ amp.}
\end{aligned}
$$

The power input to the line is

$$
\begin{aligned}
P_{SN} &= [(45{,}600)(153.5) + (3{,}900)(-105)]10^{-3}\\
&= 6{,}586 \text{ kw.}
\end{aligned}
$$

Also

$$
\begin{aligned}
P_{RN} &= \frac{22{,}000 \times 0.80}{3} = 5{,}867 \text{ kw, and}\\
\text{Efficiency} &= \frac{5{,}867}{6{,}586}100 = 89.1 \text{ per cent.}\\
V_{SN} &= U_{SN} = 45{,}750 \times 186 = 8{,}520 \text{ kva,}\\
P.F._S &= \frac{5{,}867}{8{,}520}100 = 77.3 \text{ per cent.}
\end{aligned}
$$

The voltage between lines at the sending end is

$$
\begin{aligned}
E_S &= 45{,}750\sqrt{3} = 79{,}250 \text{ volts.}\\
\dot{E}_{RN}' &= \frac{45{,}600 + j3{,}900}{1 + j(0.000125)(20 + j40)} \text{ volts}\\
&= 45{,}900 + j3{,}920 \text{ volts, and}\\
E_{RN}' &= 46{,}100 \text{ volts.}\\
\text{Regulation} &= \frac{46{,}100 - 38{,}100}{38{,}100} = 0.206 = 20.6 \text{ per cent.}
\end{aligned}
$$

This line is calculated for various loads and power factors in Table IX.

154. Nominal-π Solution Using the Per Unit Constants.—The per unit values of resistance and reactance are found in the same manner as for the short line in Chap. XIII. The per unit value of the charging current is based on the relation of a charging current that would be produced by a voltage of normal E_{RN} applied to the capacitive susceptance of the whole line, corresponding to the capacitance of one wire to neutral, to the full-load rated current of the line. Then

$$
\text{Per unit } I_C = \frac{E_{RN} b_C}{I_R}, \tag{1076}
$$

where b_C is the capacitive susceptance in mhos of one wire to neutral;

E_{RN} is the rated voltage to neutral at the load; and

I_R is the rated full-load current.

TABLE IX.—CALCULATION OF THE OPERATING CHARACTERISTICS OF A MEDIUM-LENGTH THREE-PHASE TRANSMISSION LINE BY THE NOMINAL-π METHOD USING THE EMF AT THE LOAD AS A REFERENCE

Example 91

Kind of load	Non-reactive				Inductive			Capacitive		
Per cent kilovolt-ampere output	0	50	100	150	50	100	150	50	100	150
Power factor of load	....	1.00	1.00	1.00	0.80	0.80	0.80	0.80	0.80	0.80
Reactive factor of load	....	0	0	0	−0.60	−0.60	−0.60	0.60	0.60	0.60
Line voltage, E_R	....	....	....	....	66,000					
Neutral voltage, E_{RN}	....	....	....	....	38,100					
Load current, I_R	0	96.25	192.5	288.8	96.25	192.5	288.8	96.25	192.5	288.8
Active component, i_R	0	96.25	192.5	288.8	77.0	154.0	231.0	77.0	154.0	231.0
Reactive component, i'_R	0	0	0	0	−57.75	−115.5	−173.0	57.75	115.5	173.0
$i'_{CR} = E_{RN}b_{CR}$	4.76	4.76	4.76	4.76	4.76	4.76	4.76	4.76	4.76	4.76
$i_L = i_R$	0	96.25	192.5	288.8	77.0	154.0	231.0	77.0	154.0	231.0
$i'_L = i'_R + i'_{CR}$	4.76	4.76	4.76	4.76	−53.00	−110.7	−168.5	62.50	120.3	178.1
$i_LR = 20i_L$	0	1,925	3,850	5,776	1,540	3,080	4,620	1,540	3,080	4,620
$i_L'X = 40i'_L$	190.4	190.4	190.4	190.4	−2,120	−4,428	−6,740	2,500	4,812	7,124
$e_{SN} = E_{RN} + i_LR - i'_LX$	37,910	39,840	41,760	43,690	41,760	45,610	49,460	37,140	36,370	35,600
$i'_LR = 20i'_L$	95.0	95.0	95.0	95.0	−1,060	−2,214	−3,370	1,250	2,406	3,562
$i_L X = 40i_L$	0	3,850	7,700	11,552	3,080	6,160	9,240	3,080	6,160	9,240
$e'_{SN} = i'_LR + i_LX$	95	3,945	7,795	11,647	2,020	3,946	5,870	4,330	8,566	12,802
$i_{CS} = -e'_{SN}b_{CS}$	−0.012	−0.495	−0.975	−1.45	−0.252	−0.495	−0.735	−0.541	−1.07	−1.60

$i'_{CS} = e_{SN}b_{CS}$	4.74	4.98	5.22	5.46	5.22	5.70	6.18	4.64	4.54	4.45
$i_S = i_L + i_{CS}$	−0.012	95.75	191.5	287.3	76.7	153.5	230.3	76.5	152.9	229.4
$i'_S = i'_L + i'_{CS}$	9.50	9.74	9.98	10.22	−47.77	−105.0	−162.3	67.2	124.8	182.6
$I_S = \sqrt{i_S^2 + i'^2_S}$	9.50	96.25	191.6	287.4	90.4	186.0	281.5	101.8	197.6	293.0
$E_{SN} = \sqrt{e^2_{SN} + e'^2_{SN}}$	37,910	40,050	42,500	45,250	41,780	45,750	49,800	37,390	37,360	37,810
$e_{SN}i_S \times 10^{-3}$ kilowatt	−0.425	3,816	8,000	12,560	3,206	7,000	11,400	2,843	5,550	8,160
$e'_{SN}i'_S \times 10^{-3}$ kilowatt	0.903	38.4	77.9	119	−96.5	−414	−953	291	1,068	2,335
$P_{SN} = e_{SN}i_S + e'_{SN}i'_S$ kilowatt	0.478	3,854	8,078	12,679	3,110	6,586	10,447	3,134	6,618	10,495
$E_{SN}I_S \times 10^{-3}$ kilovolt-ampere	360	3,854	8,145	13,000	3,790	8,520	14,020	3,805	7,375	11,090
$P_{RN} = E_{RN}i_R \times 10^{-3}$ kilowatt	0	3,666	7,333	11,000	2,933	5,867	8,800	2,933	5,867	8,800
Efficiency $= \frac{P_{RN}}{P_{SN}}$	....	95.2	90.6	86.8	94.3	89.1	84.2	93.5	88.7	83.8
P. F.$_S = \frac{P_{SN}}{E_{SN}I_S}$	0.13	100	99.2	97.4	82.2	77.3	74.6	82.3	89.7	94.6
$E'_{RN} = \frac{E_{SN}}{E_{SN_0}}E_{RN}$	38,100	40,400	42,790	45,550	42,100	46,100	50,300	36,300	37,550	38,100
Regulation $= \frac{E'_{RN} - E_{RN}}{E_{RN}}$	....	6.04	12.3	19.3	10.5	20.6	31.2	−1.34	−1.45	0
E_S = line voltage $= \sqrt{3}E_{SN}$	65,600	69,400	73,600	78,300	72,400	79,250	86,200	64,700	64,600	65,500

The value of b_C in mhos required to make the numerator equal to the denominator of the right-hand side of Eq. (1076) would make the charging current just equal to the rated full-load current, and it would be the base value of susceptance. Since the ratio of the actual susceptance in the circuit to this base value is the same as the ratio of the actual charging current produced by b_c to the rated current, the per unit value of b_c is equal to the per unit value of charging current, and Eq. (1076) may be written

$$\text{Per unit } b_C = \text{per unit } I_C = \frac{E_{RN}b_C}{I_R}. \qquad (1077)$$

Example 92

Calculate the operating characteristics at rated full-load 0.80 power factor inductive of the same transmission line as specified in Example 91, using the nominal-π solution and per unit values of the quantities involved. The line is designed to transmit 22,000 kva at 66,000 volts 60 cycles and has a resistance of 20 ohms per wire, an inductive reactance of 40 ohms per wire, and a capacitive susceptance of one wire to neutral of 0.00025 mho.

Using rated conditions as base values,

$$\begin{aligned} E_{RN} &= 38{,}100 \text{ volts} = 1.00 = 100 \text{ per cent } E_{RN}, \\ I_R &= 192.5 \text{ amp} = 1.00 = 100 \text{ per cent } I_R, \\ V_{RN} &= \frac{38{,}100 \times 192.5}{1{,}000} = 7{,}330 \text{ vector kva} \\ &= 1.00 = 100 \text{ per cent power output.} \\ \text{Per unit } R &= \frac{192.5 \times 20}{38{,}100} = 0.101, \\ \text{Per unit } X &= \frac{192.5 \times 40}{38{,}100} = 0.202, \text{ and} \\ \text{Per unit } b_C &= \frac{0.00025 \times 38{,}100}{192.5} = 0.0494. \end{aligned}$$

Taking E_{RN} as the reference,

$$\begin{aligned} \dot{E}_{RN} &= 1.00 + j0, \\ \dot{I}_R &= 0.80 - j0.60, \\ \text{Per-unit } b_{CR} &= \frac{0.0494}{2} = 0.0247. \end{aligned}$$

Then,

$$\begin{aligned} \dot{I}_{CR} &= j0.0247(1.00 + j0) \\ &= 0 + j0.0247. \\ \dot{I}_L &= 0.80 - j0.60 + j0.0247 \\ &= 0.80 - j0.575. \end{aligned}$$

The voltage at the sending end becomes

$$\begin{aligned}
\dot{E}_{SN} &= 1.00 + (0.80 - j0.575)(0.101 + j0.202) \\
&= 1.197 + j0.104, \text{ and} \\
E_{SN} &= 1.202. \\
\dot{I}_{CS} &= j0.0247(1.197 + j0.104) \\
&= -0.0025 + j0.0296, \text{ and} \\
\dot{I}_S &= 0.80 - j0.575 - 0.0025 + j0.0296 \\
&= 0.7975 - j0.5457, \text{ and} \\
I_S &= 0.8985.
\end{aligned}$$

For the calculation of efficiency,

$$\begin{aligned}
P_{RN} &= 0.80, \\
P_{SN} &= (1.197)(0.7975) + (0.104)(-0.5457) \\
&= 0.8985. \\
\text{Efficiency} &= \frac{0.80}{0.8985} = 0.891 = 89.1 \text{ per cent.}
\end{aligned}$$

The apparent power input is

$$\begin{aligned}
U_{SN} = V_{SN} &= (1.202)(0.966) = 1.163. \\
(P.F.)_{SN} &= \frac{0.8985}{1.163} = 0.773 = 77.3 \text{ per cent.} \\
E_{RN}' &= 1.206, \\
\text{Regulation} &= \frac{1.206 - 1.00}{1.00} = 0.206 = 20.6 \text{ per cent.}
\end{aligned}$$

This line is calculated by the per unit method for various loads and power factors in Table X.

155. Nominal-T Solution.—The circuit diagram for the nominal-T solution of a transmission line is shown in Fig. 268, the capacitance of the line being concentrated and placed at the middle of the line with one-half the resistance and one-half the reactance of one wire between it and each end of the line. The equations used in the calculation by this method of solution are derived by applying Kirchhoff's laws to the circuit diagram. The values of E_{RN} and I_R are obtained from the known load conditions and expressed to any desired reference. The voltage at the middle of the line, referring to Fig. 268, is

$$\dot{E}_{MN} = \dot{E}_{RN} + \dot{I}_R\frac{\dot{Z}}{2} \text{ volts.} \tag{1078}$$

The charging current for the line capacitance at the middle of the line becomes

$$\dot{I}_C = \dot{E}_{MN}jb_C \text{ amp.} \tag{1079}$$

TABLE X.—CALCULATION OF THE OPERATING CHARACTERISTICS OF A MEDIUM-LENGTH THREE-PHASE TRANSMISSION LINE BY THE NOMINAL-π METHOD USING PER UNIT VALUES AND THE EMF AT THE LOAD AS THE REFERENCE

Example 92

Type of load		Non-reactive			Inductive			Capacitive		
Output, per cent kilovolt-ampere	0	0.50	1.00	1.50	0.50	1.00	1.50	0.50	1.00	1.50
Power factor of load	....	1.00	1.00	1.00	0.80	0.80	0.80	0.80	0.80	0.80
Reactive factor of load	....	0	0	0	−0.60	−0.60	−0.60	0.60	0.60	0.60
Neutral voltage, E_{RN}	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Load current, I_R	0	0.50	1.00	1.50	0.50	1.00	1.50	0.50	1.00	1.50
Active component, i_R	0	0.50	1.00	1.50	0.40	0.80	1.20	0.40	0.80	1.20
Reactive component, i_R'	0	0	0	0	−0.30	−0.60	−0.90	0.30	0.60	0.90
$i'_{C_R} = E_{RN}b_{C_R}$	0.0247	0.0247	0.0247	0.0247	0.0247	0.0247	0.0247	0.0247	0.0247	0.0247
$i_L = i_R$	0	0.50	1.00	1.50	0.40	0.80	1.20	0.40	0.80	1.20
$i'_L = i'_R + i'_{C_R}$	0.0247	0.0247	0.0247	0.0247	−0.2753	−0.5753	−0.8753	0.3247	0.6247	0.9247
$i_LR = 0.101i_L$	0	0.0505	0.101	0.1515	0.0404	0.0808	0.1212	0.0404	0.0808	0.1212
$i'_LX = 0.202i'_L$	0.0050	0.0050	0.0050	0.0050	−0.0556	−0.1162	−0.1768	0.0656	0.1262	0.1868
$e_{SN} = E_{RN} + i_LR - i'_LX$	0.995	1.045	1.0960	1.1460	1.0960	1.1970	1.2980	0.9748	0.9546	0.9344
$i'_LR = 0.101i'_L$	0.0025	0.0025	0.0025	0.0025	−0.0278	−0.0581	−0.0884	0.0328	0.0631	0.0934
$i_LX = 0.202\,i_L$	0	0.1010	0.2020	0.3030	0.0808	0.1616	0.2424	0.0808	0.1616	0.2424
$e'_{SN} = i'_LR + i_LX$	0.0025	0.1035	0.2045	0.3055	0.0530	0.1035	0.1540	0.1136	0.2247	0.3358

$i_{CS} = -e'_{SN}b_{CS}$	−0.0001	−0.0026	−0.0051	−0.0076	−0.0013	−0.0025	−0.0038	−0.0028	−0.0056	−0.0083
$i'_{CS} = e_{SN}b_{CS}$	0.0246	0.0259	0.0271	0.0283	0.0271	0.0296	0.0321	0.0241	0.0236	0.0231
$i_S = i_L + i_{CS}$	−0.0001	0.4974	0.9950	1.4920	0.3987	0.7975	1.1960	0.3972	0.7944	1.1917
$i_S' = i'_L + i'_{CS}$	0.0493	0.0506	0.0518	0.0530	0.2482	−0.5457	−0.8432	0.3488	0.6483	0.9478
$I_S = \sqrt{i^2_S + i'^2_S}$	0.0493	0.5000	0.996	1.493	0.469	0.966	1.463	0.528	1.026	1.522
$E_{SN} = \sqrt{e^2_{SN} + e'^2_{SN}}$	0.995	1.050	1.116	1.188	1.096	1.202	1.308	0.982	0.980	0.993
$e_{SN}i_S$	−0.0001	0.5200	1.091	1.712	0.4375	0.955	1.555	0.388	0.757	1.114
$e'_{SN}i'_S$	0.0001	0.0052	0.0106	0.016	−0.0132	−0.0565	−0.130	0.0397	0.1457	0.318
$P_{SN} = e_{SN}i_S + e'_{SN}i'_S$	0	0.5252	1.1016	1.728	0.4243	0.8985	1.425	0.4277	0.9027	1.432
$E_{SN}I_S$	0.0491	0.5252	1.111	1.773	0.516	1.163	1.912	0.519	1.0075	1.513
$P_{RN} = E_{RN}i_R$	0	0.50	1.00	1.50	0.40	0.80	1.20	0.40	0.80	1.20
Efficiency $= \dfrac{P_{RN}}{P_{SN}}$	....	95.2	90.6	86.8	94.3	89.1	84.2	93.5	88.7	83.8
P. F.$_S = \dfrac{P_{SN}}{E_{SN}I_S}$	0.13	100.00	99.2	97.4	82.2	77.3	74.6	82.3	89.7	94.6
$E'_{RN} = E_{RN}\dfrac{E_{SN}}{E_{SN_O}}$	1.00	1.056	1.121	1.193	1.101	1.206	1.312	0.9866	0.9855	0.9975
Regulation $= \dfrac{E'_{RN} - E_{RN}}{E_{RN}}$	....	5.6	12.1	19.3	10.1	20.6	31.2	−1.34	−1.45	−0.25

The sending-end current, which passes through half the line impedance, is

$$\dot{I}_S = \dot{I}_R + \dot{I}_C \text{ amp.} \tag{1080}$$

The sending-end voltage to neutral is

$$\dot{E}_{SN} = \dot{E}_{MN} + \dot{I}_S \frac{\dot{Z}}{2} \text{ volts.} \tag{1081}$$

The operating characteristics can be calculated from these values.

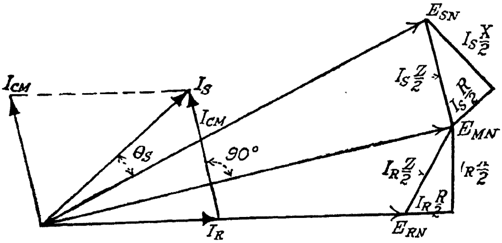

FIG. 273.—Vector diagram for the nominal-T solution of a transmission line with nonreactive load.

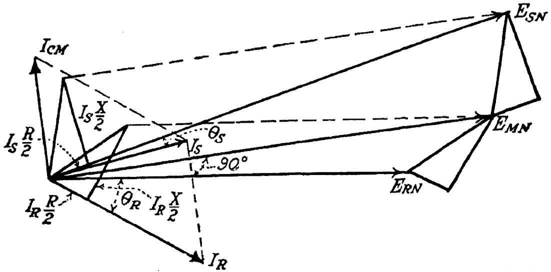

FIG. 274.—Vector diagram for a nominal-T solution of a transmission line with inductive load.

In Fig. 273 is shown the vector diagram of the nominal-T solution of a transmission line with a unity power-factor load in which case both the current and voltage coincide with the reference axis. In Fig. 274 is shown the vector diagram for an inductive load and in Fig. 275 for a capacitive load, the latter with current as the reference.

The sending-end voltage at no load differs from that at full load and also from the no-load receiving-end voltage. A calculation must be made for E_{RN}' in order to obtain the value of receiving-end voltage to be used in the calculation of regulation. Referring to Fig. 268, it is seen that with the load reduced to zero the voltage at the middle of the line is equal to that at the receiving end, so that for open circuit Eq. (1081) may be written

$$\dot{E}_{SN} = \dot{E}_{RN} + \dot{I}_C\frac{\dot{Z}}{2} \text{ volts}$$

$$= \dot{E}_{RN} + jb_C\dot{E}_{RN}\frac{\dot{Z}}{2} \text{ volts.} \qquad (1082)$$

Equation (1082) is the same as Eq. (1069), since b_{CR} is equal to $b_c/2$, so that the same method of calculating E_{RN}' for a given value of E_{SN} will apply to the nominal-T solution as to the nominal-π solution, and regulation is calculated in the same manner for both cases.

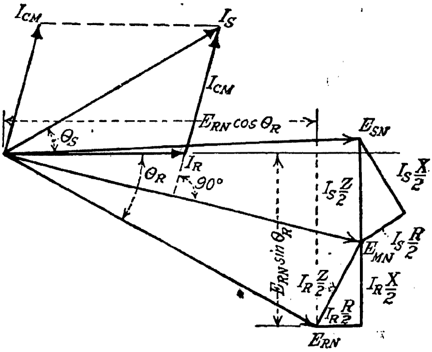

FIG. 275.—Vector diagram for the nominal-T solution of a transmission line with a capacitive load, using current as the reference vector.

Example 93

Determine the sending-end voltage and current and the regulation of the same transmission line as used in Example 91 with 150 per cent load at 0.80 power factor inductive, using the nominal-T solution. The line is designed to transmit 22,000 kva at 66,000 volts three-phase 60 cycles, and has a resistance of 20 ohms, a reactance of 40 ohms, and a capacitive susceptance of 0.0025 mho per wire.

The load current is

$$I_R = \frac{1.50 \times 22{,}000 \times 1{,}000}{66{,}000\sqrt{3}} = 288.8 \text{ amp, and}$$

$$\dot{I}_R = 288(0.80 - j0.60) = 231 - j173.3 \text{ amp.}$$

$$\dot{E}_{RN} = 38{,}100 + j0 \text{ volts.}$$

Then

$$
\begin{aligned}
\dot{E}_{MN} &= 38{,}100 + j0 + (231 - j173.3)(10 + j20) \text{ volts} \\
&= 43{,}876 + j2{,}890 \text{ volts.} \\
\dot{I}_C &= (j0.0025)(43{,}876 + j2{,}890) \text{ amp} \\
&= -0.722 + j11 \text{ amp.} \\
\dot{I}_S &= 231 - j173.3 - 0.722 + j11 \text{ amp} \\
&= 230.3 - j162.3 \text{ amp, and} \\
I_S &= 281.5 \text{ amp.} \\
\dot{E}_{SN} &= 43{,}876 + j2{,}890 + (230.3 - j162.3)(10 + j20) \text{ volts} \\
&= 49{,}425 + j5{,}870 \text{ volts, and} \\
E_{SN} &= 49{,}800 \text{ volts.}
\end{aligned}
$$

On open circuit, E_{MN} is the same as E_{RN}, and the charging current passes through one-half the line impedance; so, for rated voltage,

$$
\begin{aligned}
\dot{E}_{SN0} &= 38{,}100 + (j0.0025)(38{,}100)(10 + j20) \text{ volts} \\
&= 37{,}900 + j100 \text{ volts, and} \\
E_{SN0} &= 37{,}900 \text{ volts.} \\
E_{RN}' &= \frac{49{,}800 \times 38{,}100}{37{,}900} = 50{,}200 \text{ volts, and} \\
\text{Regulation} &= \frac{50{,}200 - 38{,}100}{38{,}100} = 0.318 = 31.8 \text{ per cent.}
\end{aligned}
$$

This line is calculated using the nominal-T method for various factors of load in Table XI.

156. Other Approximate Solutions.—A slightly more accurate solution for the longer lines may be made by breaking up the line into two lines of half the total length connected in series with each other, as indicated in Fig. 276, where one-fourth the capacitance is placed at each end of each division of the line. The equivalent diagram for solution is shown in Fig. 277, where $C/2$ is placed in the middle of the line and $C/4$ at each end, giving an impedance drop due to one-fourth the line charging current through the first half of the line impedance and three-fourths the charging current over the remainder of the impedance, the other

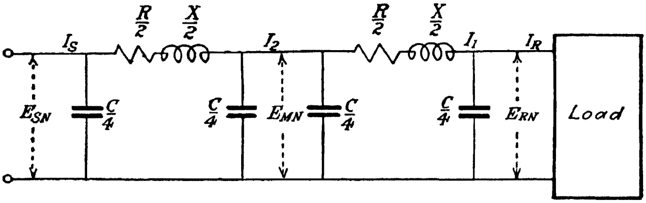

FIG. 276.—Equivalent circuit diagram for a transmission line divided into two separate nominal-π lines in series.

one-fourth of the charging current being added at the sending end of the line to give the load current. Another approximation, sometimes alluded to as the Steinmetz method, is shown in Fig. 278 where $C/6$ is placed at each end of the line and $2C/3$ in the middle, giving an impedance drop due to one-sixth the charging current over one-half the line impedance and five-sixths the charging current over the remaining half. Other methods of approximation are indicated in Figs. 279 and 280. In Fig. 279, the line is broken up into three separate lines, and in Fig. 280 into six separate lines, all in series with each other. Any other number of divisions may be used. Complications such as these, however, are seldom justified unless the emfs at substations or other definite intermediate points are desired.

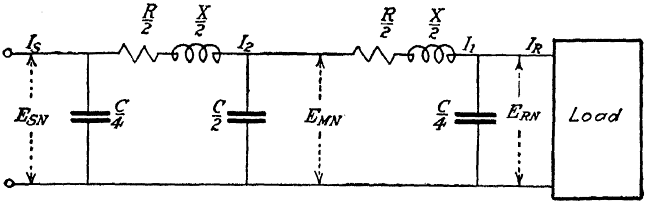

FIG. 277.—Equivalent circuit diagram for a transmission line with one-fourth the capacitance at each end and one-half in the middle.

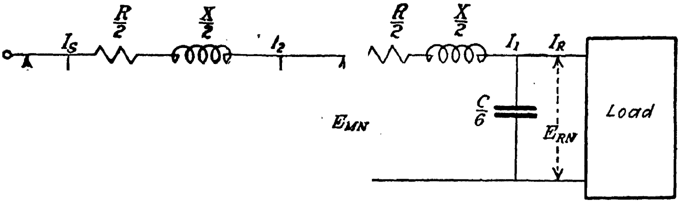

FIG. 278.—Steinmetz equivalent circuit diagram for a transmission line with one-sixth the capacitance at each end and two-thirds in the middle.

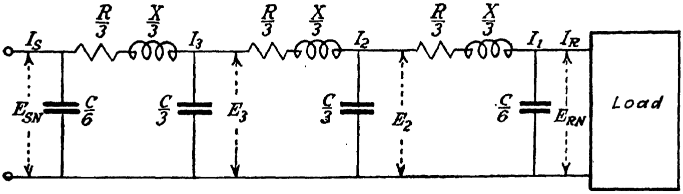

FIG. 279.—Equivalent circuit diagram for a transmission line divided into three nominal-π lines in series.

TABLE XI.—CALCULATION OF THE OPERATING CHARACTERISTICS OF A MEDIUM-LENGTH THREE-PHASE TRANSMISSION LINE BY THE NOMINAL-T METHOD USING THE EMF AT THE LOAD AS A REFERENCE

Example 93

Type of load		Non-reactive			Inductive			Capacitive		
Per cent kilovolt-ampere output	0	0.50	1.00	1.50	0.50	1.00	1.50	0.50	1.00	1.50
Power factor of load	....	1.00	1.00	1.00	0.80	0.80	0.80	0.80	0.80	0.80
Reactive factor of load	....	0	0	0	−0.60	−0.60	−0.60	0.60	0.60	0.60
Neutral voltage, E_{RN}	....	....	....	38,100						
Load current, I_R	0	96.25	192.5	288.8	96.25	192.5	288.8	96.25	192.5	288.8
Active component, i_R	0	96.25	192.5	288.8	77.0	154.0	231.0	77.0	154.0	231.0
Reactive component, i'_R	0	0	0	0	−57.75	−115.5	−173.3	57.75	115.5	173.3
$i_R \frac{R}{2} = 10 i_R$	0	962.5	1,925	2,888	770	1,540	2,310	770	1,540	2,310
$i'_R \frac{X}{2} = 20 i'_R$	0	0	0	0	−1,125	−2,310	−3,466	1,155	2,310	3,466
$e_{MN} = E_{RN} + i_R \frac{R}{2} - i'_R \frac{X}{2}$	38,100	39,063	40,025	40,998	40,025	41,950	43,876	37,715	37,330	36,944
$i'_R \frac{R}{2} = 10 i'_R$	0	0	0	0	−577	−1,155	−1,733	577	1,155	1,733
$i_R \frac{X}{2} = 20 i_R$	0	1,925	3,850	5,776	1,540	3,080	4,620	1,540	3,080	4,620
$e'_{MN} = i'_R \frac{R}{2} + i_R \frac{X}{2}$	0	1,925	3,850	5,776	963	1,925	2,887	2,117	4,235	6,353
$i_C = -e'_{MN} b_C$	0	−0.481	−0.962	−1.444	−0.241	−0.481	−0.722	−0.529	−1.059	−1.588
$i'_C = e_{MN} b_C$	9.525	9.766	10.006	10.247	10.006	10.487	10.969	9.429	9.332	9.236
$i_S = i_R + i_C$	0	95.77	191.5	287.4	76.76	153.5	230.3	76.47	152.9	229.4

$i'_S = i'_R + i'_C$	9.53	9.77	10.01	10.25	−47.74	−105.0	−162.3	67.18	124.8	182.5
$i_S \frac{R}{2} = 10i_S$	0	957.7	1,915	2,874	767.6	1,535	2,303	764.7	1,529	2,294
$i'_S \frac{X}{2} = 20i'_S$	190.6	195.4	200	205	−955	−2,100	−3,246	1,343.	2,496	3,650
$e_{SN} = e_{MN} + i_S \frac{R}{2} - i'_S \frac{X}{2}$	37,909	39,925	41,740	43,657	41,748	45,585	49,425	37,136	36,363	35,590
$i'_S \frac{R}{2} = 10i'_S$	95	98	100	103	−477	−1,050	−1,623	672	1,248	1,825
$i_S \frac{X}{2} = 20i_S$	0	1,915	3,830	5,748	1,535	3,070	4,606	1,529	3,058	4,588
$e'_{SN} = e'_{MN} + i'_S \frac{R}{2} + i_S \frac{X}{2}$	95	3,938	7,780	11,627	2,021	3,945	5,870	4,318	8,541	12,766
$E_{SN} = \sqrt{e^2_{SN} + e'^2_{SN}}$	37,910	40,100	42,450	45,200	41,780	45,750	49,800	37,380	37,360	37,800
$I_S = \sqrt{i^2_S + i'^2_S}$	9.5	96.3	191.6	287.5	90.4	186.0	281.5	101.8	197.6	293.0
$e_{SN} i_S \times 10^{-3}$ (kilowatt)	0	3,816	8,000	12,560	3,206	7,000	11,400	2,843	5,550	8,160
$e'_{SN} i'_S \times 10^{-3}$ (kilowatt)	0.905	38	78	119	−97	−414	−953	291	1,068	2,335
$P_{SN} = e_{SN} i_S + e'_{SN} i'_S$ (kilowatt)	0.905	3,854	8,078	12,679	3,109	6,586	10,447	3,134	6,618	10,495
$E_{SN} I_S \times 10^{-3}$ (kilovolt-ampere)	360	3,860	8,140	13,000	3,780	8,520	14,020	3,805	7,375	11,090
$P_{RN} = E_{RN} i_R \times 10^{-3}$ (kilowatt)	0	3,666	7,333	11,000	2,933	5,867	8,800	2,933	5,867	8,800
Efficiency $= \frac{P_{RN}}{P_{SN}}$	....	95.2	90.6	86.8	94.3	89.1	84.2	93.5	88.7	83.8
P. F.$_S = \frac{P_{SN}}{E_{SN} I_S}$	0.25	99.8	99.25	97.4	82.2	77.3	74.6	82.3	89.7	94.6
$E'_{RN} = E_{RN} \frac{E_{SN}}{E_{SN_O}}$	38,100	40,400	42,780	45,550	42,100	46,100	50,200	37,600	37,570	38,100
Regulation $= \frac{E'_{RN} - E_{RN}}{E_{RN}}$	....	6.04	12.3	19.6	10.5	21.0	31.8	−1.31	−1.40	0
$E_S = \sqrt{3} E_{SN}$	65,500	69,500	73,500	78,200	72,400	79,250	86,200	64,650	64,600	65,550

157. A, B, and C, General Circuit, or Auxiliary Constants of a Transmission Line Using the Nominal-π Solution.—Another solution of the nominal-π line may be derived in such a manner as to secure three constants for the line which may be used later in the calculation of the line characteristics for any load or reference conditions. In this derivation, neither the receiving-end voltage nor the current is used as the reference, but the reference may be substituted at any point in the solution, usually in the application of the results.

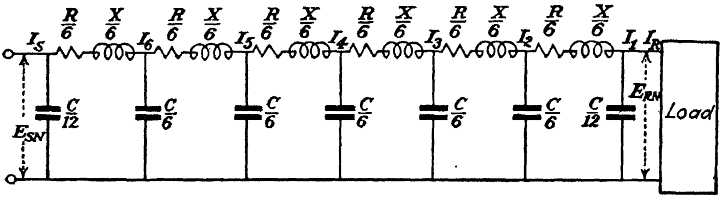

FIG. 280.—Equivalent circuit diagram for a transmission line divided into six nominal-π lines.

Using the nominal-π diagram of Fig. 267, the charging current at the load is

$$\dot{I}_{CR} = \dot{E}_{RN} j b_{CR} = \dot{E}_{RN}\frac{\dot{Y}}{2}\text{ amp,} \tag{1083}$$

and the line current becomes

$$\dot{I}_L = \dot{I}_R + \dot{I}_{CR} = \dot{I}_R + \dot{E}_{RN}\frac{\dot{Y}}{2}\text{ amp.} \tag{1084}$$

The emf at the sending-end of the line is

$$\dot{E}_{SN} = \dot{E}_{RN} + \dot{I}_L\dot{Z}\text{ volts}$$

$$= \dot{E}_{RN} + \dot{I}_R\dot{Z} + \dot{E}_{RN}\frac{\dot{Z}\dot{Y}}{2}\text{ volts} \tag{1085}$$

$$= \dot{E}_{RN}\left[1 + \frac{\dot{Z}\dot{Y}}{2}\right] + \dot{I}_R\dot{Z}\text{ volts,} \tag{1086}$$

which may be written

$$\dot{E}_{SN} = \dot{A}\dot{E}_{RN} + \dot{B}\dot{I}_R\text{ volts,} \tag{1087}$$

where

$$\dot{A} = a_1 + ja_2 = A\underline{/\theta_A} = 1 + \frac{\dot{Z}\dot{Y}}{2} \tag{1088}$$

and

$$\dot{B} = b_1 + jb_2 = B\underline{/\theta_B} = R + jX = \dot{Z}. \tag{1089}$$

The current taken by the sending-end susceptance is

$$\dot{I}_{CS} = \dot{E}_{SN}\frac{\dot{Y}}{2} \text{ amp}, \tag{1090}$$

and, substituting the value of E_{SN} from Eq. (1086),

$$\dot{I}_{CS} = \dot{E}_{RN}\frac{\dot{Y}}{2} + \dot{E}_{RN}\frac{\dot{Z}\dot{Y}^2}{4} + \dot{I}_R\frac{\dot{Y}\dot{Z}}{2} \text{ amp}. \tag{1091}$$

The total current taken by the line at the sending end is

$$\begin{aligned} \dot{I}_S &= \dot{I}_L + \dot{I}_{CS} \text{ amp} \\ &= \dot{I}_R + \dot{E}_{RN}\frac{\dot{Y}}{2} + \dot{E}_{RN}\frac{\dot{Y}}{2} + \dot{E}_{RN}\frac{\dot{Z}\dot{Y}^2}{4} + \dot{I}_R\frac{\dot{Z}\dot{Y}}{2} \text{ amp} \\ &= \dot{I}_R\left[1 + \frac{\dot{Z}\dot{Y}}{2}\right] + \dot{E}_{RN}\dot{Y}\left[1 + \frac{\dot{Z}\dot{Y}}{4}\right] \text{ amp}, \end{aligned} \tag{1092}$$

which may be written

$$\dot{I}_S = \dot{A}\dot{I}_R + \dot{C}\dot{E}_{RN} \text{ amp}, \tag{1093}$$

where

$$\dot{C} = c_1 + jc_2 = \dot{Y}\left[1 + \frac{\dot{Z}\dot{Y}}{4}\right] = C\underline{/\theta_C}, \tag{1094}$$

the value of A being the same as given in Eq. (1088).

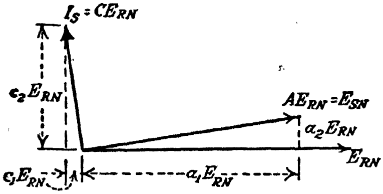

FIG. 281.—No-load vector diagram for a transmission line, using the general circuit constants.

These constants A, B, and C are complex numbers, characteristic of the line, and are independent of the load on the line or the reference used in the calculation; hence they may be calculated in advance and applied in the calculation of the line characteristics under any load conditions. They are known as the "general circuit or auxiliary constants" of the line. At no load, Eq. (1087), becomes

$$\dot{E}_{SNO} = \dot{A}\dot{E}_{RN} = a_1\dot{E}_{RN} + ja_2\dot{E}_{RN} \text{ volts,} \qquad (1095)$$

and Eq. (1093) becomes

$$\dot{I}_S = \dot{C}\dot{E}_{RN} = c_1\dot{E}_{RN} + jc_2\dot{E}_{RN} \text{ amp.} \qquad (1096)$$

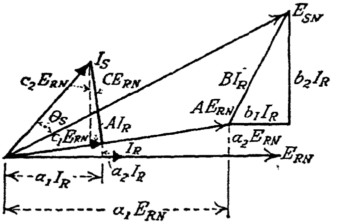

FIG. 282.—Vector diagram for a transmission line with nonreactive load, using the general circuit constants.

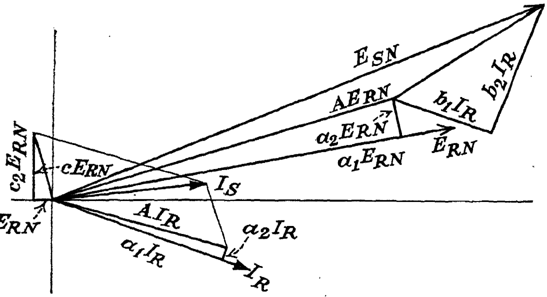

FIG. 283.—Vector diagram for a transmission line with inductive load, using the general circuit constants.

The no-load vector diagram is shown in Fig. 281 with E_{RN} as the reference. It is seen that E_{SN} leads E_{RN} by a small angle, the components of E_{SN} being a_1E_{RN} in phase with E_{RN} and a_2E_{RN} 90 deg. ahead of E_{RN}, which will be the relation with any reference axis chosen. The current taken by the line at no load is more than 90 deg. ahead of E_{RN} and slightly less than 90 deg. ahead of E_{SN}. The charging current flows through the line resistance requiring that its position will be slightly less than 90 deg. with E_{SN}. The component of I_S at no load in phase with E_{SN}, when multiplied by E_{SN}, gives the loss caused by the charging current per wire, the total loss being three times this value. The vector difference between E_{RN} and AE_{RN} at no load is the impedance drop caused by the no-load charging current. The vector diagrams for a transmission line using the A, B, and C constants are shown in Figs. 282, 283, and 284, for the conditions of unity power factor, and inductive and capacitive loads, respectively. It is seen that B is in the nature of a new line impedance corrected for the line capacitance and that C is in the nature of a line capacitance corrected for line impedance drop.

The values of sending- and receiving-end voltage needed in the calculation of regulation are very easily obtained, using the general circuit constants, when the transmission-line equations are given. From Eq. (1095) and the vector diagram, Fig. 281, it can be seen that at no load the sending-end voltage is directly proportional in magnitude to the receiving-end voltage, and

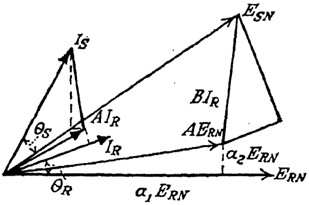

FIG. 284.—Vector diagram for a transmission line with capacitive load, using the general circuit constants.

$$\frac{E_{RN}'}{E_{RN}\text{ (rated)}} = \frac{E_{SN}\text{ (corresponding to rated }E_{RN})}{E_{SN0}}. \tag{1097}$$

Since both E_{SN} and E_{SN0} are found in using Eq. (1095), the value of E_{RN}', the receiving-end voltage when the load is removed with the sending-end voltage constant, is very conveniently calculated from Eq. (1098), which may be obtained directly from Eq. (1097), as

$$E_{RN}' = \frac{E_{SN}\text{ (corresponding to rated }E_{RN})}{E_{SN0}}E_{RN}\text{ (rated) volts.} \tag{1098}$$

Regulation is then calculated from the definition in the usual manner.

Example 94

Calculate the values of the A, B, and C constants for a 60-cycle transmission line 200 miles long constructed of 300,000-circular-mil copper spaced at the corners of a triangle 11 ft. apart. Write the corresponding general circuit equations of the line. The line constants corresponding to these line dimensions are found to be $R = 39.2$ ohms, $X = 156$ ohms, $b_c = 1{,}116 \times 10^{-6}$ mho.

For the nominal-π solution,

$$\dot{Z} = 39.2 + j156 = 161\underline{/75.9^\circ}\text{ ohms},$$
$$\dot{Y} = 0 + j1{,}116 \times 10^{-6} = 1{,}116 \times 10^{-6}\underline{/90^\circ}\text{ mho}.$$

Then

$$\begin{aligned}\dot{Z}\dot{Y} &= 161\underline{/75.9^\circ} \times 1{,}116 \times 10^{-6}\underline{/90^\circ}\\ &= 0.1795\underline{/165.9^\circ} = -0.174 + j0.0438,\end{aligned}$$

and

$$\begin{aligned}\dot{A} &= 1 + \frac{-0.174 + j0.0438}{2} = 0.913 + j0.0219,\\ \dot{B} &= \dot{Z} = 39.2 + j156,\\ \dot{C} &= j1{,}116 \times 10^{-6}\left[1 + \frac{-0.174 + j0.0438}{4}\right]\\ &= -0.0000122 + j0.001067.\end{aligned}$$

The circuit equations become

$$\begin{aligned}\dot{E}_{SN} &= (0.913 + j0.0219)\dot{E}_{RN} + (39.2 + j156)\dot{I}_R \text{ volts, and}\\ \dot{I}_S &= (0.913 + j0.0219)\dot{I}_R + (-0.0000122 + j0.001067)\dot{E}_{RN} \text{ amp.}\end{aligned}$$

158. A, B, and C Constants of a Transmission Line Using the Nominal-T Solution.—The *A*, *B*, and *C* constants of a transmission line using the nominal-T solution are slightly different from the values obtained by the nominal-π method. When Fig. 268 is used for the derivation of the equations, the emf at the middle of the line is

$$\dot{E}_{MN} = \dot{E}_{RN} + \dot{I}_R\frac{\dot{Z}}{2} \text{ volts,} \tag{1099}$$

and the charging current at this point is

$$\dot{I}_C = \dot{E}_{MN}\dot{Y} = \dot{E}_{RN}\dot{Y} + \dot{I}_R\frac{\dot{Z}\dot{Y}}{2} \text{ amp.} \tag{1100}$$

The total current taken by the line and load becomes

$$\begin{aligned}\dot{I}_S &= \dot{I}_R + \dot{I}_{CM} \text{ amp}\\ &= \dot{I}_R + \dot{E}_{RN}\dot{Y} + \dot{I}_R\frac{\dot{Z}\dot{Y}}{2} \text{ amp}\\ &= \dot{I}_R\left[1 + \frac{\dot{Z}\dot{Y}}{2}\right] + \dot{E}_{RN}\dot{Y} \text{ amp.}\end{aligned} \tag{1101}$$

The impressed emf is found by adding the impedance drop in the last half of the line to the voltage at the middle of the line, Eq. (1099), as

$$\dot{E}_{SN} = \dot{E}_{MN} + \dot{I}_S\frac{Z}{2} \text{ volts} \tag{1102}$$

$$= \dot{E}_{RN} + \dot{I}_R\frac{\dot{Z}}{2} + \dot{I}_R\frac{\dot{Z}}{2}\,\dot{E}_{RN}\frac{\dot{Z}\dot{Y}}{2} + \dot{I}_R\frac{\dot{Z}^2\dot{Y}}{4} \text{ volts}$$

$$= \dot{E}_{RN}\left[1 + \frac{\dot{Z}\dot{Y}}{2}\right] + \dot{I}_R\dot{Z}\left[1 + \frac{\dot{Z}\dot{Y}}{4}\right] \text{ volts.} \tag{1103}$$

Equation (1103) may be written

$$\dot{E}_{SN} = \dot{A}\dot{E}_{RN} + \dot{B}\dot{I}_R \text{ volts,} \tag{1087}$$

and Eq. (1101) as

$$\dot{I}_S = \dot{A}\dot{I}_R + \dot{C}\dot{E}_{RN} \text{ amp,} \tag{1093}$$

where

$$\dot{A} = a_1 + ja_2 = A\underline{/\theta_A} = 1 + \frac{\dot{Z}\dot{Y}}{2}, \tag{1094}$$

$$\dot{B} = b_1 + jb_2 = B\underline{/\theta_B} = \dot{Z}\left[1 + \frac{ZY}{4}\right], \tag{1104}$$

$$\dot{C} = c_1 + jc_2 = C\underline{/\theta_C} = \dot{Y} = g + jb_c. \tag{1105}$$

These values of the constants may be applied in Eqs. (1087) and (1093) to determine the values of E_{SN} and I_S. From these values, the regulation, efficiency, and power factor of the line may be calculated. The vector diagrams for this method of solution are the same as those for the nominal-π solution, Figs. 281, 282, 283, and 284.

Example 95

Find the A, B, and C constants for the same transmission line as in Example 94, using the nominal-T solution. Write the corresponding general circuit equations of the line.

From Example 94,

$$\dot{Z} = 39.2 + j156 = 161\underline{/75.9^\circ} \text{ ohms,}$$
$$\dot{Y} = 0 + j1{,}116 \times 10^{-6}\underline{/90^\circ} \text{ mhos.}$$
$$\dot{Z}\dot{Y} = 0.1795\underline{/165.9^\circ} = -0.174 + j0.0438.$$

Then for the nominal-T line,

$$\dot{A} = 0.913 + j0.0219,$$
$$\dot{B} = 161\underline{/75.9^\circ}\left[1 + \frac{-0.174 + j0.0438}{4}\right]$$
$$= 154\underline{/76.5^\circ} = 35.8 + j149.4,$$
$$\dot{C} = \dot{Y} = 0 + j0.001116.$$

Then

$$\dot{E}_{SN} = (0.913 + j0.0219)\dot{E}_{RN} + (35.8 + j149.4)\dot{I}_R \text{ volts, and}$$
$$\dot{I}_S = (0.913 + j0.0219)\dot{I}_R + j0.001116\dot{E}_{RN} \text{ amp.}$$

Problems

1-14. An 88,000-volt three-phase 60-cycle line is 86 miles long. The conductors are of No. 00 B. & S. stranded copper spaced 8 ft. apart horizontally. The constants per mile of conductor are: resistance, 0.4287 ohm; inductance, 2.15 millihenrys; and capacitance to neutral, 0.01406 μf. Calculate the sending-end voltage, current, and power factor when the line is delivering 15,000 kva at 0.80 power factor inductive at 88,000 volts. Use the nominal-π solution with E_{RN} as the reference. Draw the vector diagram.

2-14. A 66,000-volt 60-cycle three-phase line is 96 miles long. It is constructed of No. 000 B. & S. aluminum cable, steel reinforced, with an equivalent spacing of 7 ft. The constants per mile of conductor are: resistance, 0.579 ohm; inductive reactance, 0.790 ohm; and capacitive susceptance to neutral, 5.8×10^{-6} mho. Calculate the efficiency and regulation of the line when delivering 8,000 kva at unity power factor. Use the nominal-π solution with E_{RN} as the reference. Draw the vector diagram.

3-14. A three-phase transmission line has the following constants per wire at 60 cycles: $\dot{Z} = 27.7 + j79.7$, $\dot{Y} = 0 + j542 \times 10^{-6}$. If this line is delivering 20,000 kva at 0.707 power factor inductive load at the receiving end with a voltage of 110,000 volts, calculate the regulation and efficiency, using the nominal-π solution and E_{RN} as the reference. Draw the vector diagram.

4-14. A transmission line is constructed to transmit 66,000 hp. from a hydroelectric plant at Keokuk, Iowa, to St. Louis, Mo., a distance of 144 miles. The line voltage at St. Louis is to be maintained at 110,000 volts between lines. The system is three-phase 25-cycles, the wires being arranged in vertical planes at 10-10-20 ft. spacing, of 300,000-circular-mil B. & S. copper 19-strand cable, 0.625 in. in diameter. Calculate the efficiency, regulation, and power factor at the generating station for full load and for 50 per cent overload for a unity power factor load and for an inductive load at 0.70 power factor. Use the nominal-π solution, voltage at the receiving end of the line as the reference and tabular method of calculation. Draw the vector diagram.

5-14. A 66,000-volt 60-cycle three-phase line 80 miles long, constructed of No. 0000 B. & S. aluminum conductors, has a resistance of 39.2 ohms, an inductive reactance of 62.4 ohms, and a capacitive susceptance of 474×10^{-6} mho per conductor. Find the regulation and efficiency for a load of 10,000 kva at 0.85 power factor inductive. Use the nominal-π solution with load current as the reference. Draw the vector diagram.

6-14. Calculate the per unit constants for an 88,000-volt three-phase transmission line 92 miles long, having a resistance of 25.6 ohms, an inductive reactance of 69.6 ohms, and a capacitive susceptance to neutral of 524×10^{-6} mho per conductor, the line being rated at 25,000 kva.

7-14. Calculate the regulation and efficiency of the line of Prob. 6-14 for rated load and 0.80 power factor inductive, using the per unit constants and the nominal-π solution. Draw the vector diagram.

8-14. Calculate the regulation and efficiency for a transmission line having 10 per cent resistance, 19 per cent reactance, and 12 per cent capacitive susceptance, using the nominal-π solution and E_{RN} as the reference with 150 per cent load at unity power factor.

9-14. Find the voltage between lines, the regulation, efficiency, and the power factor at the sending end for a 10-000-kva 66,000-volt three-phase 60-cycle transmission line when delivering 5,000 and 10,000 kva at unity and 0.85 power factor inductive loads, keeping the voltage at the receiving end constant. The line has a resistance of 14 ohms, a reactance of 34 ohms, and a capacitive susceptance of 625×10^{-6} mho. Use the nominal-π solution and per unit constants.

10-14. Find the sending-end voltage, current, and power factor for a three-phase 60-cycle transmission line having a resistance of 46.4 ohms, a reactance of 84.2 ohms, and a capacitive susceptance to neutral of 554×10^{-6} mho, when delivering 35,000 kva at 110,000 volts at 0.707 power factor inductive. Use the nominal-T method with E_{RN} as the reference. Draw the vector diagram.

11-14. Calculate the sending-end power factor, efficiency, and regulation for a transmission line 100 miles long designed to transmit 22,000 kva at 88 kilovolts three-phase 60 cycles, using the nominal-T method of calculation, for the rated load at unity power factor. Use the load current as the reference, and draw the vector diagram. The resistance of the line is 27.7 ohms; the inductive reactance, 77.4 ohms; and the capacitive susceptance, 562×10^{-6} mho per conductor.

12-14. Calculate the same line as in Prob. 4-14 but use the nominal-T solution.

13-14. Find the efficiency and regulation of a 44-mile 50 kv three-phase 60-cycle transmission line using No. 1 B. & S. copper conductors, the resistance of the line being 30.8 ohms, the reactance 34.2 ohms, and the capacitive susceptance 240×10^{-6} mho per conductor, the line carrying a load of 7,500 kva at unity power factor. Find the per unit constants, and use the nominal-T solution and the per unit method with E_{RN} as the reference.

14-14. Determine the operating characteristics of a transmission line 50 miles long, designed for transmitting 18,000 kva at 66,000 volts three-phase 60 cycles, for open-circuit, half-load, and full-load output at unity power factor and 0.80 power factor inductive load. Use the nominal-T solution, per-unit values, and E_{RN} as the reference. The line is constructed of No. 0000 B. & S. copper conductors and has a resistance of 13.85 ohms, an inductive reactance of 36.4 ohms, and a total capacitive reactance to neutral of 3,320 ohms per conductor.

15-14. Derive the general circuit or auxiliary constants for a three-phase transmission line 50 miles long operating at 66,000 volts 60 cycles and constructed of No. 000 B. & S. stranded copper conductors spaced 8 ft. apart vertically, using the nominal-π solution. The resistance is 0.35 ohm per mile; the reactance, 0.792 ohm per mile; and the capacitive susceptance to neutral, 5.41×10^{-6} mho per mile of conductor.

16-14. Calculate the general circuit or auxiliary constants for a 100-mile three-phase 88,000-volt 60-cycle transmission line, having a resistance of 35 ohms, a reactance of 70 ohms, and a capacitive susceptance to neutral of 590×10^{-6} mho per conductor. Use the nominal-π solution.

17-14. Derive the auxiliary constants for a 150-mile three-phase 60-cycle transmission line for 120,000 volts, using the nominal-T solution. The conductors are 300,000-circular-mil aluminum cable steel reinforced and spaced 14 ft. vertically. The resistance per mile of conductor is 0.311 ohm; the reactance, 0.785 ohm; and the capacitive susceptance, 5.26×10^{-6} mho.

18-14. Calculate the values of the auxiliary constants for a transmission line for the nominal-T solution if the line constants are given as $\dot{Z} = 73.5 + j402$ ohms and $\dot{Y} = 0 + j2{,}690 \times 10^{-6}$ mho, expressing the values in rectangular and polar coordinates.

19-14. Calculate the sending-end voltage, current, and power factor for a 140-mile 110,000-volt 60-cycle three-phase line when delivering 25,000 kva at 0.80 power factor inductive. The general circuit constants for the line are $\dot{A} = 0.958 + j0.0141$, $\dot{B} = 37.8 + j113$, and $C = -0.000003 + j0.000728$. Draw the vector diagram.

20-14. Calculate the operating characteristics of a transmission line 75 miles long, designed for transmitting 30,000 kva at 88,000 volts three-phase 60-cycles for no load, half load and full load at unity power factor and 0.85 power factor inductive if the line has the following auxiliary constants: $\dot{A} = 0.988 + j0.00435$, $\dot{B} = 20.7 + j57.5$, and $C = -0.000001 + j0.000417$.

21-14. Find the charging kilovolt-amperes delivered by the generator and the generator voltage for the open-circuit condition for a 100,000-volt 60-cycle transmission line 105 miles long, having a resistance of 0.235 ohm, a reactance of 0.798 ohm, and a capacitive susceptance of 5.35×10^{-6} mho per mile of conductor, using the nominal-π method. Draw the vector diagram.

22-14. Calculate the operating characteristics for a 100-mile 120,000-volt three-phase 60-cycle line when delivering 40,000 kva at unity power factor and at 0.85 power factor inductive. The line is constructed of No. 0000 B. & S. copper conductors and has a resistance of 27.7 ohms, a reactance of 79.7 ohms, and a capacitive susceptance of 542×10^{-6} mho. Use the nominal-π solution and E_{RN} as the reference.

23-14. Make an analytical solution—the solution for the general case—for sending voltage and current for a transmission line, using the nominal-π solution and current at the load as the reference. Draw the vector diagram for the same reference.

24-14. Calculate the efficiency and regulation for transmission lines having 8 per cent resistance, 19 per cent reactance, and 10 per cent capacitive susceptance, using the nominal-T solution and full-load 0.80 power factor inductive.

25-14. Calculate the auxiliary constants for the line of Prob. 22-14, using the nominal-π solution.

26-14. Given a three-phase transmission line delivering 50 amp at 68,000 volts to neutral and having the auxiliary constants: $\dot{A} = 0.809 + j0.037$, $\dot{B} = 38.46 + j206.4$, and $C = 0.0017\underline{/90^\circ\ 47'}$. Calculate the sending-end voltage and current at 0.70 power factor capacitive load. Draw the vector diagram.

27-14. Derive the equations for the auxiliary constants for a transmission line using the Steinmetz approximation.

CHAPTER XV

LONG TRANSMISSION LINES

159. Distributed Constants of the Line.—Certain errors have been introduced by lumping the constants of the line in the solutions of transmission lines in the previous chapters. In this chapter, a method of solution is derived that takes into account the distributed character of these constants, *i.e.*, the fact that the resistance, inductance, and capacitance of the line are distributed uniformly along the line. It is necessary to apply this method to long lines in order to obtain accurate results. The length of a line in this sense depends upon both the distance in miles and the frequency; a line of low frequency and long in distance and one of high frequency but short in distance are both electrically long lines. The former is usually encountered in steady-state calculations of lines for power transmission, whereas the latter is encountered in the calculation of communication lines.

In addition to the constants of resistance, inductance, and capacitance of a transmission line, there is a constant to replace the loss due to corona and to leakage currents over insulators in certain portions of the system, the loss being proportional to a function of the line voltage. This consideration gives rise to four constants per unit length of line (per mile for instance), where r is the resistance per mile of wire;

x is the reactance due to inductance per mile of wire;
b is the susceptance due to the charging current per mile of wire;
g is the conductance due to leakage and corona per mile of wire.

For the whole length of line, where L is the length in miles,

R is the total resistance of a wire,
X is the total reactance of a wire,
B is the total susceptance of a wire, and
G is the total conductance of a wire.

160. Development of the Equations of a Long Line.—Let the emf impressed on the load be E_{RN} and the load current be I_R both referred to the same reference, which may be taken as E_{RN}. At the point P, Fig. 285, x miles distant from the load, the emf to neutral at the load side of P may be taken as E, which is the vector sum of the voltage E_{RN} and the impedance drop of the line for x miles. The current will be I amp, which is the vector sum of the load current I_R and the integrated current due to the line admittance for x miles. This emf and current are both referred to the same reference as are the values at the load, usually the voltage across the load.

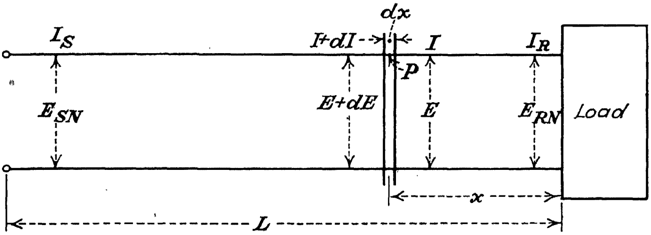

FIG. 285.—Circuit diagram for the rigorous solution of a transmission line.

Owing to the line admittance in the length dx, there will be a change in the current I by the amount

$$d\dot{I} = \dot{E}\dot{y}dx \text{ amp} \tag{1106}$$

for a constant effective value of the line emf, where ydx is the admittance of the length dx of the line. The current beyond the differential length will be increased to $\dot{I} + d\dot{I}$ by the admittance of the differential length under consideration. The current flowing in this differential length will cause a drop in emf of

$$d\dot{E} = \dot{I}\dot{z}dx \text{ volts,} \tag{1107}$$

where zdx is the impedance of the length dx. The emf beyond the point P, toward the sending end, is then $\dot{E} + d\dot{E}$. Dividing Eqs. (1106) and (1107) by dx, the differential equations of the circuit are found to be

$$\frac{d\dot{I}}{dx} - \dot{E}\dot{y} = 0 \text{ amp,} \tag{1108}$$

and

$$\frac{d\dot{E}}{dx} - \dot{I}\dot{z} = 0 \text{ volts.} \tag{1109}$$

Differentiating Eqs. (1108) and (1109) with respect to x,

$$\frac{d^2\dot{I}}{dx^2} - \dot{y}\frac{d\dot{E}}{dx} = 0, \tag{1110}$$

and

$$\frac{d^2\dot{E}}{dx^2} - \dot{z}\frac{d\dot{I}}{dx} = 0. \tag{1111}$$

Substituting the values of dI/dx and dE/dx from Eqs. (1108) and (1109) into Eqs. (1110) and (1111), respectively, new equations containing only I and E as variables in respect to x are obtained for solution, as

$$\frac{d^2\dot{I}}{dx^2} - \dot{z}\dot{y}\dot{I} = 0 \tag{1112}$$

and

$$\frac{d^2\dot{E}}{dx^2} - \dot{z}\dot{y}\dot{E} = 0, \tag{1113}$$

which are the fundamental differential equations of a transmission line. Equations (1112) and (1113) are of the form

$$\frac{d^2y}{dx^2} - ay = 0, \tag{1114}$$

which is integrated by letting

$$y = D\epsilon^{mx}. \tag{1115}$$

Differentiating Eq. (1115) with respect to x,

$$\frac{dy}{dx} = mD\epsilon^{mx}, \tag{1116}$$

and again differentiating this new equation in respect to x,

$$\frac{d^2y}{dx^2} = m^2D\epsilon^{mx}. \tag{1117}$$

Substituting the values of d^2y/dx^2 and y from Eqs. (1117) and (1115) into Eq. (1114),

$$m^2D\epsilon^{mx} - aD\epsilon^{mx} = 0,$$

or

$$m^2 - a = 0, \tag{1118}$$

where m, and D are arbitrary constants. From Eq. (1118),

$$m = \pm\sqrt{a}. \tag{1119}$$

From Eq. (1119), it is seen that m must have two roots and likewise there must be two values of D, as D_1 and D_2, so that the solution of Eq. (1114) becomes

$$y = D_1\epsilon^{\sqrt{a}x} + D_2\epsilon^{-\sqrt{a}x}, \tag{1120}$$

since the complete solution is the sum of the individual solutions, using the different arbitrary constants previously derived.

Applying this solution to Eq. (1112),

(1121)

where the integration constants D_1 and D_2 remain to be determined from particular applications of this equation. Letting

$$\dot{v} = \sqrt{\dot{z}\dot{y}}, \tag{1122}$$

to shorten the work in writing the equations, Eq. (1121) becomes

$$\dot{I} = D_1\epsilon^{\dot{v}x} + D_2\epsilon^{-\dot{v}x} \text{ amp.} \tag{1123}$$

From Eq. (1108),

$$\dot{E} = \frac{d\dot{I}}{\dot{y}dx} \text{ volts.} \tag{1108}$$

Differentiating Eq. (1123) and substituting in Eq. (1108),

$$\dot{E} = \frac{1}{\dot{y}}[\dot{v}D_1\epsilon^{\dot{v}x} - \dot{v}D_2\epsilon^{-\dot{v}x}] \tag{1124}$$

$$= \sqrt{\frac{\dot{z}}{\dot{y}}}[D_1\epsilon^{\dot{v}x} - D_2\epsilon^{-\dot{v}x}]. \tag{1125}$$

Equations (1123) and (1125) become the general solutions of the transmission line.

161. Solution for Voltage and Current at Any Point along the Line.—For the solution of Eqs. (1123) and (1125) at any point along the line, it is necessary to assume the values of E and I at some particular point. It is usual to use the voltage and current at the load where $x = 0$ as the known values. Then, for $x = 0$, $\dot{E} = \dot{E}_{RN}$ and $\dot{I} = \dot{I}_R$ and, substituting these conditions into Eqs. (1123) and (1125),

$$\dot{I}_R = \dot{D}_1 + \dot{D}_2 \text{ amp,} \tag{1126}$$

and

$$\dot{E}_{RN} = \sqrt{\frac{\dot{z}}{\dot{y}}}(\dot{D}_1 - \dot{D}_2) \text{ volts} \tag{1127}$$

$$= \dot{Z}_0(\dot{D}_1 - \dot{D}_2) \text{ volts,} \tag{1128}$$

where $\dot{Z}_0 = \sqrt{\dot{z}/\dot{y}} = \sqrt{\dot{Z}/\dot{Y}}$ is the *surge impedance* of the line, a constant independent of the length of the line and having the dimensions of an impedance.

From Eqs. (1126) and (1127),

$$\dot{D}_1 + \dot{D}_2 = \dot{I}_R, \tag{1126}$$

$$\dot{D}_1 - \dot{D}_2 = \frac{\dot{E}_{RN}}{\dot{Z}_0}, \tag{1129}$$

and, solving Eqs. (1126) and (1129),

$$\dot{D}_1 = \frac{1}{2}\left[\dot{I}_R + \frac{\dot{E}_{RN}}{\dot{Z}_0}\right] \tag{1130}$$

and

$$\dot{D}_2 = \frac{1}{2}\left[\dot{I}_R - \frac{\dot{E}_{RN}}{\dot{Z}_0}\right] \tag{1131}$$

are the particular values of the constants of integration. Substituting these values of $\dot{D}_1$ and $\dot{D}_2$ into Eqs. (1123) and (1125),

$$\dot{I} = \frac{1}{2}\left[\dot{I}_R + \frac{\dot{E}_{RN}}{\dot{Z}_0}\right]\epsilon^{\dot{v}x} + \frac{1}{2}\left[\dot{I}_R - \frac{\dot{E}_{RN}}{\dot{Z}_0}\right]\epsilon^{-\dot{v}x} \text{ amp} \tag{1132}$$

and

$$\dot{E} = \frac{1}{2}[\dot{E}_{RN} + \dot{Z}_0\dot{I}_R]\epsilon^{\dot{v}x} + \frac{1}{2}[\dot{E}_{RN} - \dot{Z}_0\dot{I}_R]\epsilon^{-\dot{v}x} \text{ volts.} \tag{1133}$$

Equations (1132) and (1133) give the value of current and emf at any point along the line in terms of the current and emf to neutral at the load, the constants of the line per unit length, and the distance from the load. Since $\dot{v}$ is a complex quantity, expressed as

$$\dot{v} = \sqrt{\dot{z}\dot{y}}, \tag{1122}$$

it may be considered as made up of two terms, real and imaginary, as

$$\dot{v} = \sqrt{\dot{z}\dot{y}} = \alpha + j\beta \tag{1134}$$

per unit length of line, and

$$\begin{aligned}\epsilon^{\pm \dot{v}x} &= \epsilon^{\pm(\alpha x + j\beta x)} = \epsilon^{\pm \alpha x}\epsilon^{\pm j\beta x} \\ &= \epsilon^{\pm \alpha x}\underline{/\beta x}.\end{aligned} \tag{1135}$$

In Eq. (1134), $\dot{v}$ is called the "propagation constant," α the "attenuation constant," and β the "wave-length constant" per unit length of line. Hence $\dot{v}$, as an exponent in $\epsilon^{\dot{v}x}$, is seen to be a measure both of the attenuation of the wave and of the change in the phase position of the vector at different distances along the line measured from the load.

Equations (1132) and (1133) may be changed by rearrangement to a more convenient form for calculation. By rearranging the terms in these equations,

$$\dot{I} = \dot{I}_R\left[\frac{\epsilon^{vx} + \epsilon^{-\dot{v}x}}{2}\right] + \frac{\dot{E}_{RN}}{\dot{Z}_0}\left[\frac{\epsilon^{\dot{v}x} - \epsilon^{-\dot{v}x}}{2}\right] \text{ amp,} \tag{1136}$$

and

$$\dot{E} = \dot{E}_{RN}\left[\frac{\epsilon^{\dot{v}x} + \epsilon^{-\dot{v}x}}{2}\right] + \dot{Z}_0\dot{I}_R\left[\frac{\epsilon^{-\dot{v}x} - \epsilon^{-\dot{v}x}}{2}\right] \text{ volts.} \tag{1137}$$

Since

$$\cosh \dot{v}x = \frac{\epsilon^{\dot{v}x} + \epsilon^{-\dot{v}x}}{2} \tag{1138}$$

and

$$\sinh \dot{v}x = \frac{\epsilon^{\dot{v}x} - \epsilon^{-\dot{v}x}}{2}, \tag{1139}$$

these equations may be written

$$\dot{E} = \dot{E}_{RN} \cosh \dot{v}x + \dot{Z}_0\dot{I}_R \sinh \dot{v}x \text{ volts} \tag{1140}$$

and

$$\dot{I} = \dot{I}_R \cosh \dot{v}x + \frac{\dot{E}_{RN}}{\dot{Z}_0} \sinh \dot{v}x \text{ amp.} \tag{1141}$$

162. Voltage and Current at the Sending End.—At the sending end of the line, the variables I and E become I_S and E_{SN}. The distance x becomes L, and the hyperbolic angle becomes $\dot{v}L$ as

$$\dot{v}L = \sqrt{\dot{z}\dot{y}}L = \sqrt{\dot{z}L\dot{y}L} = \sqrt{\dot{Z}\dot{Y}} \text{ rad.,} \tag{1142}$$

where $\dot{Z} = \dot{z}L$ is the total impedance of the line and $\dot{Y} = \dot{y}L$ is the total admittance of the line. Letting

$$\dot{v}L = \sqrt{\dot{Z}\dot{Y}} = \dot{\theta} \text{ rad.,} \tag{1143}$$

the voltage and current at the sending end of the line may be expressed as

$$\dot{E}_{SN} = \dot{E}_{RN} \cosh \dot{\theta} + \dot{Z}_0 \dot{I}_R \sinh \dot{\theta} \text{ volts,} \tag{1144}$$

and

$$\dot{I}_S = \dot{I}_R \cosh \dot{\theta} + \frac{\dot{E}_{RN}}{\dot{Z}_0} \sinh \dot{\theta} \text{ amp.} \tag{1145}$$

The hyperbolic function of the complex angle θ can be obtained from tables or charts that have been prepared for the purpose. If these charts or tables are not readily available, the hyperbolic function of the complex variable may be expanded in terms of the hyperbolic and trigonometric functions by using Eqs. (247) and (248), derived in Chap. III, as

$$\cosh (\theta_1 + j\theta_2) = \cosh \theta_1 \cos \theta_2 + j \sinh \theta_1 \sin \theta_2 \tag{248}$$

and

$$\sinh (\theta_1 + j\theta_2) = \sinh \theta_1 \cos \theta_2 + j \cosh \theta_1 \sin \theta_2, \tag{247}$$

where

$$\dot{\theta} = \theta_1 + j\theta_2. \tag{1146}$$

The hyperbolic functions of the real angle θ_1 may be found in tables, or they may be calculated from the exponential form of the function expressed as

$$\cosh \theta_1 = \frac{\epsilon^{\theta_1} + \epsilon^{-\theta_1}}{2} \tag{240}$$

and

$$\sinh \theta_1 = \frac{\epsilon^{\theta_1} - \epsilon^{-\theta_1}}{2}. \tag{239}$$

Equations (1144) and (1145) are in the same form as Eqs. (1086) and (1092) of Chap. XIV, where the relations are expressed in terms of the general circuit constants, and may be expressed as

$$\dot{E}_{SN} = \dot{A}\dot{E}_{RN} + \dot{B}\dot{I}_R \text{ volts} \tag{1087}$$

and

$$\dot{I}_S = \dot{A}\dot{I}_R + \dot{C}\dot{E}_{RN} \text{ amp,} \tag{1093}$$

where, in this case,

$$\dot{A} = \cosh \dot{\theta}, \tag{1147}$$

$$\dot{B} = \dot{Z}_0 \sinh \dot{\theta}, \tag{1148}$$

and

$$\dot{C} = \frac{1}{\dot{Z}_0} \sinh \dot{\theta}. \tag{1149}$$

Equations (1144) and (1145) may be written

$$\dot{E}_{SN} = \dot{E}_{RN} \cosh \dot{\theta} + \dot{Z}\dot{I}_R \frac{\sinh \dot{\theta}}{\dot{\theta}} \text{ volts,} \tag{1150}$$

and

$$\dot{I}_S = \dot{I}_R \cosh \dot{\theta} + \dot{Y}\dot{E}_{RN} \frac{\sinh \dot{\theta}}{\dot{\theta}} \text{ amp,} \tag{1151}$$

since

$$\dot{Z}_0 = \sqrt{\frac{\dot{Z}}{\dot{Y}}} = \dot{Z}\sqrt{\frac{1}{\dot{Z}\dot{Y}}} = \frac{\dot{Z}}{\dot{\theta}} \tag{1152}$$

and

$$\frac{1}{\dot{Z}_0} = \sqrt{\frac{\dot{Y}}{\dot{Z}}} = \dot{Y}\sqrt{\frac{1}{\dot{Z}\dot{Y}}} = \frac{\dot{Y}}{\dot{\theta}}. \tag{1153}$$

For solving the equation in the form given in Eqs. (1150) and (1151), Dr. Kennelly has published convenient tables and diagrams giving the values $\frac{\sinh \dot{\theta}}{\dot{\theta}}$, $\frac{\cosh \dot{\theta}}{\dot{\theta}}$, and $\frac{\tanh \dot{\theta}}{\dot{\theta}}$ for values of the hyperbolic angle θ as the argument.

If the starting point of writing the equations had been measured from the sending end instead of the receiving end, Eqs. (1140) and (1141) would become

$$\dot{I} = \dot{I}_R \cosh \dot{v}x - \frac{1}{\dot{Z}_0}\dot{E}_{RN} \sinh \dot{v}x \text{ amp,} \tag{1154}$$

and

$$\dot{E} = \dot{E}_{SN} \cosh \dot{v}x - \dot{Z}_0\dot{I}_R \sinh \dot{v}x \text{ volts.} \tag{1155}$$

Upon putting in the conditions that, when $x = 0$, $\dot{E} = \dot{E}_{SN}$ and $\dot{I} = \dot{I}_S$ and that, when $x = L$, $\dot{E} = \dot{E}_{RN}$ and $\dot{I} = \dot{I}_R$, the equations for the line would be

$$\dot{E}_{RN} = \dot{A}\dot{E}_{SN} - \dot{B}\dot{I}_S \text{ volts} \tag{1156}$$

and

$$\dot{I}_R = \dot{A}\dot{I}_S - \dot{C}\dot{E}_{SN} \text{ amp.} \tag{1157}$$

These equations can be used to determine the conditions at the receiving end of the line when those at the sending end are known.

Example 96

Calculate the complex angle $\dot{v} = \sqrt{\dot{z}\dot{y}}$ per mile of line for a 200,000-volt three-phase three-wire line having the following values per mile of line:

r per mile of conductor $= 0.147$ ohm,
x per mile of conductor $= 0.805$ ohm,
b per mile of conductor $= 5.38 \times 10^{-6}$ mho,
g per mile of conductor $= 0.0$.

From the above constants,

$$\dot{z} = r + jx = 0.147 + j0.805 = 0.8183\underline{/79.65^\circ} \text{ ohms,}$$
$$\dot{y} = g + jb = 0 + j5.38 \times 10^{-6} = 5.38 \times 10^{-6}\underline{/90^\circ} \text{ mho.}$$

Then

$$\dot{v} = \sqrt{\dot{z}\dot{y}} = \sqrt{zy}\,\underline{\Big/\frac{\theta y + \theta z}{2}} \text{ rad.}$$
$$= [0.8183 \times 5.38 \times 10^{-6}]^{1/2}\,\underline{\Big/\frac{79.65^\circ + 90^\circ}{2}}$$
$$= 0.002098\underline{/84.82^\circ} \text{ rad. per mile}$$
$$= \alpha + j\beta = 0.0001893 + j0.00208 \text{ rad.,}$$

where

$\alpha = 0.0001893$ hyperbolic radians,
$\beta = 0.00208$ circular radians $= 0.1197^\circ$.

Example 97

Calculate the A, B, and C constants for 100, 200, 300, 400, 500, and 600 miles of the line of Example 96. Using the 300 mile length of line as an illustration,

$$\dot{\theta} = 300\dot{v} = 0.6294\underline{/84.82^\circ} \text{ rad.}$$
$$= 0.05679 + j0.624 \text{ rad., or}$$
$$\theta_1 = 0.05679 \text{ hyperbolic rad.}$$
$$\theta_2 = 0.624 \text{ circular radians} = 35.91^\circ.$$

From tables of functions or from calculations using the definitions of the hyperbolic functions of a real variable in terms of the exponential quantities,

$$\cosh \theta_1 = \cosh 0.0567 = 1.0016,$$
$$\sinh \theta_1 = \sinh 0.0567 = 0.05682,$$
$$\cos \theta_2 = \cos 35.91^\circ = 0.8100,$$
$$\sin \theta_2 = \sin 35.91^\circ = 0.5865,$$
$$\cosh \dot{\theta} = \cosh 0.0567 \cos 35.91^\circ + j \sinh 0.0567 \sin 35.91^\circ$$
$$= 1.0016 \times 0.8100 + j0.0568 \times 0.5765$$
$$= 0.8113 + j0.03333 = 0.8120\underline{/2.35^\circ}$$
$$\sinh \dot{\theta} = \sinh 0.0567 \cos 35.91^\circ + j \cosh 0.0567 \sin 35.91^\circ$$
$$= 0.05682 \times 0.8100 + j1.0016 \times 0.5865$$
$$= 0.5893\underline{/85.52^\circ},$$

$$\begin{aligned}\dot{Z}_0 &= \sqrt{\frac{\dot{z}}{\dot{y}}} = \sqrt{\frac{z}{y}}\underline{\Big/\frac{\theta_z - \theta_y}{2}} \\ &= \sqrt{\frac{0.8183}{5.38 \times 10^{-6}}}\underline{\Big/\frac{79.65^\circ - 90^\circ}{}} \\ &= 390\underline{/-5.175^\circ}.\end{aligned}$$

The constants A, B, and C become

$$\begin{aligned}\dot{A} &= \cosh \dot{\theta} = 0.8120\underline{/2.35^\circ}, \\ \dot{B} &= \dot{Z}_0 \sinh \dot{\theta} = 390\underline{/-5.175^\circ} \times 0.5893\underline{/85.52^\circ} \\ &= 229.8\underline{/80.34^\circ}, \\ \dot{C} &= \frac{1}{\dot{Z}_0} \sinh \dot{\theta} = \frac{0.5893\underline{/85.52^\circ}}{390\underline{/-5.175^\circ}} \\ &= 1{,}511 \times 10^{-6}\underline{/90.7^\circ}.\end{aligned}$$

The calculations for all the specified lengths of line are given in Table XII.

TABLE XII.—CALCULATION OF THE A, B, AND C CONSTANTS FOR A 20,000-VOLT TRANSMISSION LINE BY THE USE OF HYPERBOLIC FUNCTIONS

Example 97

Length in miles	100	200	300	400	500	600
$\theta_1 = l\alpha$	0.01893	0.03786	0.05679	0.07572	0.09465	0.11358
$\theta_2 = l\beta$	11.97°	23.94°	35.91°	47.88°	59.85°	71.82°
$\cosh \theta_1$	1.00016	1.0007	1.0016	1.0020	1.0045	1.0064
$\sinh \theta_1$	0.01868	0.03787	0.05682	0.07756	0.09347	0.11380
$\cos \theta_2$	0.9782	0.9140	0.8100	0.6708	0.5023	0.3120
$\sin \theta_2$	0.2074	0.4058	0.5865	0.7417	0.8648	0.9501
$\cosh \theta_1 \cos \theta_2$	0.9784	0.9146	0.8113	0.6727	0.5045	0.3140
$\sinh \theta_1 \sin \theta_2$	0.003926	0.01537	0.03333	0.05622	0.08197	0.1081
θ_A	0.23°	0.962°	2.35°	4.78°	9.23°	19.0°
A	0.9785	0.9148	0.8120	0.6908	0.5110	0.3321
$\dot{A} = \cosh \dot{\theta}$	$0.9785\underline{/0.23^\circ}$	$0.9148\underline{/0.962^\circ}$	$0.8120\underline{/2.35^\circ}$	$0.6908\underline{/4.78^\circ}$	$0.5110\underline{/9.23^\circ}$	$0.3321\underline{/19^\circ}$
θ_S	84.90°	85.13°	85.52°	86.09°	86.86°	87.87°
S	0.2083	0.4076	0.5893	0.7456	0.8699	0.9569
$\dot{S} = \sinh \dot{\theta}$	$0.2083\underline{/84.9^\circ}$	$0.4076\underline{/85.13^\circ}$	$0.5893\underline{/85.52^\circ}$	$0.7456\underline{/86.09^\circ}$	$0.8699\underline{/86.86^\circ}$	$0.9569\underline{/87.87^\circ}$
$\theta_{z_0} = \frac{\theta_z - \theta_y}{2}$	−5.175°	−5.175°	−5.175°	−5.175°	−5.175°	−5.175°
$\theta_B = \theta_S - \theta_{z_0}$	79.72°	79.96°	80.34°	80.92°	81.68°	82.69°
$\dot{B}$	$81.22\underline{/79.72^\circ}$	$158.96\underline{/79.96^\circ}$	$229.8\underline{/80.34^\circ}$	$290.8\underline{/80.92^\circ}$	$339.3\underline{/81.68^\circ}$	$373.2\underline{/82.69^\circ}$
$\theta_C = \theta_S - \theta_{z_0}$	90.08°	90.305°	90.695°	91.265°	92.035°	93.045°
$\dot{C} \times 10^6$	$534\underline{/90.08^\circ}$	$1{,}045\underline{/90.305^\circ}$	$1{,}511\underline{/90.695^\circ}$	$1{,}912\underline{/91.265^\circ}$	$2{,}231\underline{/92.035^\circ}$	$2{,}453\underline{/93.045^\circ}$
a_1	0.9784	0.9146	0.8113	0.6727	0.5045	0.3140
a_2	0.003926	0.01537	0.03333	0.05622	0.08197	0.1081
b_1	14.49	27.72	38.61	45.88	49.10	47.58
b_2	79.92	156.5	226.6	287.1	335.7	370.1
c_1	0.0	-5.6×10^{-6}	-18×10^{-6}	-42×10^{-6}	-79×10^{-6}	-120×10^{-6}
c_2	534×10^{-6}	$1{,}045 \times 10^{-6}$	$1{,}511 \times 10^{-6}$	$1{,}912 \times 10^{-6}$	$2{,}230 \times 10^{-6}$	$2{,}450 \times 10^{-6}$

163. "Equivalent-π" Solution.—The constants of the circuit for the nominal-π solution have been found to be

$$\dot{A} = 1 + \frac{Z'Y'}{2}, \tag{1088}$$

$$\dot{B} = \dot{Z}', \tag{1089}$$

and

$$\dot{C} = \dot{Y}'\left[1 + \frac{\dot{Z}'\dot{Y}'}{4}\right]. \tag{1094}$$

When these values are compared with the constants of the line, as given by Eqs. (1147) to (1149), it is seen that, where the charging admittance is not large, the difference in the performance characteristics, as given by the two solutions, will not be great, but for long lines the error due to concentrating the admittance at the two ends becomes large. The constants are similar in form and may be made equal by modifying the line constants, used in the π circuit. The true constants, as expressed in Eqs. (1147) to (1149), are

$$\dot{A} = \cosh \dot{\theta} = \cosh \sqrt{\dot{Z}\dot{Y}}, \tag{1147}$$

$$\dot{B} = \dot{Z}\frac{\sinh \dot{\theta}}{\dot{\theta}} = \dot{Z}\frac{\sinh \sqrt{\dot{Z}\dot{Y}}}{\sqrt{\dot{Z}\dot{Y}}}, \tag{1148}$$

and

$$\dot{C} = \dot{Y}\frac{\sinh \dot{\theta}}{\dot{\theta}} = \dot{Y}\frac{\sinh \sqrt{\dot{Z}\dot{Y}}}{\sqrt{\dot{Z}\dot{Y}}}. \tag{1149}$$

If, in the π circuit, instead of using the values of Y and Z, a modified, or corrected, line impedance Z' is used and a modified, or corrected, admittance $Y'/2$ is concentrated at each end of the line, the constants of this circuit may be made equal to those of the actual line. The constant B for the nominal-π line will be equal to the true constant B if

$$\dot{Z}' = \dot{Z}\frac{\sinh \dot{\theta}}{\dot{\theta}} = \dot{Z}\frac{\sinh \sqrt{\dot{Z}\dot{Y}}}{\sqrt{\dot{Z}\dot{Y}}}. \tag{1158}$$

The constant A for the nominal-π line will be equal to the true constant A if

$$1 + \frac{\dot{Y}'\dot{Z}'}{2} = \cosh \dot{\theta} = \cosh \sqrt{\dot{Z}\dot{Y}} \tag{1159}$$

and

$$\dot{Y}' = \frac{2(\cosh \dot{\theta} - 1)}{\dot{Z}'} = \frac{2(\cosh \dot{\theta} - 1)}{\dot{Z}\frac{\sinh \dot{\theta}}{\sqrt{\dot{Z}\dot{Y}}}}$$

$$= \frac{2}{\sqrt{\frac{\dot{Z}}{\dot{Y}}}} \frac{\cosh \dot{\theta} - 1}{\sinh \dot{\theta}}. \tag{1160}$$

Since

$$\tanh \frac{\dot{\theta}}{2} = \frac{\cosh \dot{\theta} - 1}{\sinh \dot{\theta}}, \tag{1161}$$

$$\dot{Y}' = 2\sqrt{\frac{\dot{Y}}{\dot{Z}}} \tanh \frac{\dot{\theta}}{2}$$

$$= \dot{Y}\frac{\tanh \frac{\dot{\theta}}{2}}{\frac{\sqrt{\dot{Z}\dot{Y}}}{2}}$$

$$= \dot{Y}\frac{\tanh \frac{\dot{\theta}}{2}}{\frac{\dot{\theta}}{2}}. \tag{1162}$$

If, therefore, the impedance of the line is taken as

$$\dot{Z}' = \dot{Z}\frac{\sinh \dot{\theta}}{\dot{\theta}}$$

and the charging admittance, concentrated at each end, is taken as

$$\frac{\dot{Y}'}{2} = \frac{\dot{Y}}{2} \times \frac{\tanh \frac{\dot{\theta}}{2}}{\frac{\dot{\theta}}{2}}, \tag{1163}$$

the π circuit gives results that are exactly equivalent to those given by the rigorous solution; such a modified or corrected circuit is known as the equivalent-π circuit. This circuit is equivalent, however, only in the sense that it accurately represents the behavior of the line at the terminals, as the results given for intermediate points will differ widely from those of the actual line.

The factors,

$$\frac{\sinh \dot{\theta}}{\dot{\theta}} \text{ and } \frac{\tanh \frac{\dot{\theta}}{2}}{\frac{\dot{\theta}}{2}},$$

are called "correcting factors," and may be calculated by the use of tables of hyperbolic functions. Dr. Kennelly, who first developed this solution, has published graphical charts from which the correcting factors may be determined readily with sufficient accuracy for most transmission-line calculations. When the corrected line constants Z' and Y' are determined, the solution is obtained exactly as for the nominal-π method. The work involved in determining the correction constants, without the aid of graphical charts, and applying the equations for the nominal-π solution is fully as great as in obtaining the true constants A, B, and C from tables of hyperbolic functions.

164. Equivalent-T Solution.—The correction factors to be applied to the line constants so that the equivalent-T solution will agree with the rigorous solution are the same as those used to convert the nominal-π into the equivalent-π circuit but in the reverse order. Thus, if the line impedance is taken as

$$\dot{Z}' = \dot{Z} \frac{\tanh \frac{\dot{\theta}}{2}}{\frac{\dot{\theta}}{2}}, \tag{1164}$$

and the line admittance,

$$\dot{Y}' = \dot{Y} \frac{\sinh \theta}{\dot{\theta}}, \tag{1165}$$

is concentrated at the center of the line, the circuit is equivalent to the actual line and the equations developed for the nominal-T solution may be applied.

Example 98

Calculate the current, voltage, power, and power factor at the sending end of the lines of Example 97 under open-circuit conditions when the voltage at the receiving end is 200,000 volts between lines for lines of 100, 200, 300, 400, 500, and 600 miles.

Taking the 300-mile line as an illustration, the calculations are as follows:

$$\dot{A} = 0.812\underline{/2.35^\circ} = 0.8113 + j0.0333,$$
$$B = 229.8\underline{/80.34^\circ},$$
$$\dot{C} = 1{,}511 \times 10^{-6}\underline{/90.7^\circ} = (-18 + j1{,}511) \times 10^{-6}.$$

Taking E_{RN} as the reference,

$$\dot{E}_{RN} = 115{,}740 + j0 \text{ volts.}$$

The open-circuit sending-end voltage is

$$\dot{E}_{SN0} = 115{,}740(0.8113 + j0.0333) \text{ volts}$$
$$= 93{,}680 + j3{,}849 \text{ volts,}$$

and

$$E_{SN0} = 93{,}750 \text{ volts.}$$

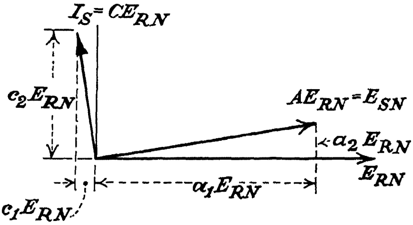

FIG. 286.—Open-circuit vector diagram for a transmission line, using the general circuit constants.

The open-circuit sending-end current is

$$\dot{I}_{S0} = 115{,}740(-18 + j1{,}511) \times 10^{-6} \text{ amp}$$
$$= -2.08 + j174.5 \text{ amp,}$$

and

$$I_{S0} = 174.5 \text{ amp.}$$
$$P_{SN0} = e_{SN0}i_{s0} + e_{SN0}'i_{S0}' \text{ watts}$$
$$= 93{,}680(-2.08) + 3{,}849 \times 174.5 \text{ watts}$$
$$= 477 \text{ kw.}$$

The vector power at the sending end is

$$V_{SN0} = E_{SN0}I_{S0} \text{ vector volt-amp}$$
$$= 93750 \times 174.5 = 16{,}360 \text{ vector kva.}$$

Then the power factor at the sending end at no load is

$$P.F._{S0} = \frac{477}{16{,}360} = 0.0292 = 2.92 \text{ per cent.}$$

The open-circuit vector diagram is shown in Fig. 286, and the calculations for the other lengths of line are given in Table XIII.

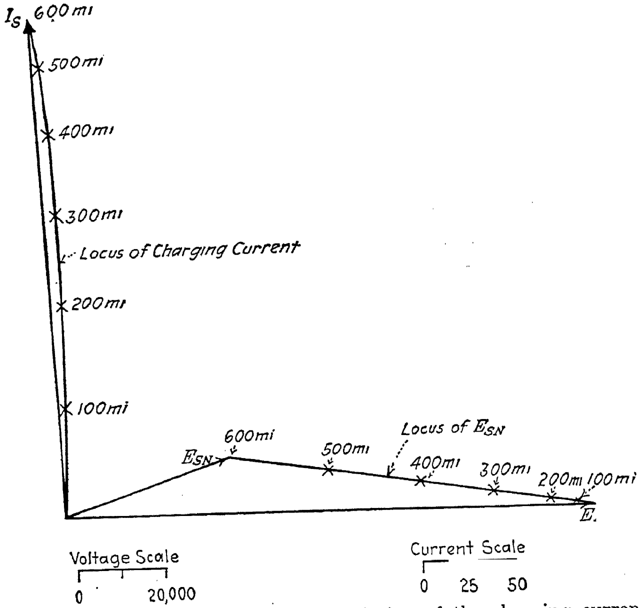

FIG. 287.—Polar diagram showing the variation of the charging current and no-load sending-end voltage with length of line, constant load voltage.

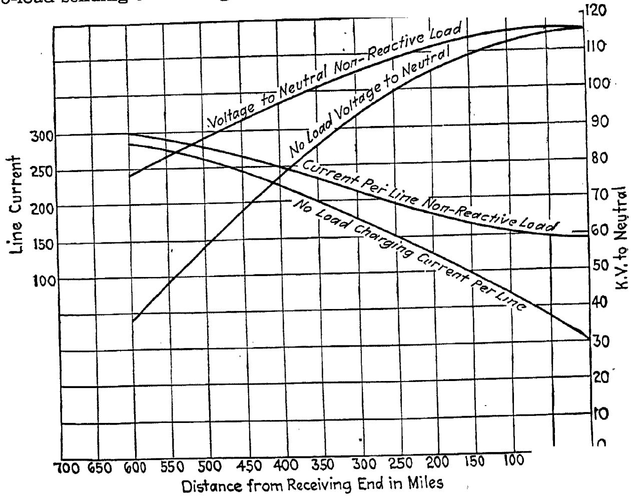

FIG. 288.—Curves showing the variation of line current and neutral voltage with distance from the load for open circuit and for nonreactive load.

TABLE XIII.—CALCULATION OF THE OPERATING CHARACTERISTICS OF A 200,000-VOLT TRANSMISSION LINE UNDER OPEN CIRCUIT

E_R = 200,000 volts, $P_{RN} = 0$
E_{RN} = 115,740 volts, $I_R = 0$

Example 98

Length, miles	100	200	300	400	500	600
a_1	0.9784	0.9146	0.8113	0.6727	0.5045	0.3140
a_2	0.003926	0.01537	0.03333	0.05622	0.08197	0.1081
b_1	14.49	27.82	38.61	45.88	49.10	47.58
b_2	79.92	156.5	226.6	287.1	335.7	370.1
c_1	0.0	-5.6×10^{-6}	-18×10^{-6}	-42×10^{-6}	-79×10^{-6}	-120×10^{-6}
c_2	534×10^{-6}	$1,045 \times 10^{-6}$	$1,511 \times 10^{-6}$	$1,912 \times 10_{-6}$	$2,230 \times 10^{-6}$	$2,450 \times 10^{-6}$
$e_{SN} = a_1E_{RN}$	112,980	105,610	93,680	77,680	58,250	36,260
$e'_{SN} = a_2E_{RN}$	453	1,775	3,849	6,491	9,465	12,480
$E_{SNO} = \sqrt{e^2_{SN} + e'^2_{SN}}$	112,980	105,620	93,750	77,930	59,010	38,350
$i_S = c_1E_{RN}$	0	−0.635	−2.08	−4.85	−9.12	−13.86
$i'S = c_2E_{RN}$	61.7	120.7	174.5	220.8	257.5	282.9
$I_{SO} = \sqrt{i^2_S + i'^2_S}$	61.7	120.7	174.5	220.8	257.7	283.1
$e_{SN}i_S \times 10^{-3}$	0	−67.0	−195.0	−377.0	−532.0	−502.0
$e'_{SN}i'_S \times 10^{-3}$	27.9	214.0	672.0	1,433	2,435	3,530
$P_{SN} = e_{SN}i_S + e'_{SN}i'_S$	27.9	147.0	477.0	1,056	1,903	3,027
$E_{SNO}I_{SO} \times 10^{-3}$	6,970	12,730	16,360	17,200	15,200	10,850
P. F.$_{SO}$, per cent	0.40	1.15	2.92	6.14	12.5	27.9

Figure 287 shows in polar form the magnitude of the neutral voltage and of the charging current and their phase position with reference to the receiving-end voltage at all points along the line at no load. In Fig. 288, the neutral voltage and charging current are shown plotted against the distance from the receiving end in miles.

Example 99

Calculate the operating characteristics of the lines of Example 97 for a nonreactive load of 50,000 kva at 200,000 volts. The calculations and results are tabulated in Table XIV.

The following calculations on the 300-mile line illustrate the method to be used:

$$\dot{E}_{RN} = 115{,}740 + j0 \text{ volts},$$

$$I_R = \frac{50{,}000}{200\sqrt{3}} = 144.3 \text{ amp, and}$$

$$\dot{I}_R = 144.3 + j0 \text{ amp.}$$

$$
\begin{aligned}
_{SN} &= 115{,}740(0.8113 + j0.03333) + 144.3(38.61 + j226.6) \text{ volts} \\
&= 99{,}250 + j36{,}549 \text{ volts, and} \\
_{SN} &= 105{,}700 \text{ volts.} \\
\dot{I}_S &= 144.3(0.8113 + j0.03333) + 115{,}740(-18 + j1{,}541) \times 10^{-6} \text{ amp} \\
&= 115 + j179.2 \text{ amp, and} \\
I_S &= 213.2 \text{ amp.} \\
_{SN} &= 99.250 \times 115 + 36.549 \times 179.2 \text{ kw} \\
&= 17{,}960 \text{ kw.}
\end{aligned}
$$

Efficiency $0 = 92.7$ per cent.

· volts

$$= \frac{105{,}700 \times 115{,}740}{93{,}750} = 130{,}200 \text{ volts.}$$

$$\text{Regulation} = \frac{130{,}200 - 115{,}740}{115{,}740}100 = 12.7 \text{ per cent.}$$

$$V_{SN} = 105.700 \times 213.2 = 22{,}520 \text{ vector kva.}$$

$0 = 79.7$ per cent

$$
\begin{aligned}
E_S &= E_{SN}\sqrt{3} = 183{,}000 \text{ volts,} \\
E_R' &= E_{RN}'\sqrt{3} = 225{,}100 \text{ volts.}
\end{aligned}
$$

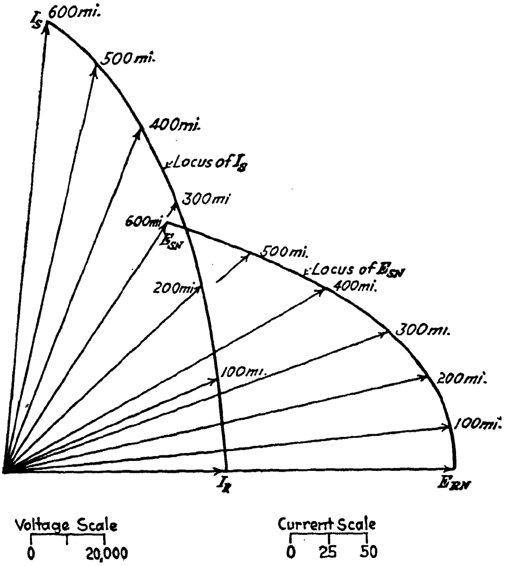

FIG. 289.—Polar diagram showing the variation of line current and neutral voltage with distance from the load for a nonreactive load.

The calculation for the other lengths of line are found in Table XIV

TABLE XIV.—CALCULATION OF THE OPERATING CHARACTERISTICS OF A 200,000-VOLT TRANSMISSION LINE UNDER NORMAL NONREACTIVE LOAD

Example 99

Length, miles	100	200	300	400	500	600
a_1	0.9784	0.9146	0.8113	0.6727	0.5045	0.3140
a_2	0.003926	0.01537	0.03333	0.05622	0.08197	0.10810
b_1	14.49	27.72	38.61	45.88	49.10	47.58
b_2	79.92	156.5	226.6	287.1	335.7	370.1
c_1	0	-5.6×10^{-6}	-18×10^{-6}	-42×10^{-6}	-79×10^{-6}	-120×10^{-6}
c_2	534×10^{-6}	$1{,}045 \times 10^{-6}$	$1{,}511 \times 10^{-6}$	$1{,}912 \times 10^{-6}$	$2{,}230 \times 10^{-6}$	$2{,}450 \times 10^{-6}$
a_1E_{RN}	112,980	105,610	93,680	77,680	58,250	36,260
a_2E_{RN}	453	1,775	3,849	6,491	9,465	12,480
b_1I_R	2,091	4,000	5,571	6,620	7,085	6,870
b_2I_R	11,530	22,580	32,700	41,430	48,440	53,410
$e_{SN} = a_1E_{RN} + b_1I_R$	115,070	109,610	99,250	84,300	65,335	43,130
$e'_{SN} = a_2E_{RN} + b_2I_R$	11,980	24,355	36,549	47,920	57,905	55,890
$E_{SN} = \sqrt{e^2_{SN} + e'^2_{SN}}$	115,700	111,700	105,700	97,080	87,300	78,800
a_1I_R	141.2	132.0	117.1	97.07	72.79	45.31
a_2I_R	0.57	2.22	4.72	8.13	11.84	15.62
c_1E_{RN}	0.0	−0.635	−2.08	−4.85	−9.12	−13.86
c^1E_{RN}	61.7	120.7	174.5	220.8	257.5	282.9
$i_S = aI_R + c_1E_{RN}$	141.2	131.4	115.0	92.22	63.67	31.45
$i'_S = a_2I_R + c_2E_{RN}$	62.27	122.9	179.2	228.9	269.3	298.5
$I_S = \sqrt{i^2_S + i'^2_S}$	154.2	179.7	213.2	246.5	276.5	300.5
$e_{SN}i_S \times 10^{-3}$	16,250	14,320	11,410	7,774	4,160	1,356
$e'_{SN}i'_S \times 10^{-3}$	746	2,994	6,550	10,970	15,590	19,670
$P_{SN} = e_{SN}i_S + e'_{SN}i'_S$	17,000	17,314	17,960	18,744	19,750	21,026
Effic. = $\frac{P_{RN}}{P_{SN}}$	98.0	96.3	92.7	88.9	84.4	78.3
$E'_{RN} = E_{RN}\frac{E_{SN}}{E_{SNO}}$	118,200	122,200	130,200	139,300	169,800	237,400
Regulation, per cent	3.36	5.82	12.7	20.6	47.0	105.5
$E_{SN}I_S \times 10^{-3}$	17,830	20,060	22,520	23,900	24,200	23,670
P. F.$_S$, per cent	95.5	86.3	79.7	78.4	81.6	88.8
$E_S = \sqrt{3}\, E_{SN}$	200,100	193,200	183,000	163,000	150,200	136,500
$E'_R = \sqrt{3}E'_{RN}$	204,600	211,500	225,100	241,000	295,500	412,000

Figure 289 shows in polar form the magnitude of the neutral voltage and of the line current and their phase position with respect to the receiving-end voltage at all points along the line. The neutral voltage and line current are plotted against the distance from the receiving end in Fig. 288.

TABLE XV.—CALCULATION OF THE OPERATING CHARACTERISTICS OF A 200,000-VOLT TRANSMISSION LINE UNDER NORMAL INDUCTIVE LOAD
Example 100

Length, miles	100	200	300	400	500	600
a_1	0.9784	0.9146	0.8113	0.6727	0.5045	0.3140
a_2	0.003926	0.01537	0.03333	0.05622	0.08197	0.10810
b_1	14.49	27.72	38.61	45.88	49.10	47.58
b_2	79.92	156.5	226.6	287.1	335.7	370.1
$c_1 \times 10^{-6}$	0	−5.6	−18	−42	−79	−120
$c_2 \times 10^{-6}$	534	1,045	1,511	1,912	2,230	2,450
a_1E_{RN}	112,980	105,610	93,680	77,680	58,250	36,260
a_2E_{RN}	453	1,775	3,849	6,491	9,465	12,480
b_1i_R	1,673	3,200	4,460	5,296	5,670	5,480
$b_2i'_R$	−6,920	−13,550	−19,620	−24,860	−29,065	−32,040
$e_{SN} = a_1E_{RN} + b_1i_R - b_2i'_R$	121,570	122,360	117,760	107,836	92,980	73,780
$b_1i'_R$	−1,255	−2,400	−3,340	−3,970	−4,250	−4,110
b_2i_R	9,226	18,070	26,160	33,140	38,750	42,720
$e'_{SN} = a_2E_{RN} + b_1i'_R + b_2i_R$	8,400	17,450	26,670	35,660	43,970	51,090
$E_{SN} = \sqrt{e^2_{SN} + e'^2_{SN}}$	121,800	123,600	120,500	113,500	102,800	89,700
c_1E_{RN}	0	−0.635	−2.08	−4.85	−9.12	−13.86
a_1i_R	112.9	105.6	93.65	77.66	58.24	36.25
$a_2i'_R$	−0.34	−1.33	−2.89	−4.87	−7.10	−9.37
$i_S = c_1E_{RN} + a_1i_R - a_2i'_R$	113.2	106.3	94.46	77.68	56.22	31.76
c_2E_{RN}	61.7	120.7	174.5	220.8	257.5	282.9
$a_1i'_R$	−84.71	−79.18	−70.24	−58.24	−43.68	−27.2
a_2i_R	0.45	1.77	3.84	6.48	9.46	12.5
$i'_S = c_2E_{RN} + a_1i'_R + a_2i_R$	−22.6	43.3	108.1	169.0	223.3	268.8
$I_S = \sqrt{i^2_S + i'^2_S}$	115.4	114.7	143.5	186.2	230.4	270.5
$e_{SN}i_S \times 10^{-3}$	13,760	13,020	11,110	8,370	5,230	2,340
$e'_{SN}i'_S \times 10^{-3}$	−190	750	2,860	6,030	9,820	13,740
$P_{SN} = e_{SN}i_S + e'_{SN}i'_S$	13,570	13,770	13,970	14,400	15,050	16,080
Effic. $= \frac{P_{RN}}{P_{SN}}$	98.3	96.8	95.5	92.6	88.6	82.9
$E'_{RN} = E_{RN}\frac{E_{SN}}{E_{SNO}}$	124,300	135,100	148,300	168,000	201,100	270,100
Per cent reg. $= \frac{E'_{RN} - E_{RN}}{E_{RN}}$	7.62	17.0	28.4	45.5	73.2	133.3
$E_{SN}I_S \times 10^{-3}$	14,050	14,160	17,300	21,130	23,650	24,260
P. F.$_S$, per cent	96.5	97.3	80.6	68.2	63.6	66.2
$E_S = \sqrt{3}\,E_{SN}$	211,000	214,000	208,900	196,500	178,000	155,500
$E'_R = \sqrt{3}\,E'_{RN}$	215,000	234,000	256,700	291,000	348,000	467,000

Example 100

Calculate the operating characteristics of the line of Example 97 for an inductive load of 50,000 kva at 0.8 power factor with 200,000 volts at the receiving end. The calculations and results are tabulated in Table XV.

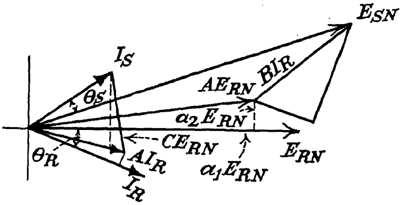

FIG. 290.—Vector diagram for a transmission line with inductive load using the general circuit constants.

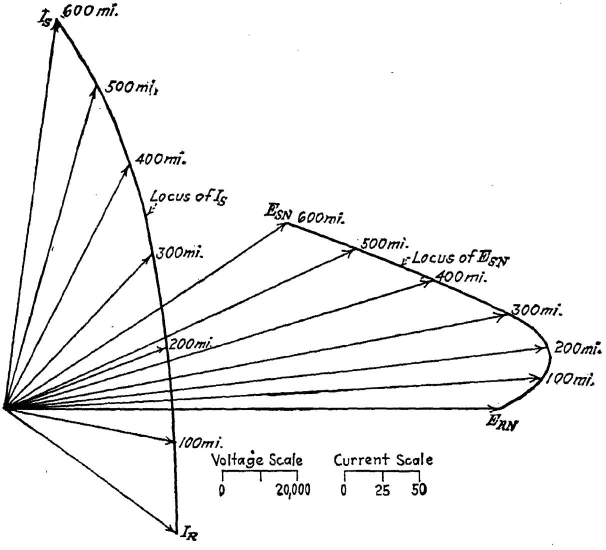

FIG. 291.—Polar diagram showing the variation of line current and neutral voltage with distance from the load for an inductive load.

Figure 290 is the vector diagram, Fig. 291 is the polar diagram, and Fig. 292 the plot of current and voltage distribution for an inductive load.

Example 101

Calculate the operating characteristics of the lines of Example 97 for a capacitive load of 50,000 kva at 0.8 power factor with 200,000 volts at the receiving end.

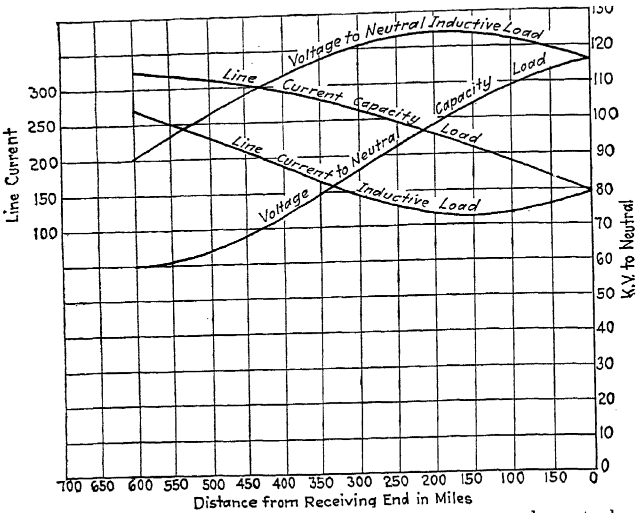

FIG. 292.—Curves showing the variation of line current and neutral voltage with distance from the load for inductive and capacitive loads.

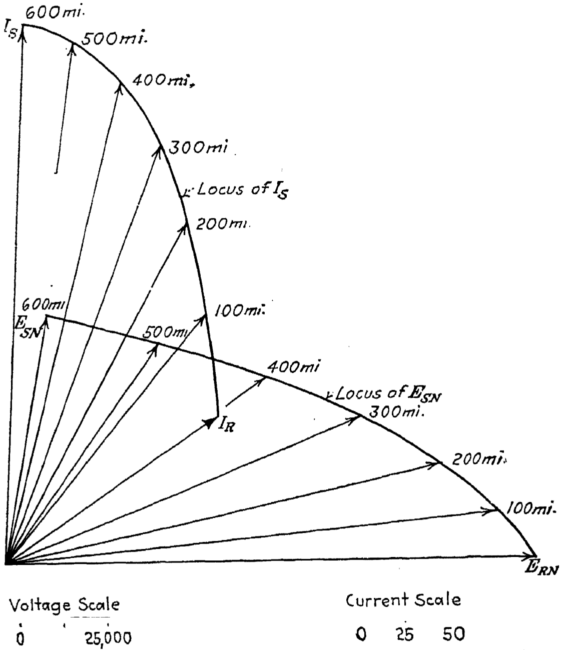

FIG. 293.—Polar diagram showing the variation of line current and neutral voltages with distance from the load for capacitive load.

The calculations and results are tabulated in Table XVI. Figure 293 is the polar diagram, and Fig. 292 shows the plot of current and voltage distribution for this load.

TABLE XVI.—CALCULATION OF THE OPERATING CHARACTERISTICS OF A 200,000-VOLT TRANSMISSION LINE UNDER NORMAL CAPACITIVE LOAD
Example 101

Length, miles	100	200	300	400	500	600
a_1	0.9784	0.9146	0.8113	0.6727	0.5045	0.3140
a_2	0.003926	0.01537	0.03333	0.05622	0.08197	0.10810
b_1	14.49	27.72	38.61	45.88	49.10	47.58
b_2	79.92	156.5	226.6	287.1	335.7	370.1
$c_1 \times 10^{-6}$	0.0	−5.6	−18	−42	−79	−120
$c_2 \times 10^{-6}$	534	1,045	1,511	1,912	2,230	2,450
a_1E_{RN}	112,980	105,610	93,680	77,680	58,250	36,260
a_2E_{RN}	453	1,775	3,849	6,491	9,465	12,480
b_1i_R	1,673	3,200	4,460	5,296	5,670	5,480
$b_2i'_R$	6,920	13,550	19,620	24,860	29,065	32,040
$e_{SN} = a_1E_{RN} + b_1i_R - b_2i'_R$	107,730	95,260	78,520	58,116	34,860	9,700
$b_1i'_R$	1,255	2,400	3,340	3,970	4,250	4,110
b_2i_R	9,226	18,070	26,160	33,140	38,750	42,720
$e'_{SN} = a_2E_{RN} + b_1i'_R + b_2i_R$	10,930	22,245	33,350	43,600	52,470	59,310
$E_{SN} = \sqrt{e^2_{SN} + e'^2_{SN}}$	108,200	97,700	85,200	72,600	63,000	60,200
c_1E_{RN}	0.0	−0.635	−2.08	−4.85	−9.12	−13.86
a_1i_R	112.9	105.6	93.65	77.66	58.24	36.25
$a_2i'_R$	0.34	1.33	2.89	4.87	7.10	9.37
$i_S = c_1E_{RN} + a_1i_R - a_2i'_R$	112.6	103.7	88.7	67.9	42.0	13.0
c_2E_{RN}	61.7	120.7	174.5	220.8	257.5	282.9
$a_1i'_R$	84.71	79.18	70.24	58.24	43.68	27.2
a_2i_R	0.45	1.77	3.84	6.48	9.46	12.5
$i'_S = c_2E_{RN} + a_1i'_R + a_2i_R$	146.9	201.6	248.6	285.5	310.6	322.6
$I_S = \sqrt{i^2_S + i'^2_S}$	185	227	264	293.5	312.0	323.0
$e_{SN}i_S \times 10^{-3}$	12,120	9,880	6,970	3,950	1,460	130
$e'_{SN}i'_S \times 10^{-3}$	1,610	4,480	8,290	12,450	16,300	19,130
$P_{SN} = e_{SN}i_S + e'_{SN}i'_S$	13,730	14,360	15,260	16,400	17,760	19,260
Effic. $= \dfrac{P_{RN}}{P_{SN}}$	97.3	92.9	82.4	81.3	75.0	69.2
$E'_{RN} = E_{RN}\dfrac{E_{SN}}{E_{SNO}}$	110,500	106,800	105,000	107,500	123,200	181,100
Regulation, per cent	−4.31	−7.5	−9.07	−6.90	6.7	56.8
$E_SI_S \times 10^{-3}$	20,000	22,190	22,500	21,300	19,670	19,450
P. F.$_S$, per cent	68.6	64.6	67.7	77.0	90.2	99.0
$E_S = \sqrt{3}\, E_{SN}$	187,200	169,100	147,500	125,800	109,100	104,300
$E'_R = \sqrt{3}\, E'_{RN}$	191,500	184,800	182,000	186,200	213,000	314,000

Problems

1-15. Calculate the complex angle $\dot{v}$ per mile of line and the surge impedance for a 250,000-volt 60-cycle transmission line constructed of 397,500-circular-mil aluminum cable, steel reinforced, having the following constants per mile of conductor: resistance, 0.235 ohm; reactance, 0.788 ohm; and capacitive susceptance to neutral, 5.24×10^{-6} mho.

2-15. Calculate the complex angle $\dot{v} = \sqrt{\dot{z}\dot{y}}$ per mile of line and the surge impedance for a 220,000-volt three-phase line that is constructed of 300,000 circular-mil copper conductors and has the following values: r per mile of conductor $= 0.196$ ohm; x per mile of conductor $= 0.83$ ohm; g per mile of conductor $= 0$; and b per mile of conductor $= 5.22 \times 10^{-6}$ mho.

3-15. A transmission line is constructed of No. 0 B. & S. stranded copper conductors having the following values per mile of conductor: resistance, 0.540 ohm; inductance, 2.32 millihenrys; and capacitance to neutral, 0.013×10^{-6} farad. Calculate the complex angle $\dot{v}$ per mile and the surge impedance for frequencies of 25, 60, 180, and 900 cycles.

4-15. Calculate the complex hyperbolic angle θ and the values of sinh $\dot{\theta}$ and cosh $\dot{\theta}$ for a 250-mile line and a 500-mile line having the following constants per mile of conductor: $r = 0.154$ ohm; $x = 0.766$ ohm; and $b = 5.38 \times 10^{-6}$ mho.

5-15. A circuit constructed of No. 10 B. & S. copper conductors has the following constants: $R = 102.8$ ohms; $L = 0.0372$ henry; and $C = 0.0787 \times 10^{-6}$ farad. Calculate the complex angle θ and the surge impedance for this circuit for a frequency of 1,100 cycles.

6-15. Evaluate the factors cosh $\dot{\theta}$ and sinh $\dot{\theta}/\dot{\theta}$ for $\dot{\theta} = 0.40\underline{/50^\circ}$.

7-15. Calculate the auxiliary constants of the line of Prob. 2-15, using the hyperbolic function solution. The total line is 200 miles long.

8-15. Calculate the auxiliary constants for a transmission line 300 miles long with the following constants per mile of conductor: $r = 0.147$ ohm; $x = 0.733$ ohm; $g = 0$; and $b = 6 \times 10^{-6}$ mho.

9-15. The transmission line of Prob. 8-15 is operated at 220,000 volts and has a load of 60,000 kva at the receiving end. Calculate the efficiency, regulation, power factor at the sending end, and sending-end current for loads of unity power factor and for 0.70 power factor inductive and capacitive. The circuit constants are $\dot{A} = 0.820 + j0.0228$, $\dot{B} = 26.3 + j214$ and $\dot{C} = -0.000013 + j0.001527$. Draw the vector diagrams.

10-15. Calculate the auxiliary constants for a 500-mile line having the line constants given in Prob. 4-15.

11-15. Calculate the efficiency and regulation of the line of Probs. 4-15 and 10-15 for a load of 50,000 kva at 0.85 power factor inductive at 240,000 volts. The circuit constants are $\dot{A} = 0.529 + j0.0854$, $\dot{B} = 52.3 + j322$, and $\dot{C} = -0.000083 + j0.00223$. Draw the vector diagram.

12-15. Calculate the voltage and current at each 100 miles from the receiving end of the line for open circuit and for a load of 60,000 kva at 0.80 power factor inductive and 250,000 volts for the line of Probs. 4-15 and 10-15.

13-15. A 220,000-volt three-phase line carrying a load of 40,000 kva has the following auxiliary constants: $\dot{A} = 0.932 + j0.00791$, $\dot{B} = 16.1 + j140$, $\dot{C} = (-3 + j941) \times 10^{-6}$. Calculate the values of the voltage and current at the sending end, regulation, and efficiency for this line with loads of unity power factor and 0.80 power factor lead and lag.

14-15. Taking the line of Example 97, using the values as given in Table XII, calculate the voltage and current at each 100 miles for a 600-mile line with the line short-circuited and 220,000 volts applied at the sending end. Plot these values against distance.

15-15. Taking the line of Example 97, using the values given in Table XII, calculate the voltage and current at each 100 miles for a 600-mile line with the line open-circuited at the receiving end, the receiver voltage being 250,000 volts. Plot these values against distance.

16-15. A No. 10 B. & S. copper open-wire telephone line has the following constants per mile: $r = 10.42$ ohms per loop mile; $L = 3.71$ millihenrys per loop mile; $C = 0.00819$ μf per mile; and $g = 0$. Find v and Z_0 for frequencies of 100 cycles and 1,000 cycles.

17-15. (*a*) Calculate the values of I_x for each 50 miles of a 200-mile line, the constants of which are given in Prob. 16-15 for a frequency of 1,000 cycles with $E_s = 20$ volts and for the line short-circuited, *i.e.*, the load impedance $Z_R = 0$. Plot these values against distance. (*b*) Calculate the values of E_x for each 50 miles of the 200-mile line of (*a*) except that the line is open-circuited, *i.e.*, the load impedance $Z_R = \infty$. Plot these values against distance.

18-15. Calculate the values of E_x and I_x for each 50 miles of the 200-mile line of Prob. 16-15 for the load impedance equal to the surge impedance of the line, $\dot{Z}_R = \dot{E}_R/\dot{I}_R = \dot{Z}_0$, and $E_s = 20$ volts. Plot these values against distance.

19-15. Calculate the equivalent-π and equivalent-T auxiliary constants for the line of Prob. 8-15.

CHAPTER XVI

NONSINUSOIDAL WAVE FORMS

165. Generation of Waves.—In Chap. I, Faraday's law applied to a conductor rotating at a uniform velocity in a uniform magnetic field yielded a sinusoidal form of emf wave. In the other chapters of this text it has been found that, when a sinusoidal wave of emf is impressed upon a circuit containing constant values of resistance, inductance, and capacitance in series or parallel or any combinations of these constants, sinusoidal waves of current and emf drop also result in each of the branches of the circuit. From this foundation and the mathematics of the vector, the vector diagrams representing the phenomena in a-c circuits have been built up. If a nonuniform field is considered, it will be found that the induced emf will not be sinusoidal. Sinusoidal waves of emf impressed on circuits with iron cores will produce distorted current waves; and load conditions in a-c machines connected to circuits will also cause distortions in the emf and current waves. It is the purpose of this chapter to develop the equations for the phenomena of a-c circuits from the point of view of distorted waves and to analyze such waves for their component harmonics.

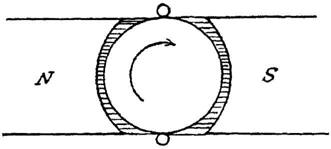

FIG. 294.—A coil rotating in a magnetic field having a greater density at the center of the poles than at the tips.

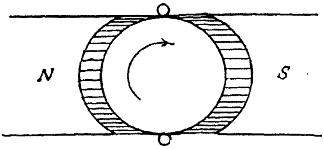

FIG. 295.—A coil rotating in a magnetic field having a greater density at the tips than at the center of the poles.

166. Distorted Flux Distribution in Generators.—If the flux distribution in the air gap of an a-c generator is as shown in Figs. 294 and 295 instead of that in Fig. 1, the emf wave induced in the coil AB will not be a sine wave. In Fig. 1, the flux distribution is uniform; in Fig. 294, it is more dense in the center of the pole face than at the tips; in Fig. 295, it is more dense at the tips than in the center; armature reactions may cause the flux to be more dense at one pole tip than at the other or than at the center of the pole. Nearly any desired flux distribution may be produced in an a-c generator by properly shaping the air gap of the machine. A study may be made of the emf wave induced in the armature coils with any wave shape of flux by the application of the methods discussed in Chap. I. It will be found that the wave form of the emf induced will differ from that of the flux linked with the coil for all wave shapes except sinusoidal waves; for example, a flat-topped wave of flux enclosed corresponds to a peaked wave of emf.

With any flux distribution the frequency of the generated emf is independent of the wave shape, being determined by the speed and the number of poles on the generator, as explained in Chap. I. The positive and negative lobes of the emf waves will be symmetrical, *i.e.*, any ordinate in the positive lobe will have an ordinate of equal value but of opposite sign at 180 deg. or π radians distant along the time axis. Owing to conditions in the circuit outside the machine, however, waves of current and emf will be found to exist occasionally with unsymmetrical lobes, but all waves will have only one value of ordinate for each value of angle or time.

167. Fourier's Series or Equation.—It is often desirable to use a general mathematical expression for a wave of any shape that gives the value of the ordinate corresponding to any instant of time. It has been found by mathematical derivation, for recurring waves having single-valued functions of y, that the wave may be represented by Fourier's equation in the form of a series, as

$$y = A + A_1 \sin \alpha + A_2 \sin 2\alpha + A_3 \sin 3\alpha + \\ \quad . \; . \; . + A_n \sin n\alpha + B_1 \cos \alpha + B_2 \cos 2\alpha \\ + B_3 \cos 3\alpha + \; . \; . \; . \; B_n \cos n\alpha, \quad (1166)$$

where A is a constant term added to all ordinates;

A_1 and B_1 are maxima of the fundamental waves, *i.e.*, waves of the lowest order or frequency;

A_2, B_2 . . . A_n, B_n are the maxima of the harmonics;

and α is the angle in degrees or radians measured from a common origin. For convenience in use, and for reference to other waves Eq. (1166) may be written

$$y = A + C_1 \sin(\alpha + \theta_1) + C_2 \sin(2\alpha + \theta_2) + C_3 \sin(3\alpha + \theta_3) + \dots + C_n \sin(n\alpha + \theta_n) \quad (1167)$$

where

$$C_1 = \sqrt{A_1^2 + B_1^2}$$
$$\dots\dots\dots\dots$$
$$C_n = \sqrt{A_n^2 + B_n^2}, \quad (1168)$$

and

$$\tan\theta_1 = \frac{B_1}{A_1}$$
$$\dots\dots$$
$$\tan\theta_n = \frac{B_n}{A_n}, \quad (1169)$$

where $C_1, C_2 \dots C_n$ are the maxima of the component waves and $\theta_1, \theta_2 \dots \theta_n$ are the angular offsets of the points of crossing from negative to positive values of the component waves from the arbitrarily chosen origin. In most a-c waves, the value of A is zero and the axis of symmetry of the wave corresponds with the reference axis of angle or time. It will be found, also, that most waves have only odd harmonics, as will be demonstrated later. The methods of analysis of waves by Fourier's series will be discussed later in this chapter.

Harmonic Component.—A harmonic component of a periodic quantity is any one of the simple sinusoidal quantities of the Fourier series into which the periodic quantity may be resolved.

Fundamental.—The fundamental of a periodic quantity is the component having the lowest frequency.

Harmonics.—The harmonics of a periodic quantity are all the harmonic components except the fundamental.

Pulsating Quantity.—A pulsating quantity is a periodic quantity of which the values are always positive (or negative). The expression for a pulsating quantity is the same as for a periodic quantity, but with the limitation that the constant term A_0 is numerically so large that y does not change sign.

Ripple Quantity.—A ripple quantity is a pulsating quantity in which the constant term is large relative to the change in amplitude during a cycle.

168. Plotting Waves.—Sometimes it becomes desirable to plot a wave corresponding to a given equation. It is usually best, for purposes of illustration, and in the study of such waves, to plot, first, the individual harmonics to the same scale of time and of ordinates and referred to the same origin and axis, and then to add the ordinates of the individual harmonics to obtain the ordinates of the required wave. In making this study, some of the following points should be noticed in regard to the waves.

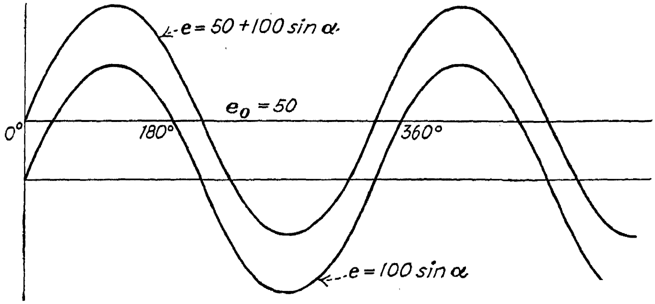

FIG. 296.—The component waves representing the equation $e = 50 + 100 \sin \alpha$.

a. The Constant Term.—To study the effect of the constant term in Eqs. (1166) and (1167), plot the wave,

$$e = 50 + 100 \sin \omega t = 50 + 100 \sin \alpha.$$

In this wave, $e = 50$ when $100 \sin \alpha = 0$, or when $\alpha = 0$, 180, 360 deg., etc., or each 180 deg. When $\alpha = 90$ deg.,

$$100 \sin \alpha = 100,$$

and

$$e = 50 + 100 = 150.$$

It will also have this value of 150 at 450 deg. and at each 360-deg. interval from $\alpha = 90$ deg.

When

$$\alpha = 270°,$$
$$100 \sin \alpha = -100,$$

and

$$e = 50 - 150 = -50.$$

It will also have this value at each 360-deg. interval from $\alpha =$ 270 deg. In Fig. 296, two component waves

$$e_1 = 100 \sin \alpha \text{ and}$$
$$e_o = 50,$$

are plotted, as well as the total wave. The component wave e_1 is symmetrical about the time axis. The component wave $e_o = 50$ is a straight horizontal line parallel to the time or angle axis and 50 units above the axis. Hence,

$$\begin{aligned} e &= e_o + e_1 \\ &= 50 + 100 \sin \alpha \end{aligned}$$

is represented by a sine wave with its axis of symmetry displaced 50 volts from the reference axis. The constant term $e_o = 50$ is the distance between the axis of symmetry of the wave,

$$e = 50 + 100 \sin \alpha,$$

and the reference axis.

b. Harmonics.—The study of the effect of harmonics may be made by plotting representative waves. In this connection, plot the harmonic components of the wave,

$$e = C_1 \sin (\alpha + \theta_1) + C_n \sin (n\alpha + \theta_n). \qquad (1170)$$

First plot the fundamental wave,

$$e_1 = C_1 \sin (\alpha + \theta_1). \qquad (1171)$$

The value of C_1 is the maximum ordinate of this wave. This fundamental wave, or wave of lowest frequency, crosses the axis of reference, passing from negative to positive values, when

$$\sin (\alpha + \theta_1) = 0$$

or when

$$(\alpha + \theta_1) = 0°,$$

and

$$\alpha = -\theta_1,$$

i.e., θ_1 deg. ahead of the origin. It will again cross the axis when

$$\alpha + \theta_1 = 180°,$$

or when

$$\alpha = 180° - \theta_1$$

180 deg. beyond the first crossing point. There will be a crossing point at each 180 deg. from the first crossing point. The wave has a positive maximum when

$$\sin (\alpha + \theta_1) = 1$$

or when

$$(\alpha + \theta_1) = 90°,$$

and

$$\alpha = 90° - \theta_1.$$

and each 360 deg. from this angle, the value of the ordinate at this angle being $e_1 = C_1$. The maximum negative value will occur when

$$\sin (\alpha + \theta_1) = -1,$$

or when

$$\alpha + \theta_1 = 270°,$$

and

$$\alpha = 270° - \theta_1,$$

and $e_1 = -C_1$ at this instant and at each succeeding 360-deg. interval. Plotting a sine wave of the proper scale through these points will locate the fundamental wave.

The nth harmonic may be located in a similar manner. Since the angle for the nth harmonic referred to the scale of the fundamental is $(n\alpha + \theta_n)$, this wave crosses the axis when

$$\sin (n\alpha + \theta_n) = 0$$

or when

$$n\alpha + \theta_n = 0°,$$

and

$$\alpha = -\frac{\theta_n}{n},$$

and again when

$$n\alpha + \theta_n = 180°,$$

or

$$\alpha = \frac{180° - \theta_n}{n},$$

or

$$\alpha = \frac{180°}{n} - \frac{\theta_n}{n}.$$

Hence, the nth harmonic crosses the axis θ/n deg. ahead of the origin and at every $180/n$ deg. of the fundamental wave scale. *The angle θ_n refers to degrees as measured on the scale of the nth harmonic, its period being $1/n$ times the period of the fundamental wave. Consequently, this angle must be divided by the order of the harmonic n to measure the offset on the scale of the fundamental.*

Example 102

Plot the component waves and the total wave for the equation

$$e = 100 \sin \alpha + 50 \sin (3\alpha - 45°).$$

The fundamental of the wave crosses the axis from negative to positive values when $\sin \alpha$ is equal to zero, *i.e.*, at 0 deg. and each succeeding 360 deg., or 0 deg., 360 deg., 720 deg. . . . The fundamental also crosses the axis from positive to negative values at 180 deg. and at each succeeding 360-deg. interval. It has a positive maximum at 90 deg. and at each succeeding 360 deg. and a negative maximum at 270 deg., 540 deg. . . .

If the third harmonic wave were to be plotted on its own time scale, *i.e.*, as

$$e_3 = 50 \sin (\beta - 45°),$$

the crossing point of this wave would be at the point at which

$$\sin (\beta - 45°) = 0,$$

i.e., where $\beta = 45°$. However, in plotting it to the scale of the fundamental, β is equal to 3α, and the wave for e_3 crosses the axis when

$$50 \sin (3\alpha - 45°) = 0$$

which crosses the fundamental axis when

$$3\alpha = 45° \text{ or}$$
$$\alpha = \frac{45°}{3} = 15°.$$

Since the period of this wave is $1/n$ or $\frac{1}{3}$ that of the fundamental, it must have a crossing point from negative to positive values at each succeeding interval of 360°/3 or 120 deg. The points on the total wave at the instant when either of the component waves passes through zero may be plotted by inspection, since it is that value of the component wave which is not passing through zero. Points between these values as needed may be found graphically by adding the instantaneous values of the component waves on the graph or by substitution of functions of the angles into the equation of the wave. The wave forms of this equation are shown in Fig. 297.

Effect of Even Harmonics.—To study the effect of even harmonics on the shape of the resultant wave, try the effect of adding a second harmonic to a fundamental at various relative phase angles. The wave

$$e = 100 \sin \alpha + 25 \sin (2\alpha + \theta_2) \tag{1172}$$

is plotted for various values of θ_2 in Figs. 298 to 301. In Fig. 298, $\theta_2 = 0$ deg.; in Fig. 299, $\theta_2 = 180$ deg.; in Fig. 300, $\theta_2 = 90$ deg.; and in Fig. 301, $\theta_2 = -90$ deg. In each of these cases, the positive and negative lobes of the resultant wave are not symmetrical. In Figs. 298 and 299, the lobes may at first appear symmetrical, but, upon application of the rule of symmetry, it is found that all ordinates in the positive lobe do not correspond to equal ordinates 180 deg. distant in the negative lobe. In Figs. 300 and 301, one lobe is flat and the next is peaked. In fact, waves cannot be symmetrical and contain even harmonics; hence, even harmonics may not exist in emf waves generated by a-c generators.

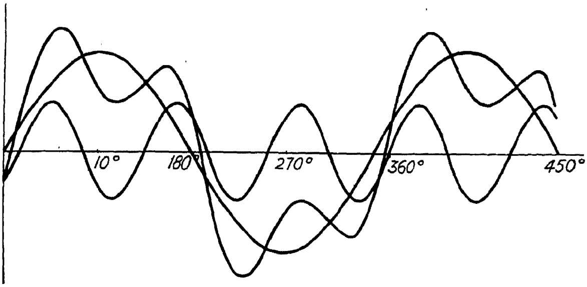

FIG. 297.—Wave form of emf for Example 102.

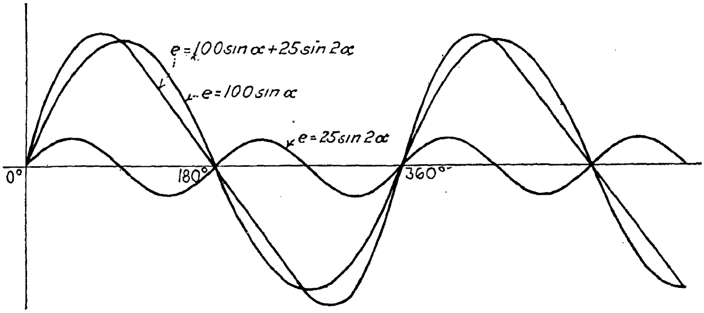

FIG. 298.—The component waves representing the equation $e = 100 \sin \alpha + 25 \sin 2\alpha$.

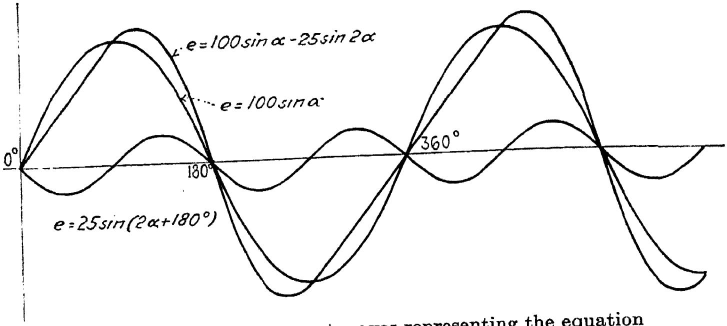

FIG. 299.—The component waves representing the equation $e = 100 \sin \alpha - 25 \sin 2\alpha$.

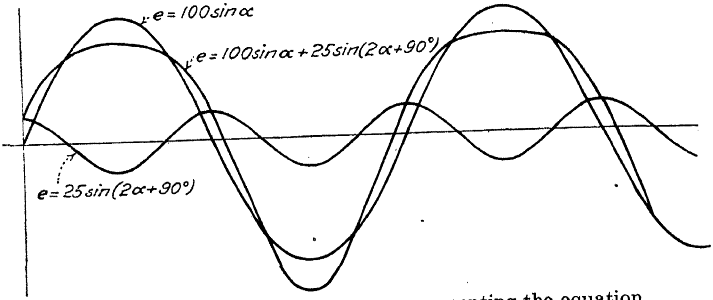

FIG. 300.—The component waves representing the equation $e = 100 \sin \alpha + 25 \sin (2\alpha + 90°)$.

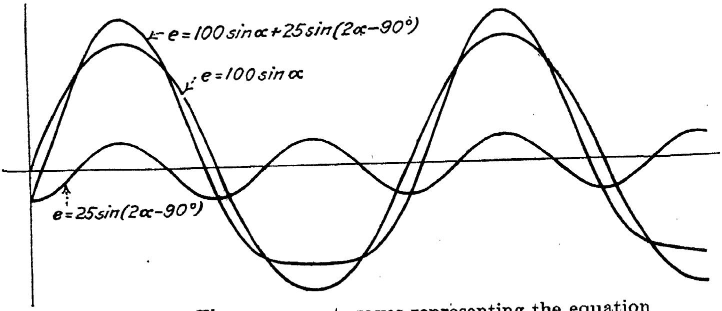

FIG. 301.—The component waves representing the equation $e = 100 \sin \alpha + 25 \sin (2\alpha - 90°)$.

d. Odd Harmonics.—A fundamental and a triple in a wave will serve to illustrate the general effect of adding odd harmonics at different phase angles. The wave,

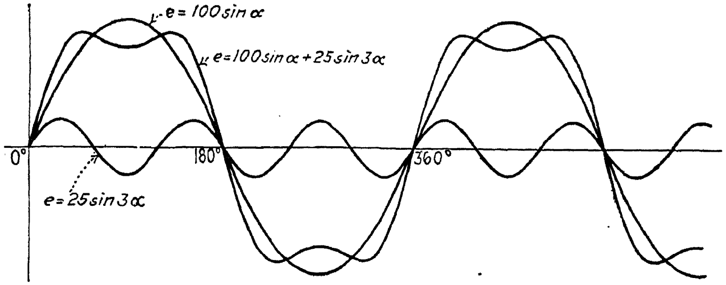

FIG. 302.—The component waves representing the equation $e = 100 \sin \alpha + 25 \sin 3\alpha$.

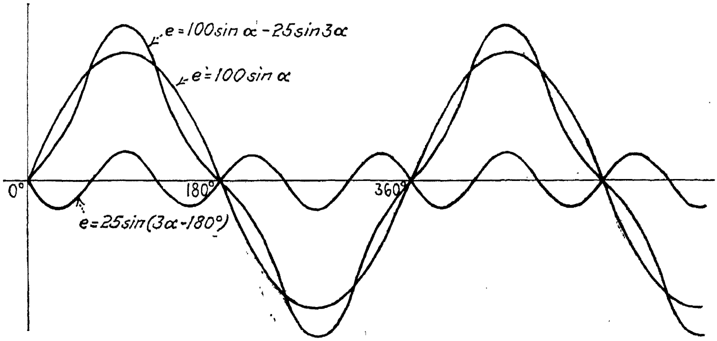

FIG. 303.—The component waves representing the equation $e = 100 \sin \alpha - 25 \sin 3\alpha$.

$$e = 100 \sin \alpha + 25 \sin (3\alpha + \theta_3), \tag{1173}$$

is plotted for various values of θ_3 in Figs. 302 to 305. In Fig. 302, $\theta_3 = 0$ deg. and

$$e = 100 \sin \alpha + 25 \sin 3\alpha. \tag{1174}$$

In Fig. 303, $\theta_3 = 180$ deg. and

$$\begin{aligned} e &= 100 \sin \alpha + 25 \sin (3\alpha + 180°) \\ &= 100 \sin \alpha - 25 \sin 3\alpha. \end{aligned} \tag{1175}$$

When θ_3 is zero, flat-topped lobes are produced, but when θ_3 is 180 deg., peaked lobes are produced in the resultant wave.

The two lobes of the resultant wave are symmetrical and of equal areas for all values of phase angle. For $\theta_3 = 90$ deg., illustrated in Fig. 304, the two lobes are symmetrical but distorted by shifting the peak toward the right. For $\theta_3 = -90$ deg., or

$$e = 100 \sin \alpha + 25 \sin (3\alpha - 90^\circ) \tag{1176}$$

illustrated in Fig. 305, the lobes are symmetrical, but with the peaks shifted toward the left. It will be found, by plotting different waves, that odd harmonics produce symmetrical waves; hence, ordinary emf and current waves from electrical apparatus may contain odd harmonics. The following study will be confined, almost entirely, to waves with odd harmonics; however, most of the equations and general theorems apply equally well to waves with even harmonics.

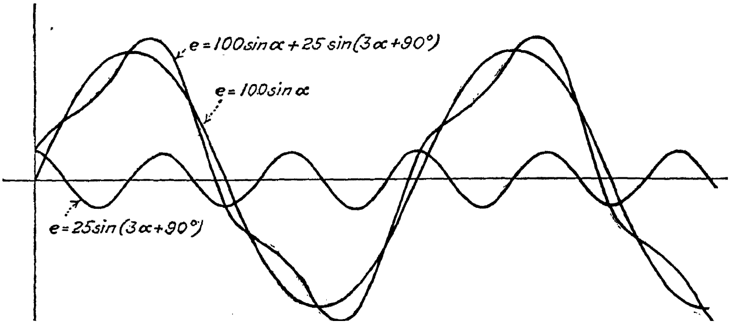

FIG. 304.—The component waves representing the equation $e = 100 \sin \alpha + 25 \sin (3\alpha + 90^\circ)^2$.

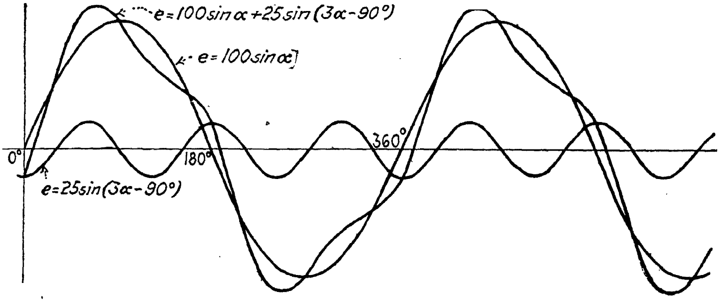

FIG. 305.—The component waves representing the equation $e = 100 \sin \alpha + 25 \sin (3\alpha - 90^\circ)$.

By inspection of a given plotted wave, it may be estimated whether there is a constant term and whether there is any appreciable even harmonic. Experience in plotting waves will teach one to estimate whether a wave has triple or fifth harmonics of appreciable magnitude. Alternator slots will sometimes produce higher odd harmonics of small magnitude in the emf waves, appearing as serrated projections and depressions on the fundamental emf, Fig. 306 representing an eleventh harmonic of small magnitude superimposed on a fundamental of much larger magnitude. These harmonics will usually produce little result in the external circuit, being filtered out by inductance. In some cases, however, these waves are greatly amplified by capacitance in parallel with the circuit or they may produce disturbances in adjacent communication circuits. These waves will be studied in more detail later.

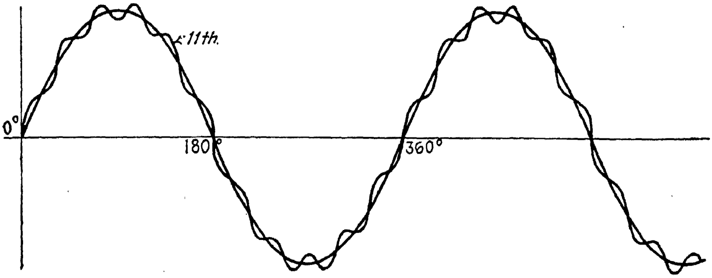

FIG. 306.—A fundamental wave and a small eleventh harmonic.

169. Half-period Average of a Wave.—*Half-period Average Value of a Symmetrical Alternating Current.—The half-period average value of a symmetrical alternating current is the algebraic average of the values of the current taken throughout a half period beginning with a zero value of the current. If the current has more than two zeros during a cycle, that zero shall be taken which gives the largest half-period average value.* For distorted waves, the half-period average of the wave may be found by taking the mean of a sufficient number of ordinates of a complete half wave by the method explained in Chap. I. In determining the half-period average value of a wave from the equation of the wave, it is necessary that the limits of integration be taken from crossing point to crossing point of the total wave, those crossing points being used which give the greatest net area. If the equation is written with this crossing point as the point of reference so that $e = 0$ when $\alpha = 0$, the limits of integration will be 0 and π. Assume an emf wave of the form

$$e = E_{m_1} \sin (\alpha + \theta_1) + E_{m_3} \sin (3\alpha + \theta_3) + \cdots + E_{m_n} \sin (n\alpha + \theta_n), \qquad (1177)$$

which has been written with the crossing point of the total wave as the reference point, *i.e.*, with the vertical axis passing through this point. The integration becomes, from the definition of the half-wave average,

$$E_{av} = \frac{1}{\pi}\int_0^{\pi} [E_{m_1} \sin (\alpha + \theta_1) + E_{m_3} \sin (3\alpha + \theta_3) + \cdots + E_{m_n} \sin (n\alpha + \theta_n)]d\alpha. \qquad (1178)$$

Choosing the kth harmonic as representing any harmonic.

$$\begin{aligned} \int_0^{\pi} \sin (k\alpha + \theta_k)d\alpha &= \int_0^{\pi}[\sin k\alpha \cos \theta_k + \cos k\alpha \sin \theta_k]d\alpha \\ &= \left|\frac{\cos \theta_k}{k}(-\cos k\alpha) + \frac{\sin \theta_k}{k} \sin k\alpha\right|_0^{\pi} \\ &= \frac{2}{k} \cos \theta_k \qquad (1179) \end{aligned}$$

when k is odd, and is equal to zero when k is even. Hence,

$$E_{av} = \frac{2}{\pi}\left[E_{m_1} \cos \theta_1 + \frac{E_{m_3}}{3} \cos \theta_3 + \frac{E_{m_5}}{5} \cos \theta_5 + \cdots + \frac{E_{m(2n-1)}}{2n-1} \cos \theta_{(2n-1)}\right] \qquad (1180)$$

$$= E_{av_1} \cos \theta_1 + \frac{E_{av_3}}{3} \cos \theta_3 + \frac{E_{av_5}}{5} \cos \theta_5 + \cdots + \frac{E_{av(2n-1)}}{(2n-1)} \cos \theta_{(2n-1)}. \qquad (1181)$$

Where the crossing point of the wave of Eq. (1177) is not known, it is necessary to place e equal to zero and to solve for the particular value of θ_1 that conforms to this condition. This cannot be done except for equations in which numerical values of the coefficients and the phase angles are known. The solution is then seldom direct, but it must be made by a "cut-and-try" process, by trying various values of θ_1 that satisfy the condition $e = 0$. The wave represented by Eq. (1177) should be plotted to see whether there is more than one value of θ_1 that satisfies the condition $e = 0$.

When the particular numerical value of θ_1 is found by the preceding method, Eq. (1177) should be rewritten, with this value of θ_1 and the other angles (θ_3, θ_5 . . . θ_n) being transformed to correspond with the new origin, or else the angle of offset must be applied in the limits of integration. Equations (1178) to (1181) then apply to this new equation. In the case of multiple values of θ_1, a solution for E_{av} should be made for each value of θ_1, and only the largest value of E_{av} retained as satisfying the conditions laid down in the definition.

An illustration of the application of this equation may be taken from the following equations and figures. Figure 302, corresponding to the equation,

$$e = 100 \sin \alpha + 25 \sin 3\alpha, \tag{1174}$$

has the area of the positive lobe increased over the area of the fundamental half wave by two triple lobes, and it is decreased by one triple lobe, or a net increase in area of one triple lobe, but, since the area of the entire wave is to be divided by the base of the fundamental, the average is seen to be the area of the fundamental plus the area of one triple lobe divided by the base of the fundamental, since $\theta_3 = 0$ and $\cos \theta_3 = 1.00$. In Fig. 303, corresponding to the equation,

$$\begin{aligned} e &= 100 \sin \alpha - 25 \sin 3\alpha \\ &= 100 \sin \alpha + 25 \sin (3\alpha + 180°), \end{aligned} \tag{1175}$$

the resultant wave has an area equal to that of the fundamental decreased by one lobe of the triple, the average agreeing with that determined by Eq. (1181), since $\cos \theta_3 = \cos 180 = -1.00$.

Example 103

Find the half-wave average of the following wave:

$$e = 100 \sin \alpha + 25 \sin (3\alpha - 90°).$$

To find the crossing points, the right-hand member of the equation is set equal to zero,

$$0 = 100 \sin \alpha + 25 \sin (3\alpha - 90°).$$

This is, in turn, satisfied when α is equal to 11°46′. Then

$$
\begin{aligned}
E_{av} &= \frac{1}{\pi}\int_{11^\circ 46'}^{191^\circ 46'} [100 \sin \alpha + 25 \sin (\\
&= \frac{1}{\pi}\Big| -100 \cos \alpha - \frac{25}{3} \cos (3\alpha - 90^\circ) \Big|_{11^\circ 46'}^{191^\circ 46'} \\
&= 65.3 \text{ volts.}
\end{aligned}
$$

Rewriting the original equation by shifting the axis to the crossing point of the wave,

$$
\begin{aligned}
E_{av} &= \frac{1}{\pi}\int_{0}^{\pi} [\quad) \sin (\alpha + 11^\circ 46') + 25 \sin (3\alpha - 54^\circ 42')] d\alpha \\
&= \frac{1}{\pi}\Big| -100 \cos (\alpha + 11^\circ 46') - \frac{25}{3} \cos (3\alpha - 54^\circ 42') \Big|_{0} \\
&= 65.3 \text{ volts.}
\end{aligned}
$$

This wave is shown in Fig. 305.

170. Effective Value of a Wave.—The definition and derivation of the expression for the effective value of a wave was given in Chap. I, and an application was there made to the simple sine wave. The following is an extension of the method to the equation of any wave, as given in the Fourier form. Consider the wave of Eq. (1177),

$$
\begin{aligned}
e = E_{m_1} \sin (\alpha + \theta_1) + E_{m_3} \sin (3\alpha + \theta_3) + \cdots \\
+ E_{m_n} \sin (n\alpha + \theta_n), \quad (1177)
\end{aligned}
$$

as expressing any emf wave. The effective value of the wave is defined as

$$
E = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} e^2 d\alpha.} \quad (29)
$$

Substituting the value of e from Eq. (1177) into Eq. (29),

$$
\begin{aligned}
E = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \Big[E_{m_1} \sin (\alpha + \theta_1) + E_{m_3} \sin (3\alpha + \theta_3) + \cdots} \\
\overline{+ E_{m_n} \sin (n\alpha + \theta_n) \Big]^2 d\alpha} \quad (1182)
\end{aligned}
$$

$$
\begin{aligned}
= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \Big\{ E_{m_1}{}^2 \cdot \sin^2 (\alpha + \theta_1) + E_{m_3}{}^2 \sin^2 (3\alpha + \theta_3) + \cdots} \\
\overline{+ E_{m_n}{}^2 \sin^2 (n\alpha + \theta_n) + \cdots + \cdots + 2E_{m_{n-1}} E_{m_n} \sin} \\
\overline{\Big[(n-1)\alpha + \theta_{n-1}\Big] \sin \Big[n\alpha + \theta_n\Big] \Big\} d\alpha.} \quad (1183)
\end{aligned}
$$

Equation (1183) contains the different kinds of terms in the entire expression representing the square of the wave of Eq. (1182), and the integration of these terms will give the integration of the equation. Integrating each term separately,

$$\int_0^{2\pi} \sin^2 (n\alpha + \theta_n) d\alpha = \int_0^{2\pi} \tfrac{1}{2}[1 - \cos 2(n\alpha + \theta_n)] d\alpha = \pi, \qquad (1184)$$

$$\int_0^{2\pi} \left\{ \sin [(n - 1)\alpha + \theta_{n-1}] \sin [n\alpha + \theta_n] \right\} d\alpha =$$
$$\int_0^{2\pi} \tfrac{1}{2} \left\{ \cos [\alpha + \theta_n - \theta_{n-1}] - \cos [(2n - 1)\alpha + \theta_n + \theta_{n-1}] \right\} d\alpha$$
$$= 0, \qquad (1185)$$

since

$$\sin x \sin y = \tfrac{1}{2} [\cos (x - y) - \cos (x + y)].$$

Hence the effective value of the emf wave becomes

$$E = \sqrt{\frac{1}{2\pi}\left\{\pi\left[E_{m_1}{}^2 + E_{m_3}{}^2 + \; . \; . \; . \; + E_{m_n}{}^2\right]\right\}}$$
$$= \sqrt{\frac{E_{m_1}{}^2 + E_{m_3}{}^2 + \; . \; . \; . \; + E_{m_n}{}^2}{2}} \qquad (1186)$$
$$= \sqrt{E_1{}^2 + E_3{}^2 + \; . \; . \; . \; + E_n{}^2}. \qquad (1187)$$

The effective value of any nonsinusoidal wave is the square root of the sum of the squares of the effective values of the component waves. It is independent of the phase relation between the component waves and of the starting point of the wave. This will be evident when it is considered that in squaring the wave all negative lobes of the fundamental and of the harmonics give positive areas, so that the area of the squared fundamental wave is increased by the same amount irrespective of the phase of the harmonics.

Amplitude of a Generalized Sinusoidal Quantity.—The amplitude of a generalized sinusoidal quantity for any value of the independent variable is the value of the modifying function for that particular value of the independent variable.

Amplitude of a Simple Sinusoidal Quantity.—The amplitude of a simple sinusoidal quantity is the largest value that the quantity attains.

Peak Value. (Crest Value.)—*The peak value of any quantity that varies with the time is the maximum value which the quantity* *attains during the interval of time under consideration. The peak value of a periodic quantity is the maximum value which the quantity attains at any time during a period.*

Crest Factor of a Periodic Quantity.—The crest factor of a periodic quantity is the ratio of the crest value to the effective value of the quantity.

Ripple Ratio.—The ripple ratio of a ripple quantity is the ratio of the difference between the maximum and minimum values of the quantity to the average value.

Form-factor of a Symmetrical Alternating Quantity.—The form factor of a symmetrical alternating quantity is the ratio of the effective value of the quantity to its half-period average value.

Distortion Factor of a Wave.—The distortion factor of a voltage wave is the ratio of the effective value of the residue after the elimination of the fundamental to the effective value of the original wave.

Since the effective value of the wave is independent of the phase of the harmonics and the average value is not, it is possible to have the same form factor, or ratio of E to E_{av}, for waves of widely different shape. It is even possible to have distorted waves with a form factor of 1.11, the same value as for a sine wave. In order, therefore, to compare a given wave with a sine wave it is necessary to consider the preceding definitions as well as other methods of comparison for the particular application in view. To illustrate some of the foregoing definitions and proofs the following examples have been chosen.

Example 104

Find the average and the effective values, the crest factor, and the form factor of the wave,

$$e = 141.4 \sin 157t - 28.2 \sin 471t.$$

This wave has a shape similar to that shown in Fig. 303, and the maximum value of the wave must be

$$E_m = 141.4 + 28.2 = 169.6 \text{ volts},$$

since the two waves have their maxima in phase at the peak of the fundamental.

$$\begin{aligned} E_{AV} &= \frac{2}{\pi}\left(141.4 - \frac{28.2}{3}\right) \\ &= \frac{2}{\pi} \times 132 = 84 \text{ volts.} \end{aligned}$$

$$E = \sqrt{\frac{\overline{141.4}^2 + \overline{28.2}^2}{2}} = 102 \text{ volts.}$$

$$\text{Crest factor} = \frac{169.6}{102} = 1.66.$$

$$\text{Form factor} = \frac{102}{84} = 1.22.$$

171. Power Equation, Active and Reactive Power.—The power consumed by a circuit having any wave of emf and current is defined in Chap. I as

$$p = ei \tag{30}$$

for its instantaneous value, and,

$$P = \frac{1}{2\pi}\int_0^{2\pi} p d\alpha = \frac{1}{2\pi}\int_0^{2\pi} ei d\alpha, \tag{49}$$

for the active power, or the reading of the wattmeter. If the waves of emf and current are expressed by

$$e = E + E_{m_1} \sin (\alpha + \theta_1) + \ . \ . \ . + E_{m_n} \sin (n\alpha + \theta_n) \tag{1177}$$

and

$$i = I + I_{m_1} \sin (\alpha + \beta_1) + \ . \ . \ . + I_{m_n} \sin (n\alpha + \beta_n), \tag{1188}$$

then,

$$p = ei = EI + \ . \ . \ . + E_{m_n} I \sin (n\alpha + \theta_n) + EI_{m_n} \sin (n\alpha + \beta_n) + \ . \ . \ . + E_{m_n} I_{m_n} \sin (n\alpha + \theta_n) \sin (n\alpha + \beta_n) + E_{m_1} I_{m_n} \sin (\alpha + \theta_1) \sin (n\alpha + \beta_n) + E_{m_n} I_{m_1} \sin (n\alpha + \theta_n) \sin (\alpha + \beta_1) + \ . \ . \ . \tag{1189}$$

The active power is

$$P = \frac{1}{2\pi}\int_0^{2\pi} \Big\{ EI + \ . \ . \ . + E_{m_n} I \sin (n\alpha + \theta_n) + EI_{m_n} \sin (n\alpha + \beta_n) + \ . \ . \ . + E_{m_n} I_{m_n} \sin (n\alpha + \theta_n) \sin (n\alpha + \beta_n) + E_{m_1} I_{m_n} \sin (\alpha + \theta_1) \sin (n\alpha + \beta_n) + E_{m_n} I_{m_1} \sin (n\alpha + \theta_n) \sin (\alpha + \beta_1) + \ . \ . \ . \Big\} d\alpha. \tag{1190}$$

Integrating Eq. (1190) by integrating the separate terms and applying the limits,

$$\int_0^{2\pi} EI\,d\alpha = EI\left|\alpha\right|_0^{2\pi} = 2\pi EI. \tag{1191}$$

$$\int_0^{2\pi} E_{m_n} I \sin (n\alpha + \theta_n)d\alpha = E_{m_n} I \left| \frac{-\cos (n\alpha + \theta_n)}{n} \right|_0^{2\pi}$$

$$= \frac{E_m I}{n} \left| -\cos n\alpha \cos \theta_n + \sin n\alpha \sin \theta_n \right|_0^{2\pi} = 0. \tag{1192}$$

$$\int_0^{2\pi} EI_{m_n} \sin (n\alpha + \beta_n)d\alpha = EI_{m_n} \left| \frac{-\cos (n\alpha + \beta_n)}{n} \right|_0^{2\pi} = 0. \tag{1193}$$

$$\int^{2\pi} E_{m_n} I_{m_1} \sin (n\alpha + \theta_n) \sin (\alpha + \beta_1)d\alpha = \int_0^{2\pi} \frac{E_{m_n} I_{m_1}}{2}$$

$$\left\{ \cos [(n - 1)\alpha + \theta_n - \beta_1] - \cos [(n + 1)\alpha + \theta_n + \beta_1] \right\} d\alpha = 0. \tag{1194}$$

$$\int_0^{2\pi} E_{m_n} I_{m_n} \sin (n\alpha + \theta_n) \sin (n\alpha + \beta_n)d\alpha = \int^{2\pi} \frac{E_{m_n} I_{m_n}}{2}$$

$$\left\{ \cos (\theta_n - \beta_n) - \cos (2n\alpha + \theta_n + \beta_n) \right\} d\alpha$$

$$= \frac{E_{m_n} I_{m_n}}{2} \left| \cos (\theta_n - \beta_n)\alpha - \frac{\sin (2n\alpha + \theta_n + \beta_n)}{2n} \right|_0^{2\pi}$$

$$= 2\pi E_n I_n \cos (\theta_n - \beta_n). \tag{1195}$$

Substituting the values of the integrals from Eqs. (1191) to (1195) into Eq. (1190),

$$P = EI + E_1 I_1 \cos (\theta_1 - \beta_1) + E_3 I_3 \cos (\theta_3 - \beta_3) + \cdots + E_n I_n \cos (\theta_n - \beta_n). \tag{1196}$$

From Eqs. (1192), (1193), and (1194), it is seen that the active power produced by current and emf waves of different frequancies is zero, and, therefore, in Eq. (1196) each term contains only emf and current factors of the same frequency or order. It is also seen that each individual harmonic produces its part of the power independent of the existence of the other harmonics. Reactive power is defined in an equation similar to Eq. (1196) as

$$Q = E_1 I_1 \sin (\theta_1 - \beta_1) + E_3 I_3 \sin (\theta_3 - \beta_3) + \cdots + E_n I_n \sin (\theta_n - \beta_n) \text{ vars.} \tag{1197}$$

The vector power is found as the square root of the sum of the squares of the active and reactive power, as stated in Eq. (329).

Example 105

If a current

$$i = 141.4 \sin \alpha + 14.1 \sin (3\alpha + 30°) + 7.07 \sin (5\alpha + 90°)$$

is passed through a circuit and the emf drop is found to be

$$e = 1{,}000 \sin (\alpha + 30°) + 500 \sin (3\alpha + 90°) + 100 \sin (5\alpha - 30°) + 50 \sin (7\alpha + 80°),$$

what are the active, the reactive, the vector, and the apparent power, and power factor?

$$I = \sqrt{100^2 + 10^2 + 5^2} = 100.6 \text{ amp},$$

$$E = \sqrt{\frac{1{,}000^2 + 500^2 + 100^2 + 50^2}{2}} = 794.5 \text{ volts},$$

$$\text{Apparent power} = U = \frac{794.5 \times 100.6}{1{,}000} = 79.9 \text{ kva},$$

$$P = \frac{1{,}000 \times 141.4}{2} \cos 30° + \frac{500 \times 14.1}{2} \cos (90° - 30°) + \frac{100 \times 7.07}{2} \cos (-30° - 90°) + \frac{50 \times 0}{2} \cos (-80°)$$

$$= 61{,}200 + 1{,}767 - 176.7 + 0 = 62{,}790 \text{ watts}$$
$$= 62.79 \text{ kw}.$$

$$\text{Power factor} = \frac{62.79}{79.9} = 0.786 = 78.6 \text{ per cent}.$$

$$Q = \frac{1{,}000 \times 141.4}{2} \sin 30° + \frac{500 \times 14.1}{2} \sin (90° - 30°) + \frac{100 \times 7.07}{2} \sin (-30° - 90°) + \frac{50 \times 0}{2} \sin (-80°)$$

$$= 35{,}350 + 3{,}060 - 306 + 0 = 38{,}104 \text{ vars}$$
$$= 38.1 \text{ kvar}.$$

$$V = \sqrt{62.79^2 + 38.1^2} = 73.4 \text{ vector kva}.$$

172. Analysis of Waves by Fourier's Series.—Since any recurring wave of y for any value of α may be expressed by the equation,

$$y = A + A_1 \sin \alpha + A_2 \sin 2\alpha + \cdots + A_n \sin n\alpha + B_1 \cos \alpha + B_2 \cos 2\alpha + \cdots + B_n \cos n\alpha, \quad (1166)$$

where n may be any positive integer, even or odd, any wave may be broken up into these component waves by the process of analysis. Equation (1166) may also be written in the form

$$y = A + C_1 \sin (\alpha + \theta_1) + C_2 \sin (2\alpha + \theta_2) + C_3 \sin (3\alpha + \theta_3) + \cdots + C_n \sin (n\alpha + \theta_n), \quad (1167)$$

as explained before, where

$$C_n = \sqrt{A_n^2 + B_n^2}, \tag{1168}$$

and

$$\tan \theta_n = \frac{B_n}{A_n}. \tag{1169}$$

In order to determine the value of C_n and θ_n for any harmonic, in respect to the chosen origin, it is necessary to obtain A_n and B_n, the coefficients of $\sin \theta_n$ and $\cos \theta_n$, respectively. This result is accomplished in several ways, one of which is as follows:

Determination of Constant Term.—To obtain the value of the constant term A, multiply Eq. (1166) by $d\alpha$ and integrate as

$$\int_0^{2\pi} y d\alpha = \int_0^{2\pi} A d\alpha + \int_0^{2\pi} A_1 \sin \alpha d\alpha + \ . \ . \ . \ + \int_0^{2\pi} A_n \sin n\alpha\, d\alpha \\ + \int_0^{2\pi} B_1 \cos \alpha d\alpha + \ . \ . \ . \ + \int_0^{2\pi} B_n \cos n\alpha d\alpha. \tag{1198}$$

In Eq. (1198), all terms integrate to zero on application of the limits except the term on the left and the first term on the right of the sign of equality, or

$$\int_0^{2\pi} y d\alpha = \int_0^{2\pi} A d\alpha = 2\pi A, \tag{1199}$$

and

$$A = \frac{1}{2\pi}\int_0^{2\pi} y d\alpha. \tag{1200}$$

From Eq. (1200), *A is seen to be the mean ordinate of the original wave*, as explained previously, since $\int_0^{2\pi} y d\alpha$ is the area of the wave and 2π is the base for 360 deg.

Determination of the Coefficient of Any Sine Term.—To determine the value of A_n, the coefficient of $\sin n\alpha$, multiply Eq. (1166) by $\sin n\alpha d\alpha$ and integrate each term as

$$\int_0^{2\pi} y \sin n\alpha d\alpha = \int_0^{2\pi} A \sin \alpha n d\alpha + \int_0^{2\pi} A_1 \sin \alpha \sin n\alpha d\alpha + \ . \ . \ . \\ . \ . \ . \ + \int_0^{2\pi} A_n \sin^2 n\alpha d\alpha + \int_0^{2\pi} B_1 \cos \alpha \sin n\alpha d\alpha + \ . \ . \ . \\ + \int_0^{2\pi} B_n \cos n\alpha \sin n\alpha d\alpha. \tag{1201}$$

After integration of the terms in the second member of Eq. (1201) it will be found that the equation reduces to the form

$$\int_0^{2\pi} y \sin n\alpha d\alpha = \frac{2\pi A_n}{2}, \tag{1202}$$

and

$$A_n = 2\left[\frac{1}{2\pi}\int_0^{2\pi} y \sin n\alpha d\alpha\right]. \tag{1203}$$

That is, A_n *is twice the mean ordinate for* 360 *deg. of the wave derived by multiplying the original wave by a unit sine wave of the nth harmonic having its origin at the starting point for the analysis of the wave.*

Determination of the Coefficient of Any Cosine Term.—To determine the value of B_n, the coefficient of cos $n\alpha$, multiply Eq. (1166) by cos $n\alpha d\alpha$ and integrate as

$$\int_0^{2\pi} y \cos n\alpha d\alpha = \int_0^{2\pi} A \cos n\alpha d\alpha + \int_0^{2\pi} A_1 \sin \alpha \cos n\alpha d\alpha + \cdots + \int_0^{2\pi} A_n \sin n\alpha \cos n\alpha d\alpha + \cdots + \int_0^{2\pi} B_1 \cos \alpha \cos n\alpha d\alpha + \cdots + \int_0^{2\pi} B_n \cos^2 n\alpha d\alpha. \tag{1204}$$

After integration of the terms in the second member of Eq. (1204) it will be found that the equation reduces to the form

$$\int_0^{2\pi} y \cos n\alpha d\alpha = \frac{2\pi B_n}{2}, \tag{1205}$$

and

$$B_n = 2\left[\frac{1}{2\pi}\int_0^{2\pi} y \cos n\alpha d\alpha\right]. \tag{1206}$$

That is, B_n *is twice the mean ordinate of the wave derived by multiplying the original wave by a unit cosine wave of the nth harmonic having its origin at the starting point for the analysis of the wave.*

By the application of the above methods, any of the coefficients may be found independent of a knowledge of the other coefficients. By a determination of each of the values, A, A_1, A_2 . . . A_n, B_1, B_2 . . . B_n, for as many harmonics as desired, the equation of the wave may be written in the form of Eq. (1167), since

$$C_1 = \sqrt{A_1^2 + B_1^2}.$$

$$\cdots$$

$$C_n = \sqrt{A_n^2 + B_n^2} \tag{1168}$$

and

$$\tan \theta_1 = \frac{B_1}{A_1}$$

$$\tan \theta_n = \frac{B_n}{A_n}. \qquad (1169)$$

In determining the values of θ_1, θ_2 . . . θ_n, the signs of the quantities A_1, B_1 . . . A_n, B_n must be noted. In the value of θ_1, if B_1 and A_1 are both positive, the angle θ_1 lies between 0 and 90 deg.; if B_1 is positive and A_1 is negative, θ_1 is between 90 and 180 deg.; if B_1 and A_1 are both negative, θ_1 lies between 180 and 270 deg.; and if B_1 is negative and A_1 is positive, θ_1 is between 270 and 360 deg., the value of θ_1 being taken in a positive or counterclockwise direction from the origin in each instance. From the relation,

$$\sin (\alpha + \theta_n) = - \sin (\alpha - 180° + \theta_n),$$

and similar trigonometrical relations, it is possible to express all phase angles in values less than 90 deg.

In making the analysis of a wave by this method, considerable time may be saved by applying the principles previously learned in regard to waves. If, by inspection, the wave has the same area above the axis as below, the value of A is zero. If the lobes of the wave are symmetrical, there are no even harmonics present in the wave and only odd harmonics need be considered in the analysis. The actual mechanical process of analysis may also be shortened by observing that the integration for all the coefficients of the harmonics may be made for 180 deg. instead of for 360 deg., whereas the value of A must be determined for 360 deg. For ordinary analysis, 10-deg. intervals are sufficient, and the start may be made at any point on the wave.

The mechanical work of integration consists of first tabulating the values of all ordinates at each 10 deg. of the wave for 360 deg., adding these ordinates, and dividing by their number, since

$$A = \frac{\sum_0^{2\pi} y}{36}. \qquad (1207)$$

Next, the values of sin α are taken for each 10 deg. for 180 deg. from a table of sines, and each ordinate is multiplied by its corresponding value of sin α. The value of A_1 is then the sum of these derived values divided by half their number, since

$$A_1 = \frac{2\sum_0^\pi y \sin \alpha}{18} = \frac{\sum_0^\pi y \sin \alpha}{9}. \tag{1208}$$

In a similar manner, cos α is applied for 180 deg., and the value of B_1 is determined as the sum of these derived values divided by half their number, since

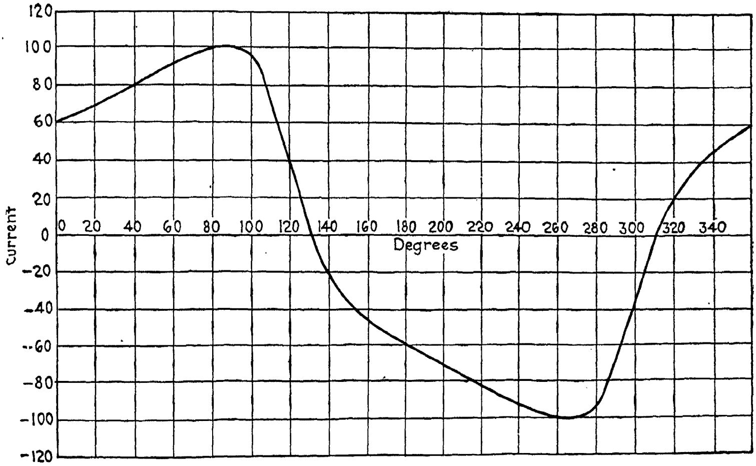

FIG. 307.—Irregular wave of Example 106.

$$B_1 = \frac{2\sum_0^\pi y \cos \alpha}{18} = \frac{\sum_0^\pi y \cos \alpha}{9}. \tag{1209}$$

By taking sin 3α and cos 3α and applying to the values of y, as explained for sin α and cos α, the values of A_3 and B_3 are determined as

$$A_3 = \frac{\sum_0^\pi y \sin 3\alpha}{9} \tag{1210}$$

and

$$B_3 = \frac{\sum_0^\pi y \cos 3\alpha}{9}. \tag{1211}$$

In a similar manner, A_5, B_5, etc., may be found and the equation of the wave written for as many harmonics as necessary.

Example 106

Make an analysis of the wave of current shown in Fig. 307 for the first, third, fifth, and seventh harmonics.

TABLE XVII.—ANALYSIS OF THE WAVE OF FIG. 307 FOR THE FUNDAMENTAL HARMONIC BY FOURIER'S SERIES
Example 106

α, degrees	i	$\sin\alpha$	$i\sin\alpha$		$\cos\alpha$	[illegible]	
			+	−		+	−
5	62.0	0.0872	5.41	.....	0.9962	61.80	
15	67.0	0.2588	17.35	.....	0.9659	64.75	
25	72.5	0.4226	30.65	.....	0.9063	65.70	
35	78.5	0.5736	45.00	.....	0.8192	64.30	
45	84.5	0.7071	59.75	.....	0.7071	59.75	
55	90.0	0.8192	73.75	.....	0.5736	51.60	
65	95.0	0.9063	86.10	.....	0.4226	40.15	
75	99.0	0.9659	95.60	.....	0.2588	25.61	
85	100.0	0.9962	99.62	.....	0.0872	8.72	
95	98.0	0.9962	97.60	.....	−0.0872	.....	8.55
105	82.0	0.9659	79.20	.....	−0.2588	......	21.22
115	50.0	0.9063	45.31	.....	−0.4226	......	21.13
125	14.0	0.8192	11.47	.....	−0.5736	......	8.03
135	− 13.0	0.7071	......	9.19	−0.7071	9.19	
145	− 29.0	0.5736	......	16.63	−0.8192	23.75	
155	− 41.0	0.4226	......	17.33	−0.9063	37.15	
165	− 50.0	0.2588	......	12.94	−0.9659	48.29	
175	− 56.5	0.0872	......	4.93	−0.9962	56.30	
...	....	......	746.81	61.02	....	617.06	58.93

$$A_1 = 2\sum\frac{i\sin\alpha}{18} = \frac{746.81 - 61.02}{9} = \frac{685.79}{9} = 76.2.$$

$$B_1 = 2\sum\frac{i\cos\alpha}{18} = \frac{617.06 - 58.93}{9} = \frac{558.13}{9} = 62.0.$$

$$C_1 = \sqrt{A_1^2 + B_1^2} = \sqrt{\overline{76.2}^2 + \overline{62.0}^2} = 98.2.$$

$$\theta_1 = \tan^{-1}\frac{62.0}{76.2} = \tan^{-1} 0.814, \quad = 39° \, 10'.$$

$$i_1 = 98.2 \sin(\alpha + 39° \, 10').$$

TABLE XVIII.—ANALYSIS OF THE WAVE OF FIG. 307 FOR THE THIRD HARMONIC BY FOURIER'S SERIES

Example 106

α, degrees	i	3α, degrees	sin 3α	i sin 3α		cos 3α	i cos 3α	
				+	−		+	−
5	62.0	15	0.2588	16.05	......	0.9659	59.89	
15	67.0	45	0.7071	47.30	......	0.7071	47.30	
25	72.5	75	0.9659	70.03	......	0.2588	18.76	
35	78.5	105	0.9659	75.82	......	−0.2588	......	20.31
45	84.5	135	0.7071	59.68	......	−0.7071	......	59.68
55	90.0	165	0.2588	23.29	......	−0.9659	......	86.93
65	95.0	195	−0.2588	......	24.59	−0.9659	......	91.76
75	99.0	225	−0.7071	......	70.00	−0.7071	......	70.00
85	100.0	255	−0.9659	......	96.59	−0.2588	......	25.88
95	98.0	285	−0.9659	......	94.66	0.2588	25.36	
105	82.0	315	−0.7071	......	57.98	0.7071	57.98	
115	50.0	345	−0.2588	......	12.94	0.9659	48.29	
125	14.0	375	0.2588	3.62	......	0.9659	13.52	
135	−13.0	405	0.7071	......	9.19	0.7071	......	9.19
145	−29.0	435	0.9659	......	28.01	0.2588	......	7.50
155	−41.0	465	0.9659	......	39.60	−0.2588	10.61	
165	−50.0	495	0.7071	......	35.35	−0.7071	35.35	
175	−56.5	525	0.2588	......	14.62	−0.9659	54.57	
...	......	...	....	295.79	483.53	....	371.63	371.25

$$A_3 = \frac{295.79 - 483.53}{9} = \frac{-187.74}{9} = -20.7.$$

$$B_3 = \frac{371.63 - 371.25}{9} = \frac{0.38}{9} = 0.042.$$

$$C_3 = \sqrt{\overline{-20.7}^2 + \overline{0.042}^2} = 20.7.$$

$$\tan \theta_3 = \frac{0.042}{-20.7} = -0.001985, \quad \theta_3 = 179^\circ\ 54'.$$

$$\begin{aligned} i_3 &= 20.7 \sin (3\alpha + 179^\circ\ 54'), \\ &= -20.7 \sin (3\alpha - 0^\circ\ 6'). \end{aligned}$$

TABLE XIX.—ANALYSIS OF THE WAVE OF FIG. 307 FOR THE FIFTH HARMONIC BY FOURIER'S SERIES

Example 106

α, degrees	i	5α, degrees	$\sin 5\alpha$	$i \sin 5\alpha$		$\cos 5\alpha$	$i \cos 5\alpha$	
				+	−		+	−
5	62.0	25	0.4226	26.19	......	0.9063	56.20	
15	67.0	75	0.9659	64.71	......	0.2588	17.34	
25	72.5	125	0.8192	59.39	......	−0.5736	......	41.59
35	78.5	175	0.0872	6.85	......	−0.9962	......	78.20
45	84.5	225	−0.7071	......	59.75	−0.7071	......	59.75
55	90.0	275	−0.9962	......	89.66	+0.0872	7.85	
65	95.0	325	−0.5736	......	54.49	0.8192	77.82	
75	99.0	375	0.2588	25.62	......	0.9659	95.62	
85	100.0	425	0.9063	90.63	......	0.4226	42.26	
95	98.0	475	0.9063	88.82	......	−0.4226	......	41.41
105	82.0	525	0.2588	21.22	......	−0.9659	......	79.20
115	50.0	575	−0.5736	......	28.68	−0.8192	......	40.96
125	14.0	625	−0.9962	......	13.95	−0.0872	......	1.22
135	−13.0	675	−0.7071	9.19	......	0.7071	......	9.19
145	−29.0	725	0.0872	......	2.53	0.9962	......	28.89
155	−41.0	775	0.8192	......	33.59	0.5736	......	23.52
165	−50.0	825	0.9659	......	48.29	−0.2588	12.94	
175	−56.5	875	0.4226		23.87	−0.9063	51.20	
...	......	....	....	392.62	354.81	....	361.23	403.93

$$A_5 = \frac{392.62 - 354.81}{9} = \frac{37.81}{9} = 4.20$$

$$B_5 = \frac{361.23 - 403.93}{9} = \frac{-42.70}{9} = -4.75.$$

$$C_5 = \sqrt{4.20^2 + 4.75^2} = 6.34.$$

$$\tan \theta_5 = \frac{-4.75}{4.20} = -1.13, \ \theta_5 = -48^\circ\ 30'.$$

$$i_5 = 6.34 \sin (5\alpha - 48^\circ\ 30').$$

TABLE XX.—ANALYSIS OF THE WAVE OF FIG. 307 FOR THE SEVENTH HARMONIC BY FOURIER'S SERIES

Example 106

α, degrees	i	7α, degrees	Sin 7α	i sin 7α		cos 7α	i cos 7α	
				+	−		+	−
5	62.0	35	0.5736	35.56	......	0.8192	50.79	
15	67.0	105	0.9659	64.72	......	−0.2588	......	17.34
25	72.5	175	0.0872	6.32	......	−0.9962	......	72.22
35	78.5	245	−0.9063	......	71.14	−0.4226	......	33.17
45	84.5	315	−0.7071	......	59.75	0.7071	59.75	
55	90.0	385	0.4226	38.03	......	0.9063	81.57	
65	95.0	455	0.9962	94.64	......	−0.0872	......	8.28
75	99.0	525	0.2588	25.62	......	−0.9659	......	95.62
85	100.0	595	−0.8192	......	81.92	−0.5736	......	57.36
95	98.0	665	−0.8192	......	80.28	0.5736	56.21	
105	82.0	735	0.2588	21.22	......	0.9659	79.20	
115	50.0	805	0.9962	49.81	......	0.0872	4.36	
125	14.0	875	0.4226	5.92	......	−0.9063	......	12.69
135	−13.0	945	−0.7071	9.19	......	−0.7071	9.19	
145	−29.0	1,015	−0.9063	26.28	......	0.4226	......	12.26
155	−41.0	1,085	0.0872	......	3.58	0.9962	......	40.84
165	−50.0	1,155	0.9659	......	48.29	0.2588	......	12.94
175	−56.5	1,225	0.5736	......	32.41	−0.8192	46.28	
...	......	....	....	377.31	377.37	....	387.35	362.72

$$A_7 = \frac{377.31 - 377.37}{9} = \frac{-0.06}{9} = -0.0067.$$

$$B_7 = \frac{387.35 - 362.72}{9} = \frac{24.63}{9} = 2.73.$$

$$C_7 = \sqrt{0.0067^2 + 2.73^2} = 2.73.$$

$$\tan \theta_7 = \frac{2.73}{-0.0067} = -408, \quad \theta_7 = 90^\circ\ 8.5'$$

$$i_7 = 2.73 \sin (7\alpha + 90^\circ\ 8.5').$$

The complete equation of the wave is, therefore,

$$i = 98.2 \sin (\alpha + 39^\circ\ 10') - 20.7 \sin (3\alpha - 0^\circ\ 6') + 6.34 \sin (5\alpha - 48^\circ\ 30') + 2.73 \sin (7\alpha + 90^\circ\ 8.5').$$

The values obtained for the phase displacement angles of the various harmonics are measured from the starting point of the analysis or from $\alpha = 0$. If the analysis had been begun at some other point on the wave, the values obtained for A_1, B_1, A_3, B_3, A_5, B_5, and A_7, B_7 would have been different, but the values of C_1, C_3, C_5, and C_7 would have remained unchanged, and the two equations would have differed only in the values of the phase-displacement angles of the harmonics. If the starting point for the analysis is shifted by β deg., the phase displacement angle for the fundamental will be changed by β deg., that for the third harmonic by 3β deg., and that for the nth harmonic by $n\beta$ deg. The analysis may, therefore, be started at any point on the wave and the resulting equation rewritten with reference to any other point on the wave by subtracting $n\beta$ deg. from the phase displacement angle of each harmonic. If the reference point is shifted to the left, or ahead, the sign of β will be positive, and if the shift is to the right, or backward, the sign of β will be negative.

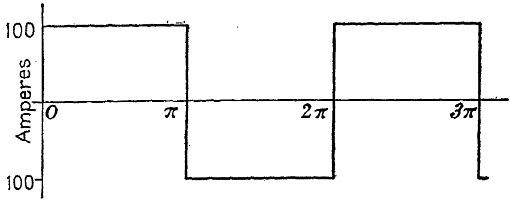

FIG. 308.—Rectangular wave of current for Example 107.

The usual practice is to consider the equation of a wave with reference to the crossing point of the total wave. For the preceding analysis, this would mean shifting the reference point 50 deg. ahead of the point used in the analysis, and the new equation would be obtained by subtracting 50 deg. from the phase of the fundamental, 150 deg. from the phase of the third harmonic, 250 deg. from the phase of the fifth harmonic, and 350 deg. from the phase of the seventh harmonic. If only angles less than 90 deg. are used, the equation becomes

$$i = 98.2 \sin (\alpha - 10^\circ\ 50') + 20.7 \sin (3\alpha + 29^\circ\ 54') + 6.34 \sin (5\alpha + 61^\circ\ 30') + 2.73 \sin (7\alpha + 110^\circ\ 8.5').$$

Example 107

Derive the equation for a rectangular current wave having a maximum value of 100 amp and crossing the axis at $\alpha = 0^\circ, 180^\circ, 360^\circ$, etc.

Figure 308 illustrates the wave for which the analysis is to be made. From Eqs. (1203) and (1206), the values of the coefficients of the sine and cosine terms of the kth harmonic may be derived when the equation of the wave is known. Since the wave is a series of straight lines parallel to the axis, it is easy to write the equation for each part of the discontinuous wave. The equations for integration are written

$$A_k = \frac{1}{\pi}\int_0^{2\pi} i \sin k\alpha d\alpha \tag{1203}$$

and

$$B_k = \frac{1}{\pi}\int_0^{2\pi} i \cos k\alpha d\alpha, \tag{1206}$$

where A_k and B_k are the magnitudes of the sine and cosine waves of the kth harmonic, respectively. For the case of a wave with discontinuities, an integration must be made for each interval between discontinuities.

For the wave of Fig. 308, it is seen that for

$$\begin{aligned} &0<\alpha<\pi, \ i = 100, \text{ and for} \\ &\pi<\alpha<2\pi, \ i = -100. \end{aligned}$$

Then for the kth harmonic, Eqs. (1203) and (1206) become,

$$A_k = \frac{1}{\pi}\left[\int_0^{\pi} i \sin k\alpha d\alpha + \int_{\pi}^{2\pi} i \sin k\alpha d\alpha\right],$$

and

$$B_k = \frac{1}{\pi}\left[\int_0^{\pi} i \cos k\alpha d\alpha + \int_{\pi}^{2\pi} i \cos k\alpha d\alpha\right].$$

Substituting the values of i, since the wave is a straight line parallel to the axis of α,

$$\begin{aligned} A_k &= \frac{100}{\pi}\left[\int_0^{\pi} \sin k\alpha d\alpha - \int_{\pi}^{2\pi} \sin k\alpha d\alpha\right] \\ &= \frac{400}{k\pi} \text{ for } k \text{ odd and 0 for } k \text{ even,} \\ B_k &= \frac{100}{\pi}\left[\int_0^{\pi} \cos k\alpha d\alpha - \int^{2\pi} \cos k\alpha d\alpha\right] \\ &= 0 \text{ for } k \text{ odd or even.} \end{aligned}$$

Then, substituting the various values of k from 1 to n, the equation for the wave may be written as

$$i = \frac{400}{\pi}\left[\sin \alpha + \frac{1}{3}\sin 3\alpha + \frac{1}{5}\sin 5\alpha + \cdots + \frac{1}{n}\sin n\alpha\right].$$

For any square-top current wave having I_m as its height,

$$i = \frac{4}{\pi}I_m\left[\sin \alpha + \frac{1}{3}\sin 3\alpha + \frac{1}{5}\sin 5\alpha + \cdots + \frac{1}{n}\sin n\alpha\right],$$

when the analysis is started at the crossing point of the wave from negative to positive values.

It is seen from this analysis that all cosine terms drop to zero for a wave that is symmetrical about the axis at $\alpha = \pi/2$ as well as about the horizontal axis.

173. Fisher-Hinnen Method of Wave Analysis.

—A more convenient and rapid method of determining the constants

A and B for the various harmonics, particularly when their number is limited to only two or three, is based upon the simple trigonometrical relation by which a circle is divided into m equal parts, or angles. The sum of the sines is zero and the sum of the cosines is also zero for all integral values of m greater than one. Then the sum of m ordinates equally spaced over a complete cycle or 360 deg. of a sine or cosine wave is equal to zero irrespective of the starting point. This may be seen from Fig. 309 where the successive ordinates are displaced from each other by $360/m$ deg., and, starting at $\alpha = \delta$, their values are

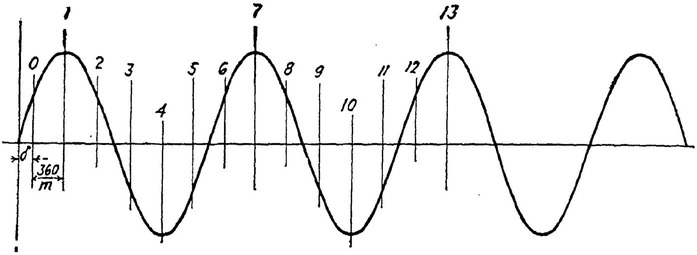

FIG. 309.—A sine wave cut by six coordinates equally spaced over one complete cycle.

$$e_1 = E_{m_1} \sin\left(\delta + \frac{360°}{m}\right) = E_{m_1} \sin\delta \cos\frac{360°}{m} + E_{m_1} \cos\delta \sin\frac{360°}{m}, \quad (1212)$$

$$e_2 = E_{m_1} \sin\left(\delta + 2\frac{360}{m}\right) = E_{m_1} \sin\delta \cos 2\frac{360°}{m} + E_{m_1} \cos\delta \sin 2\frac{360°}{m}, \quad (1213)$$

$$e_3 = E_{m_1} \sin\left(\delta + 3\frac{360°}{m}\right) = E_{m_1} \sin\delta \cos 3\frac{360°}{m} + E_{m_1} \cos\delta \sin 3\frac{360°}{m} \quad (1214)$$

. . . .

$$e_m = E_{m_1} \sin\left(\delta + m\frac{360°}{m}\right) = E_{m_1} \sin\delta \cos m\frac{360°}{m} + E_{m_1} \cos\delta \sin m\frac{360°}{m}. \quad (1215)$$

Since $\cos x + \cos 2x + \cos 3x + \ldots + \cos mx = 0, \quad (1216)$ when $mx = 360$ deg., or a multiple of 360 deg., then $\Sigma e = 0$.

The $2n$ half waves of the nth harmonic will also be divided into m equal parts by these same ordinates spaced $n\ 360/m$ degrees apart on the scale of the nth harmonic. These ordinates will be

$$e_1 = E_{m_n} \sin\left(n\delta + n\frac{360^\circ}{m}\right) = E_{m_n} \sin n\delta \cos n\frac{360^\circ}{m} + E_{m_n} \cos n\delta \sin n\frac{360^\circ}{m}, \tag{1217}$$

$$e_2 = E_{m_n} \sin\left(n\delta + 2n\frac{360^\circ}{m}\right) = E_{m_n} \sin n\delta \cos 2n\frac{360^\circ}{m} + E_{m_n} \cos n\delta \sin 2n\frac{360^\circ}{m} \tag{1218}$$

. . . .

$$e_m = E_{m_n} \sin\left(n\delta + mn\frac{360^\circ}{m}\right) = E_{m_n} \sin n\delta \cos mn\frac{360^\circ}{m} + E_{m_n} \cos n\delta \sin mn\frac{360^\circ}{m}. \tag{1219}$$

These ordinates will add to zero except in the limiting case when $n/m = k$, a whole number, when the angle between successive ordinates becomes a multiple of 360 deg., or when

$$n\frac{360^\circ}{m} = k\ 360^\circ$$

and

$$\sin(n\delta + k\ 360^\circ) = \sin n\delta,$$

giving these ordinates the values

$$e_1 = E_{m_n} \sin(n\delta + k\ 360^\circ) = E_{m_n} \sin n\delta, \tag{1220}$$

$$e_2 = E_{m_n} \sin(n\delta + 2k\ 360^\circ) = E_{m_n} \sin n\delta \tag{1221}$$

. . . .

$$e_m = E_{m_n} \sin(n\delta + mk\ 360^\circ) = E_{m_n} \sin n\delta. \tag{1222}$$

Hence,

$$\Sigma e = mE_{m_n} \sin n\delta, \tag{1223}$$

or, in this limiting case, the sum of the m ordinates is equal to m times the ordinate at the starting point, $\alpha = \delta$. Figure 310 shows three complete cycles divided into m parts where $k = \frac{3}{4}$, and Fig. 311 shows the same relation when $k = 1.00$.

To apply the above principles to the analysis of a wave, assume the wave to have the ordinary Fourier's form,

$$e = A_1 \sin \alpha + A_3 \sin 3\alpha + A_5 \sin 5\alpha + \ . \ . \ . \ + A_n \sin n\alpha + B_1 \cos \alpha + B_3 \cos 3\alpha + B_5 \cos 5\alpha + \ . \ . \ . \ + B_n \cos n\alpha. \qquad (1166)$$

If, starting at any point $\alpha = \delta$, three ordinates are erected at intervals of 120 deg., *i.e.*, at $(\delta + 120°)$, $(\delta + 240°)$, and $(\delta + 360°)$, thus dividing the wave into three equal parts, the value of the ordinates will be

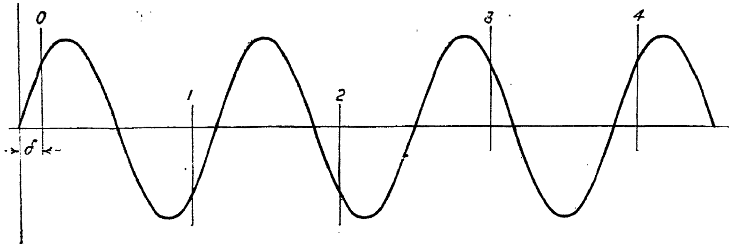

FIG. 310.—A sine wave showing three complete cycles cut by ordinates spaced so that $k = 3/4$.

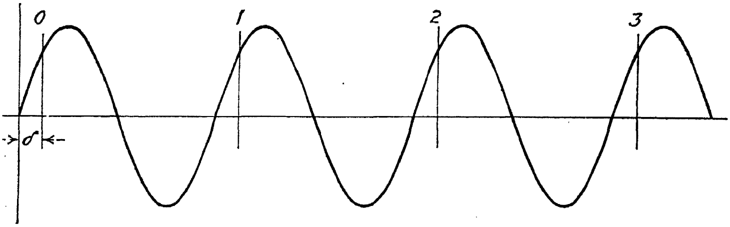

FIG. 311.—A sine wave showing three complete cycles cut by ordinates spaced so that $k = 1$ on the basis of the wave shown being a third harmonic.

$$e_1 = A_1 \sin (\delta + 120°) + A_3 \sin (3\delta + 360°) + A_5 \sin (5\delta + 240°) + \ . \ . \ . \ + A_9 \sin (9\delta + 360°) + \ . \ . \ . \ + A_n \sin (n\delta + n120°) + B_1 \cos (\delta + 120°) + B_3 \cos (3\delta + 360°) + B_5 \cos (5\delta + 240°) + \ . \ . \ . \ + B_9 \cos (9\delta + 360°) + \ . \ . \ . \ + B_n \cos (n\delta + n\,120°), \qquad (1224)$$

$$e_2 = A_1 \sin (\delta + 240°) + A_3 \sin (3\delta + 360°) + A_5 \sin (5\delta + 120°) + \ . \ . \ . \ + A_9 \sin (9\delta + 360°) + \ . \ . \ . \ + A_n \sin (n\delta + n240°) + B_1 \cos (\delta + 240°) + B_3 \cos (3\delta + 360°) + B_5 \sin (5\delta + 120°) + \ . \ . \ . \ + B_9 \cos (9\delta + 360°) + \ . \ . \ . \ + B_n \cos (n\delta + n\,240°), \qquad (1225)$$

and

$$e_3 = A_1 \sin (\delta + 360°) + A_3 \sin (3\delta + 360°) + A_5 \sin (5\delta + 360°) + \dots + A_9 \sin (9\delta + 360°) + \dots + A_n \sin (n\delta + n360°) + B_1 \cos (\delta + 360°) + B_3 \cos (3\delta + 360°) + B_5 \cos (5\delta + 360°) + \dots + B_9 \cos (9\delta + 360°) + \dots B_n \cos (n\delta + n\,360°). \quad (1226)$$

The sum of these ordinates becomes

$$e_1 + e_2 + e_3 = 3A_3 \sin 3\delta + 3B_3 \cos 3\delta + 3A_9 \sin 9\delta + 3B_9 \cos 9\delta + 3A_{15} \sin 15\delta + 3B_{15} \cos 15\delta + \dots, \quad (1227)$$

all harmonics disappearing except the third and multiples of the third. In order to secure values of the coefficients A_3, B_3, etc., it is necessary to make the values of sin 3δ, cos 3δ, etc., either unity or zero. Choosing the value of δ such that $\alpha = \delta = 0$, erecting the ordinates at 120, 240, and 360 deg., and since cos 0 deg. = 1.00,

$$e_{120°} + e_{240°} + e_{360°} = 3B_3 + 3B_9 + 3B_{15} + \dots, \quad (1228)$$

the A terms dropping out. Again, taking a value of $\alpha = \delta = {}^{90}\!/_{3} = 30$ deg., erecting ordinates at 30, 150, and 270 deg.,

$$e_{30°} + e_{150°} + e_{270°} = 3A_3 - 3A_9 + 3A_{15} - 3A_{21} + \dots, \quad (1229)$$

since sin 90 deg. = 1, sin 270 deg. = −1, etc. It is evident that the preceding proof must be correct since all component sine waves of the third harmonic and each odd multiple of the third harmonic will pass through zero each 120 deg. of the fundamental scale, beginning at the chosen starting point of the analysis. Also each cosine wave of the third harmonic will have a maximum positive or negative value at zero and each 120 deg. of the fundamental scale. The cosine waves will pass through zero and the sine waves through maxima at 90 deg. from the starting point on the scale of the triple or 30 deg. on the scale of the fundamental, and each 120 deg. of the fundamental from this point or at 30, 150, and 270 deg., as chosen.

In the same manner, if the wave is divided into five parts by ordinates spaced ${}^{360}\!/_{5} = 72°$ apart,

$$e_1 + e_2 + e_3 + e_4 + e_5 = 5A_5 \sin 5\delta + 5B_5 \cos 5\delta + 5A_{15} \sin 15\delta + 5B_{15} \cos 15\delta + \dots \quad (1230)$$

and if $\alpha = \delta = 0$, or if ordinates are erected at 72, 144, 216, 288, and 360 deg.,

$$e_{72^\circ} + e_{144^\circ} + e_{216^\circ} + e_{288^\circ} + e_{360^\circ} = 5B_5 + 5B_{15} + \ . \ . \ . \ , \qquad (1231)$$

and if $\delta = {}^{90}\!/_{5} = 18^\circ$, $\sin 5\delta = 1$, $\sin 15\delta = -1$, etc.,

$$e_{18^\circ} + e_{90^\circ} + e_{162^\circ} + e_{234^\circ} + e_{306^\circ} = 5A_5 - 5A_{15} + \ . \ . \ . \ . \qquad (1232)$$

This will give the coefficients of the fifth harmonic and each higher odd multiple of the fifth harmonic.

Putting $\alpha = 0$ deg. in Eq. (1166),

$$e_0 = B_1 + B_3 + B_5 + \ . \ . \ . \ + B_n, \qquad (1233)$$

or

$$B_1 = e_0 - (B_3 - \ B_5 + \ . \ . \ . \ + B_n), \qquad (1234)$$

and when $\alpha = 90$ deg., $\sin \alpha$ 1, $\sin 3\alpha = -1$, $\sin 5\alpha = 1$, etc., and

$$e_{90^\circ} = A_1 - A_3 + A_5 - A_7 + \ . \ . \ . \ A_n, \qquad (1235)$$

or

$$A_1 = e_{90^\circ} + A_3 - A_5 + A_7 - A_9 + \ . \ . \ . \ A_n. \qquad (1236)$$

From Eqs. (1234) and (1236), it is seen that the values of the coefficients of the first harmonic may be found only after all the other coefficients have been determined. It is also seen that none of the coefficients for the lower harmonics may be found by this method until those for all the higher odd multiples are known. It is advisable, therefore, to determine by inspection or test the highest harmonics of appreciable magnitude and to analyze for the constants in the reverse order of their frequencies. Since, in most cases, the magnitude of the harmonic will decrease with the increase in its order or frequency, it is seldom necessary to carry the analysis farther than the ninth harmonic.

Example 108

Analyze the wave shown in Fig. 307 by the Fischer-Hinnen method.

When the two lobes of the wave are symmetrical, the half lobe may be divided into $2n$ parts and the ordinates at angles greater than 180 deg. read at (α − 180 deg.) but with the sign reversed. Thus, for the third harmonic, the values may be read at 0, 120, and 60 deg. instead of at 120, 240, and 360 deg. The ordinates as read from the curve of Fig. 307 are shown in Table XXI.

The results of the analysis by this method are seen to compare favorably with those obtained by the Fourier series form of solution for the same wave. The accuracy of this method is limited by the accuracy of locating and measuring the different ordinates on the wave. Since the values of A and B depend upon the difference between values in the table that are very nearly equal, a slight mistake in measuring any of the ordinates of the wave may cause a considerable percentage error in the values of these coefficients. Ordinary errors of measurement will have the greatest effect on the values of the coefficients for the lower harmonics, where only a few ordinates are used, as the errors do not compensate, as they tend to do for the higher harmonics. The percentage error, however, will not be greater since the maximum value of the lower harmonics is much greater than that of

TABLE XXI.—ANALYSIS OF THE WAVE OF FIG. 307 BY THE FISCHER-HINNEN METHOD

Fifteenth harmonic

α	$i+$	$i-$	α	$i+$	$i-$
24	72.0	.....	6	62.3	
48	86.3	.....	30	75.5	
72	98.0	.....	54	89.5	
96	97.2	.....	78	99.8	
120	34.5	.....	102	89.5	
144	.....	28.0	126	10.0	
168	.....	51.7	150	.....	35.8
192	.....	65.3	174	.....	55.8
216	.....	79.0	198	.....	68.8
240	.....	92.4	222	.....	82.7
264	.....	100.0	246	....	95.6
288	.....	75.5	270	.....	99.7
312	7.0	.....	294	.....	55.5
336	41.9	.....	318	19.0	
360	59.5	.....	342	47.0	
...	496.4	491.9	...	492.6	493.9

$$B_{15} = \frac{496.4 - 491.9}{15} = 0.300. \qquad A_{15} = \frac{492.6 - 493.9}{15} = -0.087$$

TABLE XXI.—ANALYSIS OF THE WAVE OF FIG. 307 BY THE FISCHER-HINNEN METHOD.—(*Continued*)

Thirteenth harmonic

α	$i+$	$i-$	α	$i+$	$i-$
28	74.0	.....	7	63.0	
55	90.0	.....	35	78.5	
83	100.0	.....	62	93.5	
111	65.0	.....	90	99.7	
138	.....	19.0	118	40.5	
166	.....	50.5	145	.....	29.0
194	.....	66.5	173	.....	55.0
222	.....	82.7	201	.....	70.0
249	.....	97.0	229	.....	86.7
277	.....	96.5	256	.....	99.2
305	.....	18.0	284	.....	85.5
332	38.0	.....	312	6.0	
360	59.5	.....	339	44.6	
...	426.5	430.2	...	425.8	425.4

$$B_{13} = \frac{426.5 - 430.2}{13} = -0.284. \qquad A_{13} = \frac{425.8 - 425.4}{13} = 0.03.$$

Eleventh harmonic

α	$i+$	$i-$	α	$i+$	$i-$
33	77.0	.....	8	63.5	
65	95.2	.....	41	82.0	
98	95.5	.....	74	98.6	
131	.....	4.0	106	80.0	
164	.....	48.5	139	.....	20.0
196	.....	67.5	172	.....	54.3
229	.....	86.7	204	.....	72.0
262	.....	99.9	237	.....	91.0
294	.....	55.5	270	.....	99.7
327	32.0	.....	303	.....	25.0
360	59.5	.....	335	41.0	
...	359.2	362.1	...	365.1	362.0

$$B_{11} = \frac{359.2 - 362.1}{11} = -0.26. \qquad A_{11} = \frac{365.1 - 362.0}{11} = 0.29.$$

TABLE XXI.—ANALYSIS OF THE WAVE OF FIG. 307 BY THE FISCHER-HINNEN METHOD.—(*Continued*)

Ninth harmonic

α	$i+$	$i-$	α	$i+$	$i-$
40	81.3	.....	10	64.3	
80	99.8	.....	50	87.4	
120	34.5	.....	90	99.7	
160	.....	45.5	130	.....	1.5
200	.....	69.5	170	.....	53.0
240	.....	92.3	210	.....	75.3
280	.....	93.0	250	.....	97.0
320	21.7	.....	290	.....	67.5
360	59.5	.....	330	35.7	
...	296.8	300.3	.....	287.0	294.3

$B_9 = \frac{296.8 - 300.3}{9} = -0.39.$ $A_9 = \frac{287.0 - 294.3}{9} = -0.81.$

Seventh harmonic

α	$i+$	$i-$	α	$i+$	$i-$
51	88.1	.....	13	66.0	
103	87.5	.....	64	95.0	
154	.....	40.3	116	51.0	
206	.....	73.0	167	.....	51.0
257	.....	99.5	219	.....	80.7
308	.....	3.0	270	.....	99.7
360	59.5	.....	320	25.2	
...	235.1	215.8	...	237.2	231.4

$B_7 = \frac{235.1 - 215.8}{7} = 2.76.$ $A_7 = \frac{237.2 - 231.4}{7} = 0.83.$

TABLE XXI.—ANALYSIS OF THE WAVE OF FIG. 307 BY THE FISCHER-HINNEN METHOD.—(*Continued*)

Fifth harmonic

α	$i+$	$i-$	α	$i+$	$i-$
72	97.8	.....	18	68.8	
144	.....	28.0	90	99.7	
216	.....	79.0	162	.....	47.0
288	.....	75.0	234	.....	89.5
360	59.5	.....	306	.....	10.0
...	157.3	182.0	...	168.5	146.5

$$B_5 + B_{15} = \frac{157.3 - 182.0}{5} = -4.95. \quad A_5 - A_{15} = \frac{168.5 - 146.5}{5} = 4.4$$

$$B_5 = -4.95 - 0.30 = -5.25. \quad A_5 = 4.40 - 0.09 = 4.31.$$

Third harmonic

α	$i+$	$i-$	α	$i+$	$i-$
120	34.5	....	30	75.4	
240	....	92.3	150	....	35.8
360	59.5	....	270	....	99.7
	94.0	92.3		75.4	135.5

$$B_3 + B_9 + B_{15} = \frac{94 - 92.3}{3} = 0.56.$$

$$B_3 = 0.56 + 0.39 - 0.30 = 0.65.$$

$$A_3 - A_9 + A_{15} = \frac{75.4 - 135.5}{3} = -20.03.$$

$$A_3 = -20.03 - 0.81 + 0.09 = -20.75.$$

Fundamental Harmonic

$\alpha = 0. \quad i_0 = 59.5. \quad \alpha = 90. \quad i_{90} = 99.7.$

$i_0 = B_1 + B_3 + B_5 + B_7 + \ . \ . \ . \ + B_{15}.$

$B_1 = 59.5 - 0.65 + 5.25 - 2.76 + 0.39 + 0.26 + 0.28 - 0.30 = 61.97.$

$i_{90} = A_1 - A_3 + A_5 - A_7 + \ . \ . \ . \ A_{15}.$

$A_1 = 99.7 - 20.75 - 4.31 + 0.83 + 0.81 + 0.29 - 0.03 - 0.09 = 76.45.$

TABLE XXI.—ANALYSIS OF THE WAVE OF FIG. 307 BY THE FISCHER-HINNEN METHOD.—(*Continued*)

$B_1 = 61.97,$	$A_1 = 76.45,$	$C_1 = \sqrt{A_1^2 + B_1^2}$	$= 98.3,$	$\theta_1 = 39° 10'$
$B_3 = 0.65,$	$A_3 = -20.75,$	$C_3 = \sqrt{A_3^2 + B_3^2}$	$= 20.8,$	$\theta_3 = 178° 10'$
$B_5 = -5.25,$	$A_5 = 4.31,$	$C_5 = \sqrt{A_5^2 + B_5^2}$	$= 6.8,$	$\theta_5 = -50°40'$
$B_7 = 2.76,$	$A_7 = 0.83,$	$C_7 = \sqrt{A_7^2 + B_7^2}$	$= 2.9,$	$\theta_7 = 73° 15'$
$B_9 = -0.39,$	$A_9 = -0.81,$	$C_9 = \sqrt{A_9^2 + B_9^2}$	$= 0.895,$	$\theta_9 = 205° 50'$
$B_{11} = -0.26,$	$A_{11} = 0.29,$	$C_{11} = \sqrt{A_{11}^2 + B_{11}^2}$	$= 0.39,$	$\theta_{11} = -41°50'$
$B_{13} = -0.28,$	$A_{13} = 0.03,$	$C_{13} = \sqrt{A_{13}^2 + B_{13}^2}$	$= 0.281,$	$\theta_{13} = -83°50'$
$B_{15} = 0.30,$	$A_{15} = -0.09,$	$C_{15} = \sqrt{A_{15}^2 + B_{15}^2}$	$= 0.31,$	$\theta_{15} = 106° 45'.$

The wave is then expressed by

$$
\begin{aligned}
i &= 98.3 \sin (\alpha + 39° 10') - 20.8 \sin (3\alpha - 1° 50') \\
&+6.8 \sin (5\alpha - 50° 40') + 2.9 \sin (7\alpha + 73° 15') \\
&- 0.895 \sin (9\alpha + 25° 50') + 0.39 \sin (11\alpha - 41° 50') \\
&+ 0.281 \sin (13\alpha - 83° 50') - 0.31 \sin (15\alpha - 73° 15').
\end{aligned}
$$

the upper harmonics for ordinary waves. As given in the preceding analysis the values of the ordinates are taken at the nearest whole angle in degrees. If a device is available for reading the ordinates at equal divisions of the base, the accuracy is increased.

It is to be noted that this analysis has been carried to the fifteenth harmonic with the coefficients for the last three higher harmonics nearly equal. This was done so as to make the values of the components of the lower harmonics more accurate since the coefficients of these higher harmonics must be used in determining the coefficients for the first, third, and fifth harmonics. It is possible that any investigation of the seventeenth, nineteenth, and twenty-first harmonics would bring the values for the first and third as well as the fifth and seventh more nearly like those of the first method of analysis.

174. Electromotive Force Relations in a Circuit Having Resistance, Inductance, and Capacitance in Series with a Distorted Wave of Current Flowing.—To study the effect of resistance, inductance, and capacitance in series in a circuit with any wave of current flowing, assume that the current wave,

$$
i = I_{m_1} \sin (\alpha + \theta_1) + I_{m_3} \sin (3\alpha + \theta_3) + \cdots + I_{m_n} \sin (n\alpha + \theta_n), \quad (1237)
$$

is passed through the circuit. The resistance will produce a drop proportional to the resistance as

$$e_r = ri \tag{6}$$
$$= rI_{m_1}\ \sin\ (\alpha + \theta_1) + rI_{m_3}\ \sin\ (3\alpha + \theta_3) + \ .\ .\ .\ + rI_{m_n}\ \sin\ (n\alpha + \theta_n), \tag{1238}$$

having an effective value

$$E_r = \sqrt{\frac{(rI_{m_1})^2 + (rI_{m_3})^2 + \ .\ .\ .\ + (rI_{m_n})^2}{2}} = rI, \tag{1239}$$

the effective value of the current wave of Eq. (1237) being

$$I = \sqrt{\frac{I_{m_1}{}^2 + I_{m_3}{}^2 + \ .\ .\ .\ + I_{m_n}{}^2}{2}}. \tag{1240}$$

From Eq. (1238), it is seen that the resistance-drop wave has each component of the current wave multiplied by the resistance and that the emf wave must be of the same shape as, and in phase with, the current wave. The effective value of the resistance drop is equal to the resistance times the effective value of the current wave.

The inductance effect will be found by an application of Faraday's law to Eq. (1237) to obtain the induced emf wave, since

$$e_i = -L\frac{di}{dt}. \tag{57}$$

The wave of emf consumed by inductance from a passage of the current wave through the circuit will be

$$e_i' = L\frac{di}{dt}, \tag{67}$$

or

$$e_i' = \omega LI_{m_1} \cos\ (\alpha + \theta_1) + 3\omega LI_{m_3} \cos\ (3\alpha + \theta_3) + \ .\ .\ .\ + n\omega LI_{m_n} \cos\ (n\alpha + \theta_n) \tag{1241}$$
$$= xI_{m_1} \cos\ (\alpha + \theta_1) + 3xI_{m_3} \cos\ (3\alpha + \theta_3) + \ .\ .\ .\ + nxI_{m_n} \cos\ (n\alpha + \theta_n). \tag{1242}$$

It is seen from Eq. (1242) that each component wave of current through a coil produces its reactance drop independent of the presence of the other harmonics of current; that the reactance of the coil for each harmonic is equal to its reactance for the fundamental multiplied by the order of the harmonic; and that the emf-drop wave due to inductance is not of the same shape as, nor in phase with, the current wave. The effective value of this wave is

$$E_L = \sqrt{\frac{(xI_{m_1})^2 + (3xI_{m_3})^2 + \; . \; . \; . \; + (nxI_{m_n})^2}{2}} \qquad (1243)$$

$$= \sqrt{(xI_1)^2 + (3xI_3)^2 + \; . \; . \; . \; + (nxI_n)^2}. \qquad (1244)$$

The maximum value of each harmonic of the emf wave is amplified in respect to its relative value in the current wave by the order of that harmonic. The equivalent reactance of the coil is taken as

$$x' = \frac{E_L}{I} = \sqrt{\frac{(xI_1)^2 + (3xI_3)^2 + \; . \; . \; . \; + (nxI_n)^2}{I_1^2 + I_3^2 + \; . \; . \; . \; + I_n^2}}, \qquad (1245)$$

which is greater than the reactance of the coil for the fundamental frequency. The value of the equivalent reactance depends upon the order and the relative value of the harmonics; hence, it can have no definite value for a given coil independent of the shape and the fundamental frequency of the current wave. Its value is, however, independent of the relative phase shift of the component harmonics in the current wave.

From previous equations, the effect of capacitance may be studied by the application of the equation,

$$i = C\frac{de_c}{dt}, \qquad (112)$$

to the emf wave across the condenser, or, solving for the emf drop,

$$e_c = \frac{1}{C}\int idt \qquad (119)$$

may be applied to the equation of the current wave (Eq. (1237)) as

$$e_c = \frac{1}{C}\int [I_{m_1} \sin (\alpha + \theta_1) + I_{m_3} \sin (3\alpha + \theta_3) + \; . \; . \; . \; + I_{m_n} \sin (n\alpha + \theta_n)]d\alpha \qquad (1246)$$

$$= -\frac{I_{m_1}}{\omega C}\cos(\alpha+\theta_1) - \frac{I_{m_3}}{3\omega C}\cos(3\alpha+\theta_3) - \cdots - \frac{I_{m_n}}{n\omega C}\cos(n\alpha+\theta_n) \quad (1247)$$

$$= x_cI_{m_1}\cos(\alpha+\theta_1) + \frac{x_cI_{m_3}}{3}\cos(3\alpha+\theta_3) + \cdots + \frac{x_cI_{m_n}}{n}\cos(n\alpha+\theta_n) \quad (1248)$$

for the steady condition of the circuit after the starting transient terms have died away. From Eqs. (1247) and (1248), it is seen that the emf-drop wave due to capacitance is similar in some respects to the emf-drop wave due to inductance, each harmonic of emf being 90 deg. out of phase with its corresponding harmonic of current; however, the shift of 90 deg. is in the opposite direction, being 90 deg. behind the current instead of 90 deg. ahead. In other words, each harmonic of emf drop due to capacitance is 180 deg. out of phase with the corresponding harmonic of emf drop due to the inductance. It is also seen that each harmonic of the emf drop due to capacitance is smaller in relative magnitude than the corresponding harmonic in the current wave. The effective value of the emf drop due to capacitance is

$$E_c = \sqrt{(x_cI_1)^2 + \left(\frac{x_cI_3}{3}\right)^2 + \cdots + \left(\frac{x_cI_n}{n}\right)^2}, \quad (1249)$$

and the equivalent reactance is

$$x_c' = \frac{E_c}{I} = \sqrt{\frac{(x_cI_1)^2 + \left(\frac{x_cI_3}{3}\right)^2 + \cdots + \left(\frac{x_cI_n}{n}\right)^2}{I_1^2 + I_3^2 + \cdots + I_n^2}}, \quad (1250)$$

which must be less than the reactance of the condenser for the fundamental frequency. The value of this equivalent reactance depends also upon the order and the relative value of the harmonics; hence, it can have no definite value for any particular condenser independent of the shape of the current wave. As for inductive reactance, the value of the equivalent reactance is independent of the relative phase angle or shift of the component harmonics.

When the effects of resistance, inductance, and capacitance are combined into the same equation by the application of the fundamental equation,

$$e_o = e_r + e_L + e_c \tag{124}$$

$$= Ri + L\frac{di}{dt} + \frac{1}{C}\int idt, \tag{125}$$

and passing the current wave of Eq. (1237) through this circuit,

$$\begin{aligned} e_o = {} & rI_{m_1} \sin (\alpha + \theta_1) + x_L I_{m_1} \cos (\alpha + \theta_1) + x_c I_{m_1} \cos (\alpha + \theta_1) \\ & + rI_{m_3} \sin (3\alpha + \theta_3) + 3x_L I_{m_3} \cos (3\alpha + \theta_3) + \frac{x_c}{3} I_{m_3} \cos \\ & (3\alpha + \theta_3) + \ . \ . \ . + rI_{m_n} \sin (n\alpha + \theta_n) + nx_L I_{m_n} \cos \\ & (n\alpha + \theta_n) + \frac{x_c}{n} I_{m_n} \cos (n\alpha + \theta_n). \end{aligned} \tag{1251}$$

Consolidating the terms in Eq. (1251),

$$\begin{aligned} e_0 = {} & \sqrt{r^2 + (x_L + x_c)^2} I_m \sin (\alpha + \theta_1 + \beta_1) + \\ & \sqrt{r^2 + \left(3x_L + \frac{x_c}{3}\right)^2} I_{m_3} \sin (3\alpha + \theta_3 + \beta_3) + \ . \ . \ . \\ & + \sqrt{r^2 + \left(nx_L + \frac{x_c}{n}\right)^2} I_{m_n} \sin (n\alpha + \theta_n + \beta_n) \end{aligned} \tag{1252}$$

$$\begin{aligned} = {} & \sqrt{r^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2} I_{m_1} \sin (\alpha + \theta_1 + \beta_1) + \\ & \sqrt{r^2 + \left(6\pi fL - \frac{1}{6\pi fC}\right)^2} I_{m_3} \sin (3\alpha + \theta_3 + \beta_3) + \ . \ . \ . \\ & + \sqrt{r^2 + \left(2n\pi fL - \frac{1}{2n\pi fC}\right)^2} I_{m_n} \sin (n\alpha + \theta_n + \beta_n), \end{aligned} \tag{1253}$$

$$\begin{aligned} e_o = {} & z_1 I_{m_1} \sin (\alpha + \theta_1 + \beta_1) + z_3 I_{m_3} \sin (3\alpha + \theta_3 + \beta_3) \\ & + \ . \ . \ . + z_n I_{m_n} \sin (n\alpha + \theta_n + \beta_n), \end{aligned} \tag{1254}$$

where

$$z_1 = \sqrt{r^2 + (x_L + x_c)^2} = \sqrt{r^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}$$

. . . .

$$z_n = \sqrt{r^2 + \left(nx_L + \frac{x_c}{n}\right)^2} = \sqrt{r^2 + \left(2n\pi fL - \frac{1}{2n\pi fC}\right)^2}, \tag{1255}$$

and

$$\tan \beta_1 = \frac{x_L + x_c}{r} = \frac{2\pi f L - \dfrac{1}{2\pi f C}}{r}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tan \beta_n = \frac{n x_L + \dfrac{x_c}{n}}{r} = \frac{2n\pi f L - \dfrac{1}{2n\pi f C}}{r}. \qquad (1256)$$

The effective value of this emf is

$$E_o = \sqrt{(z_1 I_1)^2 + (z_3 I_3)^2 + \ . \ . \ . \ + (z_n I_n)^2}, \qquad (1257)$$

and the equivalent impedance is

$$z' = \frac{E_o}{I} = \sqrt{\frac{(z_1 I_1)^2 + (z_3 I_3)^2 + \ . \ . \ . \ (z_n I_n)^2}{I_1^2 + I_3^2 + \ . \ . \ . \ I_n^2}}, \qquad (1258)$$

which depends upon the relative values of r, x_L, and x_c as well as upon the relative effective values of the component current harmonics. The power factor of the circuit is

$$P.F._o = \frac{I^2 r}{E_o I} = \frac{E_r}{E_o} = \frac{r}{z'}, \qquad (1259)$$

and the equivalent phase angle is found from

$$\theta_o = \cos^{-1} \frac{E_r}{E_o} = \cos^{-1} \frac{r}{z'}. \qquad (1260)$$

175. Current Wave Produced by a Distorted Wave of Emf Impressed on a Circuit Having R, L, and C in Series.—When a wave of emf represented by Eq. (1177) is impressed upon a circuit having R, L, and C in series, the current wave flowing will have the form

$$i = I_{m_1} \sin (\alpha + \gamma_1) + I_{m_3} \sin (3\alpha + \gamma_3) + \ . \ . \ . \\ + I_{m_n} \sin (n\alpha + \gamma_n), \qquad (1261)$$

the current being taken as the reference wave. Passing this current through the circuit with r, L, and C in series, the emf-drop wave will be

$$e = rI_{m_1} \sin(\alpha + \gamma_1) + \left(\omega L - \frac{1}{\omega C}\right) I_{m_1} \cos(\alpha + \gamma_1) +$$
$$rI_{m_3} \sin(3\alpha + \gamma_3) + \left(3\omega L - \frac{1}{3\omega C}\right) I_{m_3} \cos(3\alpha + \gamma_3) + \dots$$
$$rI_{m_n} \sin(n\alpha + \gamma_n) + \left(n\omega L - \frac{1}{n\omega C}\right) I_{m_n} \cos(n\alpha + \gamma_n) \qquad (1262)$$
$$= \sqrt{r^2 + (x_L + x_c)^2} I_{m_1} \sin(\alpha_1 + \gamma_1 + \beta_1) +$$
$$\sqrt{r^2 + \left(3x_L + \frac{x_c}{3}\right)^2}\, I_{m_3} \sin(3\alpha + \gamma_3 + \beta_3) + \dots$$
$$\sqrt{r^2 + \left(nx_L + \frac{x_c}{n}\right)^2}\, I_{m_n} \sin(n\alpha + \gamma_n + \beta_n) \qquad (1263)$$
$$= z_1 I_{m_1} \sin(\alpha + \gamma_1 + \beta_1) + z_3 I_{m_3} \sin(3\alpha + \gamma_3 + \beta_3) +$$
$$\dots\ z_n I_{m_n} \sin(n\alpha + \gamma_n + \beta_n), \qquad (1264)$$

where

$$z_1 = \sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{r^2 + (x_L + x_c)^2},$$
$$z_n = \sqrt{r^2 + \left(n\omega L - \frac{1}{n\omega C}\right)^2} = \sqrt{r^2 + \left(nx_L + \frac{x_c}{n}\right)^2}, \qquad (1255)$$
$$\tan\beta_1 = \frac{\omega L - \frac{1}{\omega C}}{r} = \frac{x_L + x_c}{r},$$
$$\dots$$
$$\tan\beta_n = \frac{nx_L + \frac{x_c}{n}}{r} = \frac{n\omega L - \frac{1}{n\omega C}}{r}. \qquad (1256)$$

Equation (1264) must be identical with the impressed emf Eq. (1177), since there must be a dynamic balance of emf components at every instant. Hence,

$$E_{m_1} = z_1 I_{m_1}; \quad I_{m_1} = \frac{E_{m_1}}{z_1}$$
$$\dots$$
$$E_{m_n} = z_n I_{m_n}; \quad I_{m_n} = \frac{E_{m_n}}{z_n}, \qquad (1265)$$
$$\alpha + \gamma_1 + \beta_1 = \alpha + \theta_1; \quad \gamma_1 = \theta_1 - \beta_1$$
$$\dots$$
$$n\alpha + \gamma_n + \beta_n = n\alpha + \theta_n; \quad \gamma_n = \theta_n - \beta_n. \qquad (1266)$$

The equation of current must become

$$i = \frac{E_{m_1}}{z_1}\sin(\alpha + \theta_1 - \beta_1) + \frac{E_{m_3}}{z_3}\sin(3\alpha + \theta_3 - \beta_3) + \dots \frac{E_{m_n}}{z_n}\sin(n\alpha + \theta_n - \beta_n), \quad (1267)$$

the effective value being

$$I = \sqrt{\left(\frac{E_1}{z_1}\right)^2 + \left(\frac{E_3}{z_3}\right)^2 + \dots \left(\frac{E_n}{z_n}\right)^2}. \quad (1268)$$

The equivalent impedance of each branch is

$$z_o' = \sqrt{\frac{E_1^2 + E_3^2 + \dots E_n^2}{\left(\frac{E_1}{z_1}\right)^2 + \left(\frac{E_3}{z_3}\right)^2 + \dots \left(\frac{E_n}{z_n}\right)^2}}, \quad (1269)$$

and the power factor

$$P.F._o = \cos\theta_o = \frac{I^2 r}{EI} = \frac{Ir}{E}, \quad (1270)$$

the equivalent phase angle being $\cos^{-1}(P.F._o)$.

176. Resonance in a Circuit Containing R, L, and C in Series with a Distorted Wave of Impressed Emf.—If an emf expressed by Eq. (1177),

$$e = E_{m_1}\sin(\alpha + \theta_1) + E_{m_3}\sin(3\alpha + \theta_3) + \dots + E_{m_n}\sin(n\alpha + \theta_n), \quad (1177)$$

be impressed on a circuit having a constant resistance and inductance and a variable capacitance the current flowing at any instant will be given by Eq. (1267) as

$$i = \frac{E_{m_1}}{\sqrt{r^2 + (x_L + x_c)^2}}\sin(\alpha + \theta_1 - \beta_1) + \frac{E_{m_3}}{\sqrt{r^2 + \left(3x_L + \frac{x_c}{3}\right)^2}}\sin(3\alpha + \theta_3 - \beta_3) + \dots + \frac{E_{m_n}}{\sqrt{r^2 + \left(nx_L + \frac{x_c}{n}\right)^2}}\sin(n\alpha + \theta_n - \beta_n). \quad (1267)$$

With some value of C the value of x_c will be such that

$$nx_L = \frac{-x_c}{n},$$

or a condition of resonance will exist for the nth harmonic. The effective value of I_n will, therefore, be

$$I_n = \frac{E_n}{r},$$

and, if r is small relative to x_c and x_L, the nth harmonic of current will be very prominent in the total current wave, producing also a large nth harmonic in the voltage drops across the condenser as well as across the inductance.

A slight increase in C will now make $nx_L > -x_c/n$ and thus produce a marked reduction in the nth harmonic of current. As the other components of the current wave will be only slightly affected by a small increase in C, the effective value of the total current will be decreased at first. As C is further increased, I_n will continue to decrease but all the other components of current will increase, the total value of the current reaching a minimum and then increasing with increasing values of C until, at some larger value of capacitance, a resonant condition will exist for the harmonic of the next lower order. This harmonic of current will have an effective value of

$$I_{(n-1)} = \frac{E_{(n-1)}}{r}$$

and will be large relative to the other components of the current wave, producing a second maximum point in the total current, usually higher than for the nth harmonic. It will thus be seen that as C is increased the effective value of the current will reach maxima at each of the values of C that produces resonance for some harmonic of emf. Thus a curve plotted between capacitance and current will, in general, have as many maxima as there are harmonics in the impressed emf wave. Since the magnitudes of the harmonics in the emf wave usually decrease as the order of the harmonic increases, the magnitudes of the maxima of current will increase as C is decreased, and the current will reach its greatest value when $x_L = -x_c$, or at the point of resonance for the fundamental frequency.

TABLE XXII, EXAMPLE 109

$$e = \sqrt{2}\,150 \sin 377t + \sqrt{2}\,50 \sin 1{,}131t + \sqrt{2}\,30 \sin 1{,}885t.$$

$R = 10$ ohms, $L = 0.0796$ henry, $x_{L_1} = 30$ ohms, $x_{L_3} = 90$ ohms, $x_{L_5} = 150$ ohms.

Capacitance (microfarad)	133	106	88.5	66.4	26.6	13.3	11.05	9.83	8.85	5.31	3.54	2.65
$x_{c_1} = \frac{-10^6}{377C}$	−20	−25	−30	−40	−100	−200	−240	−270	−300	−500	−750	−1,000
$x_{c_3} = \frac{1}{3} x_{c_1}$	−6.67	−8.33	−10	−13.3	−33.3	− 66.7	−80	−90	−100	−167	−250	−333
$x_{c_5} = \frac{1}{5} x_{c_1}$	−4	−5	−6	−8	−20	−40	−48	−54	−60	−100	−150	−200
$x_1 = x_{L_1} + x_{c_1}$	10	5	0	−10	−70	−170	−210	−240	−270	−470	−720	−970
$x_3 = x_{L_3} + x_{c_3}$	83.3	81.7	80.0	76.7	56.7	23.3	10	0	−10	−77	−160	−243
$x_5 = x_{L_5} + x_{c_5}$	146	145	144	142	130	110	102	96	90	50	0	−50
$z_1 = \sqrt{r^2 + x_1^2}$	14.14	11.2	10.0	14.14	70.7	170	210	240	270	470	720	970
$z_3 = \sqrt{r^2 + x_3^2}$	83.9	82.3	80.6	77.2	68.3	34.8	14.14	10.0	14.14	77.2	160	243
$z_5 = \sqrt{r^2 + x_5^2}$	146.5	145.3	144.5	142.4	130.7	110.4	102.7	96.5	90.5	51.0	10.0	51.0
$I_1 = \frac{E_1}{z_1}$	10.6	13.4	15.0	10.6	2.12	0.882	0.715	0.625	0.555	0.319	0.208	0.155
$I_3 = \frac{E_3}{z_3}$	0.595	0.608	0.62	0.648	0.730	1.44	3.54	5.00	3.54	0.648	0.312	0.206
$I_5 = \frac{E_5}{z_5}$	0.205	0.207	0.208	0.211	0.229	0.272	0.292	0.311	0.332	0.588	3.00	0.588
$I = \sqrt{I_1^2 + I_3^2 + I_5^2}$	10.77	13.54	15.13	10.8	2.24	1.71	3.61	5.05	3.58	0.933	3.02	0.642
$E_{L_1} = I_1 x_{L_1}$	318	402	450	318	63.6	26.5	21.5	18.75	16.65	9.57	6.24	4.65
$E_{L_3} = I_3 x_{L_3}$	53.6	54.8	56.8	58.3	65.7	130	319	450	319	58.3	28.1	18.5
$E_{L_5} = I_5 x_{L_5}$	30.8	31.1	31.3	31.7	34.8	40.8	43.8	46.7	50	88.5	450	88.5
$E_L = \sqrt{E^2_{L_1} + E^2_{L_3} + E^2_{L_5}}$	324	408	455	323	97.5	160	323	454	324	107	451	90.6
$E_{C_1} = I_1 x_{c_1}$	212	338	453	416	212	176.4	172	169	167	159.5	156	155
$E_{C_3} = I_3 x_{c_3}$	3.97	5.07	6.2	8.62	24.4	94.1	283	450	354	108.2	78	68.7
$E_{C_5} = I_5 x_{c_5}$	0.82	1.04	1.25	1.69	4.58	10.9	14.0	16.8	19.9	58.8	450	117.6
$E_C = \sqrt{E^2_{c1} + E^2_{c3} + E_{c5}^2}$	212	338	453	416	212.5	200	331	481	392	202	482	206

From the above analysis, it is seen that a distorted emf wave may produce abnormal voltages as well as currents in a circuit that is in resonance for any harmonic higher than the fundamental wave of the circuit as well as when a sine wave is impressed and the circuit is "tuned" for this frequency. This is of considerable importance in transmission-line operation where lightning or other transient conditions may superpose a higher frequency emf on the normal voltage of the line.

Example 109

An emf expressed by

$$e = 150\sqrt{2} \sin 377t + 50\sqrt{2} \sin 1{,}131t + 30\sqrt{2} \sin 1{,}885t$$

is impressed on a circuit having a resistance of 10 ohms, an inductance of 0.0796 henry, and a variable capacitance. Calculate the effective value of the current for the different values of capacitance.

$$x_{c1} = \frac{-1}{377C}, \; x_{c3} = \frac{-1}{1{,}131C}, \; x_{c5} = \frac{-1}{1{,}885C},$$

$$x_{L1} = 377 \times 0.0796 = 30, \; x_{L3} = 1{,}131 \times 0.0796 = 90, \; x_{L5} = 1{,}885 \times 0.0796 = 150.$$

The effective value of the harmonics of current will be

$$I_1 = \frac{150}{\sqrt{10^2 + \left(30 - \frac{1}{377C}\right)^2}},$$

$$I_3 = \frac{50}{\sqrt{10^2 + \left(90 - \frac{1}{1{,}131C}\right)^2}},$$

$$I_5 = \frac{30}{\sqrt{10^2 + \left(150 - \frac{1}{1{,}885C}\right)^2}},$$

and

$$I_o = \sqrt{I_1^2 + I_3^2 + I_5^2}.$$

The values of I_o for the different values of C are tabulated in Table XXII. Figure 312 shows the curve between I_o and C in which the maxima at the points of resonance for the different harmonics of current are evident.

177. Current Relations in a Circuit Containing Resistance, Inductance, and Capacitance in Parallel with Distorted Wave of Impressed Emf.

—In parallel circuits, the sum of the instantaneous values of the currents in the different branches is equal to the instantaneous value of the total current, or

$$i_O = i_A + i_B + i_C + i_D + \cdots \qquad (279)$$

The impressed-emf wave will be the same for all parallel branches. Representing the impressed emf by

$$e = {}_{m_1} \sin(\alpha + \theta_1) + E_{m_3} \sin(3\alpha + \theta_3) \ . \ . \ . + E_{m_n} \sin(n\alpha + \theta_n) \qquad (1177)$$

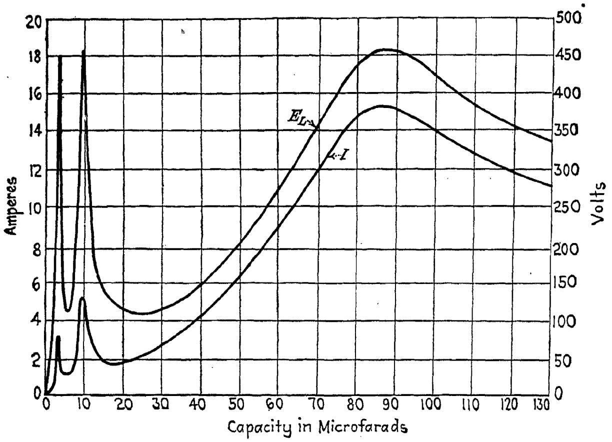

FIG. 312.—Resonant curves for a series circuit with an irregular wave of impressed emf.

and assuming the circuit to consist of the branches A, B, and C (Fig. 313), the current in branch A will be

$$i_A = \frac{e}{r} = \frac{E_{m_1}}{r} \sin(\alpha + \theta_1) + \frac{E_{m_3}}{r} \sin(3\alpha + \theta_3) + \ . \ . \ . \ + \frac{E_{m_n}}{r} \sin(n\alpha + \theta_n), \qquad (1271)$$

a current wave having harmonics of the same relative magnitude and in phase with the harmonics of the emf wave. The current in the path B will be found from

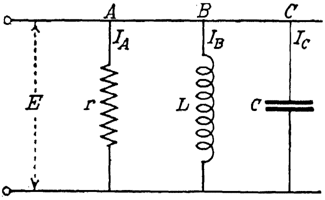

FIG. 313.—Resistance, inductance, and capacitance in parallel.

$$e_x = L\frac{di}{dt} \qquad (67)$$

or

$$i = \frac{1}{r}\int e_x dt \qquad (1272)$$

where $e = e_x$, since the impressed emf must overcome the induced emf at every instant. Hence,

$$i_B = -\frac{E_{m_1}}{\omega L}\cos(\alpha + \theta_1) - \frac{E_{m_3}}{3\omega L}\cos(3\alpha + \theta_3) - \cdots - \frac{E_{m_n}}{n\omega L}\cos(n\alpha + \theta_n) \quad (1273)$$

$$= -\frac{E_{m_1}}{x_L}\cos(\alpha + \theta_1) - \frac{E_{m_3}}{3x_L}\cos(3\alpha + \theta_3) \cdots - \frac{E_{m_n}}{nx_L}\cos(n\alpha + \theta_n). \quad (1274)$$

The current in path C will be

$$i_c = C\frac{de_c}{dt} = C\frac{de}{dt} \quad (112)$$

since $e = e_c$. Hence,

$$i_c = \omega CE_{m_1}\cos(\alpha + \theta_1) + 3\omega CE_{m_3}\cos(3\alpha + \theta_3) + \cdots + n\omega CE_{m_n}\cos(n\alpha + \theta_n) \quad (1275)$$

$$= -\frac{E_{m_1}}{x_c}\cos(\alpha + \theta_1) - \frac{3E_{m_3}}{x_c}\cos(3\alpha + \theta_3) - \cdots - \frac{nE_{m_n}}{x_c}\cos(n\alpha + \theta_n). \quad (1276)$$

The effective values of these three currents are

$$I_A = \sqrt{\left(\frac{E_1}{r}\right)^2 + \left(\frac{E_3}{r}\right)^2 + \cdots + \left(\frac{E_n}{r}\right)^2}, \quad (1277)$$

$$I_B = \sqrt{\left(\frac{E_1}{x_L}\right)^2 + \left(\frac{E_3}{3x_L}\right)^2 + \cdots + \left(\frac{E_n}{nx_L}\right)^2}, \quad (1278)$$

$$I_C = \sqrt{\left(\frac{E_1}{x_c}\right)^2 + \left(\frac{3E_c}{x_c}\right)^2 + \cdots + \left(\frac{nE_n}{x_c}\right)^2}. \quad (1279)$$

From Eq. (1278), it is seen that the effective value of the current through path B is affected by the reactance of the circuit at fundamental frequency, by the relative maximum value of each emf harmonic, and, inversely, by the order of the harmonic. From Eq. (1274), it is seen that each harmonic of emf produces a component harmonic of current in the coil of circuit B at 90 deg. behind the emf wave of that harmonic and independent of the other harmonics present in the emf wave.

From Eq. (1279), it is seen that the effective value of the current through the path C is affected by the reactance of the circuit at fundamental frequency, by the relative maximum values of each emf harmonic, and directly by the order of the harmonic. From Eq. (1276), it is seen that each harmonic of emf produces a component harmonic of current in the condenser 90 deg. ahead of the emf wave of that harmonic and independent of the other harmonics. It is also seen from Eqs. (1274) and (1276) that the current in path B is less distorted and in path C more distorted than the impressed-emf wave.

The total current taken by the three branches in parallel is then

$$\begin{aligned} i_0 = {} & \frac{E_{m_1}}{r} \sin (\alpha + \theta_1) - \frac{E_{m_1}}{x_L} \cos (\alpha + \theta_1) - \frac{E_{m_1}}{x_c} \cos (\alpha + \theta_1) \\ & + \frac{E_{m_3}}{r} \sin (3\alpha + \theta_3) - \frac{E_{m_3}}{3x_L} \cos (3\alpha + \theta_3) \\ & - \frac{3E_{m_3}}{x_c} \cos (3\alpha + \theta_3) + \cdots \\ & + \frac{E_{m_n}}{r} \sin (n\alpha + \theta_n) - \frac{E_{m_n}}{nx} \cos (n\alpha + \theta_n) \\ & - \frac{nE_{m_n}}{x_c} \cos (n\alpha + \theta_n) \qquad (1280) \end{aligned}$$

$$\begin{aligned} = {} & E_{m_1}\left[\frac{1}{r} \sin (\alpha + \theta_1) - \left(\frac{1}{x_L} + \frac{1}{x_c}\right) \cos (\alpha + \theta_1)\right] \\ & + E_{m_3}\left[\frac{1}{r} \sin (3\alpha + \theta_3) - \left(\frac{1}{3x_L} + \frac{3}{x_c}\right) \cos (3\alpha + \theta_3)\right] \\ & + \cdots + E_{m_n}\left[\frac{1}{r} \sin (n\alpha + \theta_n) \right. \\ & \left. - \left(\frac{1}{nx_L} + \frac{n}{x_c}\right) \cos (n\alpha + \theta_n)\right] \qquad (1281) \end{aligned}$$

$$\begin{aligned} = {} & E_{m_1} [g \sin (\alpha + \theta_1) + (b_L + b_c) \cos (\alpha + \theta_1)] \\ & + E_{m_3}\left[g \sin (3\alpha + \theta_3) + \left(\frac{b_L}{3} + 3b_c\right) \cos (3\alpha + \theta_3)\right] \\ & + \cdots + E_{m_n} \left[g \sin (n\alpha + \theta_n) \right. \\ & \left. + \left(\frac{b_L}{n} + nb_c\right) \cos (n\alpha + \theta_n) \right], \qquad (1282) \end{aligned}$$

$$i_0 = \sqrt{g^2 + (b_L + b_c)^2}E_{m_1} \sin (\alpha + \theta_1 + \beta_1) + \sqrt{g^2 + \left(\frac{b_L}{3} + 3b_c\right)^2} E_{m_3} \sin (3\alpha + \theta_3 + \beta_3) + \dots + \sqrt{g^2 + \left(\frac{b_L}{n} + nb_c\right)^2} E_{m_n} \sin (n\alpha + \theta_n + \beta_n), \quad (1283)$$

or

$$i_0 = y_1E_{m_1} \sin (\alpha + \theta_1 + \beta_1) + y_3E_{m_3} \sin (3\alpha + \theta_3 + \beta_3) + \dots + y_nE_{m_n} \sin (n\alpha + \theta_n + \beta_n), \quad (1284)$$

where

$$g = \frac{1}{r}, \quad (1285)$$

$$b_L = -\frac{1}{x_L}, \quad (1286)$$

$$b_c = -\frac{1}{x_c}, \quad (1287)$$

$$\tan \beta_1 = \frac{b_L + b_c}{g}$$

$$\dots\dots\dots$$

$$\tan \beta_n = \frac{\frac{b_L}{n} + nb_c}{g}. \quad (1288)$$

It is seen from Eq. (1283) that the total current is entirely different in wave shape from the impressed emf and also different from the currents in the branches A, B, or C. The effective value of the total current is

$$I_o = \sqrt{E_1^2[g^2 + (b_L + b_c)^2] + E_3^2\left[g^2 + \left(\frac{b_L}{3} + 3b_c\right)^2\right] + \dots + E_n^2\left[g^2 + \left(\frac{b_L}{n} + nb_c\right)^2\right]} \quad (1289)$$

or

$$I_0 = \sqrt{E_1^2y_1^2 + E_3^2y_3^2 + \dots + E_n^2y_n^2} \quad (1290)$$

which is neither the same as the sum of the effective values of the currents I_A, I_B, I_C . . . nor the square root of the sum of the squares of these values. The reason for this is that each component wave in B is partly neutralized by the corresponding harmonic of the current wave in C. This neutralization varies for the different frequencies, being greatest for the frequency which is most nearly that of resonance for the circuits *B* and *C* in parallel.

178. Distorted Wave of Emf Impressed on a Parallel Circuit Each Branch of Which Contains Resistance, Inductance, and Capacitance in Series.—The general parallel circuit will have *R*, *L*, and *C* in series in each branch. In such a circuit, with the emf of Eq. (1177) impressed, the current in each branch will be given by Eq. (1267) as

$$i = \frac{E_{m_1}}{z_1}\sin(\alpha + \theta_1 - \beta_1) + \frac{E_{m_3}}{z_3}\sin(3\alpha + \theta_3 - \beta_3) + \cdots + \frac{E_{m_n}}{z_n}\sin(n\alpha + \theta_n - \beta_n). \quad (1267)$$

Expanding Eq. (1267),

$$\begin{aligned} i = {} & E_{m_1}\frac{\cos\beta_1}{z_1}\sin(\alpha + \theta_1) - E_{m_1}\frac{\sin\beta_1}{z_1}\cos(\alpha + \theta_1) \\ & + E_{m_3}\frac{\cos\beta_3}{z_3}\sin(3\alpha + \theta_3) \\ & - E_{m_3}\frac{\sin\beta_3}{z_3}\cos(3\alpha + \theta_3) + \cdots \\ & + E_{m_n}\frac{\cos\beta_n}{z_n}\sin(n\alpha + \theta_n) \\ & - E_{m_n}\frac{\sin\beta_n}{z_n}\cos(n\alpha + \theta_n). \quad (1291) \end{aligned}$$

Replacing $\cos\beta_1$, $\sin\beta_1$, $\cos\beta_3$, $\sin\beta_3$ . . . $\cos\beta_n$, and $\sin\beta_n$ by their values in terms of the constants of the circuit,

$$\begin{aligned} \frac{\cos\beta_1}{z_1} &= \frac{\frac{r}{z_1}}{z_1} = \frac{r}{z_1^2} = g_1, \\ \frac{\sin\beta_1}{z_1} &= \frac{\frac{x_1}{z_1}}{z_1} = \frac{x_1}{z_1^2} = -b_1, \\ & \cdots\cdots\cdots\cdots \\ \frac{\cos\beta_n}{z_n} &= \frac{\frac{r}{z_n}}{z_n} = \frac{r}{z_n^2} = g_n, \end{aligned} \quad (1292)$$

$$\frac{\sin \beta_n}{z_n} = \frac{\frac{x_n}{z_n}}{z_n} = \frac{x_n}{z_n^{\,2}} = -b_n, \qquad (1293)$$

$$\begin{aligned} i = {} & g_1 E_{m_1} \sin (\alpha + \theta_1) + b_1 E_{m_1} \cos (\alpha + \theta_1) \\ & + g_3 E_{m_3} \sin (3\alpha + \theta_3) + b_3 E_{m_3} \cos (3\alpha + \theta_3) \\ & + \cdots + g_n E_{m_n} \sin (n\alpha + \theta_n) \\ & + b_n E_{m_n} \cos (n\alpha + \theta_n), \qquad (1294) \end{aligned}$$

where $g_1, g_3 \ldots g_n$ are the conductances of the circuit for the different frequencies and $b_1, b_3 \ldots b_n$ are the susceptances of the circuit for the given frequencies.

When several circuits, each containing either R, L, or C or combinations of these in series, are connected in parallel, the currents taken by the several branches may be added by adding the coefficients of the like sine and cosine components of the different harmonics of the wave. Thus, if the circuits A, B, C, etc., are in parallel (Fig. 314),

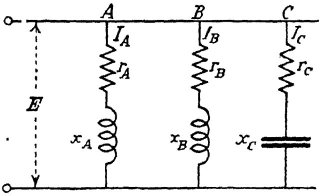

FIG. 314.—A circuit showing three general branches in parallel.

$$i_o = i_A + i_B + i_C + \cdots \qquad (279)$$

The equations of the currents in the several branches may be written from Eq. (1294),

$$\begin{aligned} i_A = {} & g_{A_1} E_{m_1} \sin (\alpha + \theta_1) + b_{A_1} E_{m_1} \cos (\alpha + \theta_1) \\ & + g_{A_3} E_{m_3} \sin (3\alpha + \theta_3) \\ & + b_{A_3} E_{m_3} \cos (3\alpha + \theta_3) + \cdots + g_{A_n} E_{m_n} \sin (n\alpha + \theta_n) \\ & + b_{A_n} E_{m_n} \cos (n\alpha + \theta_n), \qquad (1295) \end{aligned}$$

$$\begin{aligned} i_B = {} & g_{B_1} E_{m_1} \sin (\alpha + \theta_1) + b_{B_1} E_{m_1} \cos (\alpha + \theta_1) \\ & + g_{B_3} E_{m_3} \sin (3\alpha + \theta_3) \\ & + b_{B_3} E_{m_3} \cos (3\alpha + \theta_3) + \cdots + g_{B_n} E_{m_n} \sin (n\alpha + \theta_n) \\ & + b_{B_n} E_{m_n} \cos (n\alpha + \theta_n), \qquad (1296) \end{aligned}$$

$$\begin{aligned} i_C = {} & g_{C_1} E_{m_1} \sin (\alpha + \theta_1) + b_{C_1} E_{m_1} \cos (\alpha + \theta_1) \\ & + g_{C_3} E_{m_3} \sin (3\alpha + \theta_3) \\ & + b_{C_3} E_{m_3} \cos (3\alpha + \theta_3) + \cdots + g_{C_n} E_{m_n} \sin (n\alpha + \theta_n) \\ & + b_{C_n} E_{m_n} \cos (n\alpha + \theta_n). \qquad (1297) \end{aligned}$$

The total line current is

$$
\begin{aligned}
i_O = {} & (g_{A_1} + g_{B_1} + g_{C_1})E_{m_1} \sin(\alpha + \theta_1) \\
& + (b_{A_1} + b_{B_1} + b_{C_1})E_{m_1} \cos(\alpha + \theta_1) \\
& + (g_{A_3} + g_{B_3} + g_{C_3})E_{m_3} \sin(3\alpha + \theta_3) \\
& + (b_{A_3} + b_{B_3} + b_{C_3})E_{m_3} \cos(3\alpha + \theta_3) + \cdots \\
& + (g_{A_n} + g_{B_n} + g_{C_n})E_{m_n} \sin(n\alpha + \theta_n) \\
& + (b_{A_n} + b_{B_n} + b_{C_n})E_{m_n} \cos(n\alpha + \theta_n) \qquad (1298) \\
= {} & g_{O_1}E_{m_1} \sin(\alpha + \theta_1) + b_{O_1}E_{m_1} \cos(\alpha + \theta_1) \\
& + g_{O_3}E_{m_3} \sin(3\alpha + \theta_3) + b_{O_3}E_{m_3} \cos(3\alpha + \theta_3) \\
& + \cdots + g_{O_n}E_{m_n} \sin(n\alpha + \theta_n) \\
& + b_{O_n}E_{m_n} \cos(n\alpha + \theta_n) \qquad (1299) \\
= {} & y_{O_1}E_{m_1} \sin(\alpha + \theta_1 + \beta_{O_1}) + y_{O_3}E_{m_3} \sin(3\alpha + \theta_3 + \beta_{O_3}) \\
& + \cdots + y_{O_n}E_{m_n} \sin(n\alpha + \theta_n + \beta_{O_n}), \qquad (1300)
\end{aligned}
$$

where

$$y_{O_1} = \sqrt{g_{O_1}^2 + b_{O_1}^2} \qquad \tan \beta_{O_1} = \frac{b_{O_1}}{g_{O_1}}$$

. . . .

$$y_{O_n} = \sqrt{g_{O_n}^2 + b_{O_n}^2}, \qquad (1301)$$

$$\tan \beta_{O_n} = \frac{b_{O_n}}{g_{O_n}}. \qquad (1302)$$

It is seen that each component of the current is calculated from the constants of the circuit for that frequency by exactly the same method as was used in Chap. III for simple sine waves. These values of i contain coefficients of $\sin(\alpha + \theta_1)$ and $\sin(3\alpha + \theta_3)$, etc., and of $\cos(\alpha + \theta_1)$ and $\cos(3\alpha + \theta_3)$, etc., of the total current. From these values, the magnitude and phase positions of the component harmonics of the total current wave may be found, and from this the effective value, the power factor, and the equivalent phase angle.

Example 110

An emf expressed by

$$e = 200 \sin 314t + 50 \sin 942t$$

is impressed upon three parallel branches (Fig. 314), having the following constants:

Circuit A, $r = 20$, $L = 0$,
Circuit B, $r = 20$, $L = 0.05$,
Circuit C, $r = 0$, $C = 50\ \mu$f.

Find the current in each branch and the total current.

$$x_{A_1} = 0 \qquad \dot{z}_{A_1} = 20 + j0$$

$$x_{B_1} = 314 \times 0.05 = 15.7 \qquad \dot{z}_{B_1} = 20 + j15.7$$

$$x_{C_1} = \frac{-10^6}{314 \times 50} = -63.7 \qquad \dot{z}_{C_1} = 0 - j63.7$$

$$g_{A_1} = \frac{1}{20} = 0.05 \qquad b_{A_1} = 0$$

$$g_{B_1} = \frac{20}{\overline{20}^2 + \overline{15.7}^2} = 0.031 \qquad b_{B_1} = \frac{-15.7}{\overline{20}^2 + \overline{15.7}^2} = -0.0243$$

$$g_{C_1} = 0 \qquad b_{C_1} = \frac{-1}{-63.7} = 0.0157$$

$$g_{O_1} = 0.05 + 0.031 + 0 = 0.081 \qquad b_{O_1} = 0 - 0.0243 + 0.0157 = -0.0086$$

$$x_{A_3} = 0 \qquad \dot{z}_{A_3} = 20 + j0.$$

$$x_{B_3} = 942 \times 0.05 = 47.1 \qquad \dot{z}_{B_3} = 20 + j47.1$$

$$x_{C_3} = \frac{-10^6}{942 \times 50} = -21.2 \qquad \dot{z}_{C_3} = 0 - j21.2$$

$$g_{A_3} = \frac{1}{20} = 0.05 \qquad b_{A_3} = 0.$$

$$g_{B_3} = \frac{20}{\overline{20}^2 + \overline{47.1}^2} = 0.0076 \qquad b_{B_3} = \frac{-47.1}{\overline{20}^2 + \overline{47.1}^2} = -0.018$$

$$g_{C_3} = 0 \qquad b_{C_3} = \frac{-1}{-21.2} = 0.0472$$

$$g_{O_3} = 0.05 + 0.0076 + 0 = 0.0576 \qquad b_{O_3} = 0 - 0.018 + 0.0472 = 0.0292$$

$$i_A = 0.05 \times 200 \sin 314t + 0.05 \times 50 \sin 942t = 10 \sin 314t + 2.5 \sin 942t$$

$$\begin{aligned} i_B &= 0.031 \times 200 \sin 314t - 0.0243 \times 200 \cos 314t + 0.0076 \times 50 \sin 942t - 0.018 \times 50 \cos 942t \\ &= 6.2 \sin 314t - 4.86 \cos 314t + 0.38 \sin 942t - 0.9 \cos 942t \end{aligned}$$

$$\begin{aligned} i_C &= 0.0157 \times 200 \cos 314t + 0.0472 \times 50 \cos 942t \\ &= 3.14 \cos 314t + 2.36 \cos 942t \end{aligned}$$

$$\begin{aligned} i_O &= 16.2 \sin 314t - 1.72 \cos 314t + 2.88 \sin 942t + 1.46 \cos 942t \\ &= 16.3 \sin (314t - 6^\circ\, 5') + 3.23 \sin (942t + 26^\circ\, 53') \end{aligned}$$

$$I_O = \sqrt{\frac{\overline{16.3}^2 + \overline{3.23}^2}{2}} = 11.8 \text{ amp}$$

$$I_A = \sqrt{\frac{\overline{10}^2 + \overline{2.5}^2}{2}} = 7.29 \text{ amp}$$

$$I_B = \sqrt{\frac{\overline{6.2}^2 + \overline{4.86}^2 + \overline{0.38}^2 + \overline{0.9}^2}{2}} = 5.58 \text{ amp}$$

$$I_C = \sqrt{\frac{\overline{3.14}^2 + \overline{2.36}^2}{2}} = 2.78 \text{ amp}$$

$$E = \sqrt{\frac{200^2 + 50^2}{2}} = 145.5 \text{ volts}$$

$$\cos\theta_A = 1.00$$

$$\cos\theta_B = \frac{5.58 \times 20}{145.5} = 0.766$$

$$\cos\theta_3 = 0$$

$$P_O = \frac{16.2 \times 200}{2} + \frac{2.88 \times 50}{2} = 1{,}692 \text{ watts}$$

$$\cos\theta_O = \frac{1692}{145.5 \times 11.8} = 0.985.$$

179. Nonsinusoidal Wave Forms in Three-phase Circuits.—In a three-phase generator represented by Fig. 315, the voltages induced in the coil aA will have, in general, an equation of the form

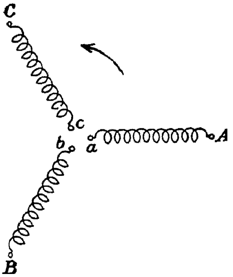

FIG. 315.—Three separate phase windings for a three-phase generator.

$$e_{aA} = E_{m1}\sin(\alpha + \theta_1) + E_{m3}\sin(3\alpha + \theta_3) + E_{m5}\sin(5\alpha + \theta_5) + E_{m7}\sin(7\alpha + \theta_7) + E_{m9}\sin(9\alpha + \theta_9) + \cdots \quad (1303)$$

The direction of rotation being considered as shown in Fig. 315, the voltages induced in coils bB and cC will be 120 deg. behind and 120 deg. ahead of e_{aA}, respectively, and since they have the same wave form as e_{aA}, their equations are obtained by replacing α in Eq. (1303) by $(\alpha - 120°)$ and $(\alpha + 120°)$, respectively, as

$$e_{bB} = E_{m1}\sin[(\alpha - 120°) + \theta_1] + E_{m3}\sin[3(\alpha - 120°) + \theta_3] + E_{m5}\sin[5(\alpha - 120°) + \theta_5] + E_{m7}\sin[7(\alpha - 120°) + \theta_7] + E_{m9}\sin[9(\alpha - 120°) + \theta_9] + \cdots, \quad (1304)$$

$$e_{cC} = E_{m1}\sin[(\alpha + 120°) + \theta_1] + E_{m3}\sin[3(\alpha + 120°) + \theta_3] + E_{m5}\sin[5(\alpha + 120°) + \theta_5] + E_{m7}\sin[7(\alpha + 120°) + \theta_7] + E_{m9}\sin[9(\alpha + 120°) + \theta_9] + \cdots \quad (1305)$$

When these equations are simplified and expressed in terms of angles less than 180 deg., Eqs. (1303), (1304), and (1305) become

$$e_{aA} = E_{m1}\sin(\alpha + \theta_1) + E_{m3}\sin(3\alpha + \theta_3) + E_{m5}\sin(5\alpha + \theta_5) + E_{m7}\sin(7\alpha + \theta_7) + E_{m9}\sin(9\alpha + \theta_9) + \cdots, \quad (1303)$$

$$e_{bB} = E_{m1} \sin (\alpha + \theta_1 - 120°) + E_{m3} \sin (3\alpha + \theta_3) + E_{m5} \sin (5\alpha + \theta_5 + 120°) + E_{m7} \sin (7\alpha + \theta_7 - 120°) + E_{m9} \sin (9\alpha + \theta_9) + \cdots, \quad (1306)$$

$$e_{cC} = E_{m1} \sin (\alpha + \theta_1 + 120°) + E_{m3} \sin (3\alpha + \theta_3) + E_{m5} \sin (5\alpha + \theta_5 - 120°) + E_{m7} \sin (7\alpha + \theta_7 + 120°) + E_{m9} \sin (9\alpha + \theta_9) + \cdots. \quad (1307)$$

When the corresponding harmonics in Eqs. (1303), (1306), and (1307) are compared, it is seen that

a. The fundamentals form a three-phase system having the positive phase sequence e_{aA}, e_{bB}, e_{cC}.

b. The third harmonics are all in phase, forming a zero-phase sequence system.

c. The fifth harmonics form a three-phase system having the negative phase sequence e_{aA}, e_{cC}, e_{bB}.

d. The seventh harmonics form a three-phase system having the same positive phase sequence as the fundamentals.

e. The ninth harmonics are all in phase, forming a zero-phase sequence system with the third harmonics.

f. In general, the harmonics of orders divisible by three form a zero-phase sequence system, and all other harmonics have alternately positive- and negative-phase sequences, beginning with a positive-phase sequence for the fundamental.

180. Coils Connected in Y.—The most common connection for the coils of an alternator is that of the Y, illustrated in Fig. 316, where the three coil ends *a*, *b*, *c* of the coils in Fig. 315 are connected together. In this case, the line voltages are

$$e_{AB} = e_{bB} - e_{aA}, \quad (1308)$$
$$e_{BC} = e_{cC} - e_{bB}, \quad (1309)$$
$$e_{CA} = e_{aA} - e_{cC}. \quad (1310)$$

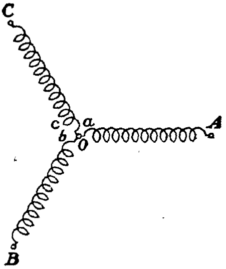

FIG. 316.—Coils of the generator of Fig. 315 connected in Y.

The values of the coil voltages being substituted from Eqs. (1303), (1306), and (1307) into Eqs. (1308), (1309), and (1310) and these expressions being simplified,

$$\sin(\alpha + \theta_1 - 150°) + \sqrt{3}Em_5 \sin(5\alpha + \theta_5 + 150°) + \sqrt{3}Em_7 \sin(7\alpha + \theta_7 - 150°) + \cdots, \quad (1311)$$

$$e_{BC} = \sqrt{3}Em_1 \sin(\alpha + \theta_1 + 90°) + \sqrt{3}Em_5 \sin(5\alpha + \theta_5 - 90°) + \sqrt{3}Em_7 \sin(7\alpha + \theta_7 + 90°) + \cdots, \quad (1312)$$

$$e_{CA} = \sqrt{3}Em_1 \sin(\alpha + \theta_1 - 30°) + \sqrt{3}Em_5 \sin(5\alpha + \theta_5 + 30°) + \sqrt{3}Em_7 \sin(7\alpha + \theta_7 - 30°) + \cdots. \quad (1313)$$

From an inspection of these equations, it is seen that there are no harmonics of triple frequency in the line voltages, Eqs. (1311) to (1313). Figure 317 illustrates the fundamentals and third harmonics of the voltages e_{aA}, e_{bB}, and e_{cC} for the case of θ_3 equal to zero. The third harmonics are all in phase for these coil voltages; therefore, when e_{aA} is reversed and added to e_{bB} to obtain e_{AB}, these third harmonics cancel, eliminating this frequency from the voltages between lines. This is also true for all harmonics whose frequency is a multiple of the third-harmonic frequency. All other harmonics of the coil voltages appear in the line voltages with magnitudes equal to $\sqrt{3}$ times their respective magnitudes in the neutral voltages but with altered phase relations with respect to the fundamentals. The wave shape of the line voltage is, therefore, quite different from that of the neutral voltage. The elimination of the triple harmonics from the line voltages reduces the effective value of this voltage, so that

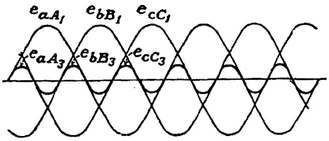

FIG. 317.—Fundamental and third-harmonic emf waves for the coils of Fig. 315 and θ_3 equal to zero.

$$E_{AB} < \sqrt{3}E_{OA}.$$

181. Current Relations in a Y-connected Three-phase System. Assume that the neutral point O (Fig. 316) is brought out, forming a four-wire three-phase generator, and that a balanced load is connected between the three lines and the neutral wire. Since all harmonics, including the triples, appear in the voltages between lines and neutral wire, the coil and line currents will be nonsinusoidal, having the general form

$$i_{OA} = I_{m1} \sin(\alpha + \beta_1) + I_{m3} \sin(3\alpha + \beta_3) + I_{m5} \sin(5\alpha + \beta_5) + I_{m7} \sin(7\alpha + \beta_7) + I_{m9} \sin(9\alpha + \beta_9) + \cdots, \quad (1314)$$

$$\begin{aligned} i_{OB} = {} & I_{m1} \sin (\alpha + \beta_1 - 120°) + I_{m3} \sin (3\alpha + \beta_3) + I_{m5} \sin \\ & (5\alpha + \beta_5 + 120°) + I_{m7} \sin (7\alpha + \beta_7 - 120°) + I_{m9} \sin \\ & (9\alpha + \beta_9) + \cdots, \end{aligned} \tag{1315}$$

$$\begin{aligned} i_{OC} = {} & I_{m1} \sin (\alpha + \beta_1 + 120°) + I_{m3} \sin (3\alpha + \beta_3) + I_{m5} \sin \\ & (5\alpha + \beta_5 - 120°) + I_{m7} \sin (7\alpha + \beta_7 + 120°) + I_{m9} \sin \\ & (9\alpha + \beta_9) + \cdots. \end{aligned} \tag{1316}$$

The current in the neutral wire is

$$i_{NO} = i_{OA} + i_{OB} + i_{OC}. \tag{1317}$$

When the values of the currents on the right-hand side of this equation are substituted from Eqs. (1314) to (1316) and when it is remembered that

$$\sin \alpha + \sin (\alpha + 120°) + \sin (\alpha - 120°) = 0,$$

the neutral current becomes

$$i_{NO} = 3I_{m3} \sin (3\alpha + \beta_3) + 3I_{m9} \sin (9\alpha + \theta_9) + \cdots. \tag{1318}$$

The third harmonic and multiples of the third harmonic each form a zero-phase sequence, and the neutral wire carries the arithmetic sum of the line currents for each of these frequencies. The neutral current is not zero, even with a balanced load, unless these frequencies are absent in the voltage and current waves. If the neutral connection for the system, consisting of generator and distribution lines, has a ground return, currents of the triple-frequency system of harmonics may flow in the same direction over the line wires in parallel and return through the ground. The magnetic-flux waves set up by these higher frequency currents produce very troublesome interference with near-by communication lines.

If the neutral wire is omitted, forming a three-wire three-phase system, there can be no neutral current, or

$$i_{NO} = i_{OA} + i_{OB} + i_{OC} = 0.$$

This condition can be true only if there are no triple harmonics in the line currents. The line currents are

$$\begin{aligned} i_{OA} = {} & I_{m1} \sin (\alpha + \beta_1) + I_{m5} \sin (5\alpha + \beta_5) + I_{m7} \sin \\ & (7\alpha + \beta_7) + \cdots, \end{aligned} \tag{1319}$$

$$\begin{aligned} i_{OB} = {} & I_{m1} \sin (\alpha + \beta_1 - 120°) + I_{m5} \sin (5\alpha + \beta_5 + 120°) \\ & + I_{m7} \sin (7\alpha + \beta_7 - 120°) + \cdots, \end{aligned} \tag{1320}$$

$$i_{oC} = I_{m1}\sin(\alpha+\beta_1+120^\circ) + I_{m5}\sin(5\alpha+\beta_5-120^\circ) + I_{m7}\sin(7\alpha+\beta_7+120^\circ) + \cdots \quad (1321)$$

No harmonics of triple frequency orders can exist in the line voltages of such a balanced Y-connected three-phase generator, and in a balanced three-wire three-phase system no third harmonics can exist in the line-wire currents.

182. Voltage and Current Relations in a Delta-connected Three-phase System.

182. Voltage and Current Relations in a Delta-connected Three-phase System.—If the coils of the generator of Fig. 315 are connected as shown in Fig. 318, the voltage across the open corner is

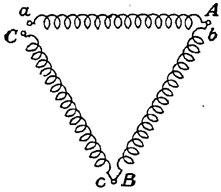

FIG. 318.—Coils of the generator of Fig. 315 connected in delta with one corner left open.

$$e_{aC} = e_{aA} + e_{bB} + e_{cC}.$$

When the values of these coil voltages are substituted from Eqs. (1303), (1306), and (1307) into this equation,

$$e_{aC} = 3E_{m3}\sin(3\alpha+\theta_3) + 3E_{m9}\sin(9\alpha+\theta_9) + \cdots \quad (1322)$$

From Eq. (1322), it is seen that there are no harmonics except those of the triple-frequency family across the corner of this delta before the corner is closed. However, when these voltages are of considerable magnitude, as is the case in some Y-delta transformer connections, the effective value of this voltage may be greater than any coil voltage. If, instead of the voltage being taken from *a* to *C* in Fig. 318, it were taken from *a* to *B*, it would contain two of each of the triples and one fundamental voltage. This is the condition in some open-delta or *V* connections of three-phase transformers or generators.

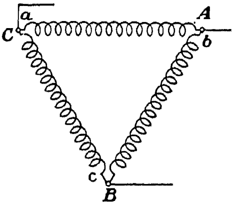

FIG. 319.—Delta-connected three-phase generator formed from the coils of Fig. 315.

If *a* and *C* are connected, as shown in Fig. 319, forming a delta-connected generator, the voltage e_{aC} will produce a circulating current within the closed delta. If the impedance of each coil is Z_3 to the third harmonic and Z_9 to the ninth harmonic of current, the circulating current will be

$$i_{abc} = \frac{3E_{m3}}{3Z_3}\sin(3\alpha+\beta_3) + \frac{3E_{m9}}{3Z_9}\sin(9\alpha+\beta_9) + \cdots \quad (1323)$$

$$= I_{m3}\sin(3\alpha+\beta_3) + I_{m9}\sin(9\alpha+\beta_9) + \cdots \quad \mathbf{(1324)}$$

This current will produce a voltage drop in each coil, as

$$e_z = I_{m3}Z_3 \sin(3\alpha + \theta_3) + I_{m9}Z_9 \sin(9\alpha + \theta_9) + \cdots \qquad (1325)$$

$$= E_{m3} \sin(3\alpha + \theta_3) + E_{m9} \sin(9\alpha + \theta_9) + \cdots. \qquad (1326)$$

The terminal voltages are

$$\begin{aligned} e_{AB} &= e_{bB} - e_Z, \\ e_{BC} &= e_{cC} - e_Z, \\ e_{CA} &= e_{aA} - e_Z. \end{aligned}$$

When the values of the voltages on the right are substituted from Eqs. (1303), (1306), (1307), and (1326),

$$\begin{aligned} e_{AB} &= E_{m1} \sin(\alpha + \theta_1 - 120°) + E_{m5} \sin(5\alpha + \theta_5 + 120°) \\ &\quad + E_{m7} \sin(7\alpha + \theta_7 - 120°) + \cdots, \qquad (1327) \\ e_{BC} &= E_{m1} \sin(\alpha + \theta_1 + 120°) + E_{m5} \sin(5\alpha + \theta_5 - 120°) \\ &\quad + E_{m7} \sin(7\alpha + \theta_7 + 120°) + \cdots, \qquad (1328) \\ e_{CA} &= E_{m1} \sin(\alpha + \theta_1) + E_{m5} \sin(5\alpha + \theta_5) \\ &\quad + E_{m7} \sin(7\alpha + \theta_7) + \cdots. \qquad (1329) \end{aligned}$$

The voltage harmonics of triple frequency orders are short-circuited in the delta and hence do not appear in the line voltages.

Under a balanced load, the currents in the coils of the delta are of the form given in Eqs. (1314) to (1316). The line currents are

$$\begin{aligned} i_A &= i_{aA} - i_{bB}, \\ i_B &= i_{bB} - i_{cC}, \\ i_C &= i_{cC} - i_{aA}. \end{aligned}$$

Substituting values from Eqs. (1314) to (1316) and simplifying,

$$\begin{aligned} i_A &= \sqrt{3}I_{m1} \sin(\alpha + \beta_1 + 30°) + \sqrt{3}I_{m5} \sin(5\alpha + \beta_5 - 30°) \\ &\quad + \sqrt{3}I_{m7} \sin(7\alpha + \beta_7 + 30°) + \cdots, \qquad (1330) \\ i_B &= \sqrt{3}I_{m1} \sin(\alpha + \beta_1 - 90°) + \sqrt{3}I_{m5} \sin(5\alpha + \beta_5 + 90°) \\ &\quad + \sqrt{3}I_{m7} \sin(7\alpha + \beta_7 - 90°) + \cdots, \qquad (1331) \\ i_c &= \sqrt{3}I_{m1} \sin(\alpha + \beta_1 + 150°) + \sqrt{3}I_{m5} \sin(5\alpha + \beta_5 - 150°) \\ &\quad + \sqrt{3}I_{m7} \sin(7\alpha + \beta_7 + 150°) + \cdots. \qquad (1332) \end{aligned}$$

The only harmonics of triple frequency and its multiples that can appear in the coil currents are those circulating in the delta and produced by short-circuiting the corresponding harmonics of the coil voltages. *With a delta-connected generator there can be no harmonics of triple frequency either in the line currents or in the line voltages.*

183. Power Relations in Polyphase Circuits.—The following definitions have recently been adopted by the American Institute of Electrical Engineers covering power in polyphase circuits:

Instantaneous Power in a Polyphase Circuit.—The instantaneous power at the points of entry of a polyphase circuit is equal to the algebraic sum of the products obtained by multiplying the instantaneous current at each point of entry by the potential difference between that point of entry and some arbitrarily selected base point (which may be the neutral point of entry) from which all the potential differences are measured. From the convention here used that a current is positive when its direction is into a delimited region, it follows that instantaneous power is positive when the instantaneous flow of energy is into the region delimited by the points of entry. The value of the instantaneous power is given in watts when the currents are in amperes and the potential differences are in volts.

Active Power of a Polyphase Circuit.—The active power at the points of entry of a polyphase circuit is the time average of the values of the instantaneous power at the points of entry, the average power being taken over a complete cycle of the alternating current. The value of active power is given in watts when the effective currents are in amperes and the effective potential differences are in volts. The active power can also be stated in terms of the effective values of the symmetrical components of the current and emf as

$$P = n\sum_{k=1}^{n}\sum_{r=1}^{\infty}E_{kr}I_{kr}\cos(\theta_{kr} - \beta_{kr}),$$

where k is the number of the symmetrical component and r is the number of the harmonic component. The convention for direction of current, which is used in these definitions, makes active power positive when the flow of energy is into the delimited region and negative when out of this region.

Reactive Power of a Polyphase Circuit.—The reactive power at the points of entry of a polyphase circuit is the algebraic sum of the reactive powers for the individual points of entry when the potential differences are all determined with respect to the same base point (which may be the neutral point of entry). In terms of the effective values of the symmetrical components of the currents and potential differences,

$$Q = n\sum_{k=1}^{n}\sum_{r=1}^{\infty}E_{kr}I_{kr}\sin(\theta_{kr} - \beta_{kr}).$$

The convention here used gives a positive sign to the reactive power of a harmonic component at any point of entry when the phase relation of current and of potential difference is that which would exist if, within the delimited region, a capacitive circuit connected the point of entry to the base point. The value of reactive power is given in vars when the effective currents are in amperes and the effective potential differences are in volts.

The most favorable method of connecting n wattmeters to indicate the reactive power of a polyphase circuit is to connect the current coil of each wattmeter in series with one line conductor and its voltage coil between a pair of line terminals that are symmetrically located with respect to the line conductor in which the current coil is connected.

Vector Power of a Polyphase Circuit.—Vector power at the points of entry of a polyphase circuit is equal to the square root of the sum of the squares of the active power and the reactive power. The unit of vector power is the vector volt-ampere.

Arithmetic Apparent Power of a Polyphase Circuit.—The arithmetic apparent power at the points of entry of a polyphase circuit is equal to the arithmetic sum of the products obtained by multiplying the effective current at each point of entry by the effective potential difference between the point of entry and the neutral point of entry, or, if one does not exist, an artificial neutral, the potential of which is established by joining it to each of the line points of entry by a set of equal resistances. The unit of arithmetic apparent power is the volt-ampere.

184. Power Relations in Three-phase Circuits with Distorted Waves of Voltage and Current.—In a four-wire three-phase system, with coil voltages and currents as given in Eqs. (1303), (1306), and (1307) and in Eqs. (1314) to (1316), the active power developed is

$$P_O = 3E_1I_1 \cos(\beta_1 - \theta_1) + 3E_3I_3 \cos(\beta_3 - \theta_3) + 3E_5I_5 \cos(\beta_5 - \theta_5) + \cdots \quad (1333)$$

Each harmonic of voltage and current supplies power to the load. If the neutral wire is removed from the Y-connected system, the currents are given in Eqs. (1319) to (1321), and the power is

$$P_O = 3E_1I_1 \cos(\beta_1 - \theta_1) + 3E_5I_5 \cos(\beta_5 - \theta_5) + 3E_7I_7 \cos(\beta_7 - \theta_7) + \cdots \quad (1334)$$

The triple harmonics still exist in the coil voltages, but they produce no power, since there is no triple harmonic in the coil currents.

With delta connection of the coils, the current and voltage in each coil have the same form as for the four-wire system, and the power developed is given by Eq. (1333). Since the triple harmonic and its multiples are suppressed in both line voltage and line current, the power represented by these harmonics is not delivered to the load. The expressions

$$3E_3I_3 \cos (\beta_3 - \theta_3),$$
$$3E_9I_9 \cos (\beta_9 - \theta_9),$$

and similar expressions for all multiples of the third harmonic must, therefore, represent power loss due to the current circulating in the delta-connected armature. These circulating currents of higher frequencies may reach appreciable magnitudes and may produce an excess heating in the armature. They are one of the objectionable features of the use of delta connection for generator armatures.

Problems

1-16. Draw to scale all component waves and the resultant wave for the emf $e = 800 \sin \alpha + 200 \sin 3\alpha$.

2-16. Draw to scale all component waves and the resultant wave for the current $i = 25 \sin \alpha + 15 \sin 5\alpha$.

3-16. Draw to scale all component waves and the resultant wave for the emf $e = 120 \sin 157t + 30 \sin 471t$.

4-16. Draw to scale all component waves and the resultant wave for the emf $e = 800 \sin (\alpha + 20°) + 200 \sin (3\alpha - 60°)$.

5-16. Plot the component waves of the emf $e = 100 \sin (\alpha + 10°) + 75 \sin (3\alpha - 30°) + 50 \sin 5\alpha$. Shift the axis of reference 10 deg. to the left so that the fundamental wave passes through the origin of the coordinate system. Rewrite the equation with the new axis of reference.

6-16. Find the half-wave average of the emf wave $e = 210 \sin \alpha + 40 \sin 3\alpha$.

7-16. Find the half-wave average of the current wave $i = 15 \sin 25t + 12 \sin 125t + 6 \sin 175t$.

8-16. Find the effective value of the wave of emf $e = 1{,}800 \sin (\alpha + 10°) + 350 \sin 3\alpha + 610 \sin (5\alpha - 60°) + 148 \sin (7\alpha + 90°)$.

9-16. Derive by the use of the calculus the effective value of the wave of current $i = 4 + 5 \sin \alpha$.

10-16. Find the form factor of the wave $i = 1{,}000 \sin \alpha + 3C0 \sin (5\alpha + 20°)$.

11-16. Find the form factor of the wave of emf $e = 460 \sin 157t + 120 \sin (471t + 60°)$.

12-16. Find the half-wave average, the effective value, the crest factor, the form factor, and the deviation factor of the wave $e = 800 \sin \alpha - 200 \sin 3\alpha$.

13-16. Find the half-wave average, the effective value, and the form factor for the wave of current $i = 2.6 \sin (377t + 10°) + 0.6 \sin (1131t - 30°)$.

14-16. Calculate the effective values of the voltage and current, and the active and reactive power in a circuit having the following waves of current and emf:

$$\begin{aligned} e &= 10{,}000 \sin (157t + 20°) + 3{,}000 \sin (471t + 30°), \\ i &= 400 \sin (157t - 10°) + 200 \sin (471t + 170°). \end{aligned}$$

15-16. Calculate the active and reactive power and the power factor in a circuit having the waves of emf and current:

$$\begin{aligned} e &= 1{,}000 \sin (314t + 30°) + 100 \sin (942t - 30°), \\ i &= 300 \sin (314t - 30°) + 100 \sin (942t + 20°). \end{aligned}$$

16-16. When two currents of the same frequency are combined in the same conductor, is the I^2R loss greater or less than when the currents are of different frequencies? Suppose that one of the currents is continuous.

17-16. Find the equation for the harmonics in an emf wave having a rectangular form for its lobes and with a maximum value of 2,000 volts, starting the analysis for an axis midway between the crossing points of the wave.

18-16. Make an analysis of an emf wave having its lobes equilateral triangles with maxima of 2,000 volts. Start the analysis at the crossing point of the wave.

19-16. A current wave has repeating cycles of the following characteristics: from zero to 90 deg., its value is 200 amp; from 90 to 180 deg., it is zero; from 180 to 270 deg., it is −200 amp; and from 270 to 360 deg., it is zero. Derive the equation for this wave.

20-16. Make an analysis of the following emf wave by Fourier's series method:

Angle	e	Angle	e	Angle	e
0°	−0.50	60°	0.10	120°	0.73
10°	−0.12	70°	0.15	130°	0.90
20°	0	80°	0.22	140°	1.00
30°	0.005	90°	0.31	150°	0.98
40°	0.01	100°	0.42	160°	0.91
50°	0.04	110°	0.58	170°	0.73
				180°	0.50

21-16. A current represented by the equation, $i = 200 \sin 377t + 50 \sin (1{,}131t + 20°) + 25 \sin (1{,}885t + 90°)$, is passed through a coil having a resistance of 15 ohms and an inductance of 0.02 henry. (*a*) What is the equation of the impressed-emf wave? (*b*) What active and reactive power are consumed?

22-16. A wave of current $i = 8 \sin (\alpha + 30°) + 4.6 \sin (3\alpha - 30°)$ is passed through a circuit having a resistance of 44 ohms and a capacitive reactance of 18.6 ohms on the fundamental frequency. What is the equation of the impressed emf?

23-16. A wave of current $i = 24 \sin (157t - 60°) + 15 \sin 471t$ is passed through a circuit having a resistance of 8 ohms, an inductance of 0.1 henry, and a capacitance of 80 μf connected in series. What is the equation of the impressed-emf wave?

24-16. An emf wave represented by the equation $e = 1{,}000 \sin 377t + 200 \sin 1131t$ is impressed on a circuit having a resistance of 10 ohms and a capacitor of 50μf capacitance connected in series. (*a*) What is the equation of the current wave? (*b*) Of the emf across the capacitor? (*c*) What is the effective value of the emf and current? (*d*) What are the apparent values of the reactance and impedance of the circuit?

25-16. An emf wave whose equation is expressed by $e = 1{,}000 \sin 314t + 400 \sin (942t + 60°)$ is impressed on a coil having a resistance of 25 ohms and an inductance of 0.04 henry. What is the equation of the current wave? Find the total active and reactive power, and the power factor.

26-16. The wave of emf $e = 300 \sin (377t + 30°) + 100 \sin (1{,}131t - 40°)$ is impressed on a series circuit having a resistance of 10 ohms, an inductive reactance of 24 ohms, and a capacitive reactance of 24 ohms at 60 cycles. Determine the equation for the current wave, its effective value, the active and reactive power and the power factor.

27-16. A voltage wave represented by $e = 500 \sin 377t + 200 \sin (1{,}131t - 30°)$ is impressed on two impedances in parallel. The first impedance has a resistance of 10 ohms and an inductance of 0.03 henry, the second a resistance of 10 ohms. (*a*) What is the equation for the current flowing in each impedance and for the total current? (*b*) What are the readings of an ammeter and a wattmeter used to indicate the total current and the total active power? (*c*) What is the total power factor?

28-16. A voltage wave $e = 125 \sin (157t + 45°) + 63 \sin 471t$ is impressed on a parallel circuit having three branches. The first branch has a resistance of 14 ohms, the second branch has an inductance of 0.1 henry, and the third branch has a capacitance of 485 μf. What is the equation of current flowing in each branch and the total current?

29-16. A voltage wave $e = 263 \sin \alpha + 106 \sin (3\alpha + 90°)$ is impressed on a parallel circuit, one branch having a resistance of 28 ohms and an inductive reactance of 46 ohms at fundamental frequency and the other having a capacitive reactance of 33 ohms on the fundamental frequency. What is the equation of current flowing in each branch and the total current?

30-16. The generated emf of one phase of a three-phase generator is $e_{1a} = 100 \sin \alpha + 50 \sin 3\alpha + 25 \sin 5\alpha$. The emfs e_{2b} and e_{3c} are, respectively, 120 and 240 deg. behind e_{1a}, the three emfs forming a symmetrical three-phase system. Write the equations of the phase and line voltages at no load, the coils being connected in star by joining the ends 1, 2, and 3.

31-16. (a) If the generator impedance per coil of the generator of Prob. 30-16 is $\dot{Z} = 1 + j1$ and the machine is loaded with three impedances, each $\dot{Z} = 9 + j9$ on the fundamental frequency, and connected in star to form a three-wire three-phase load, find the current in the lines. (b) If now the neutral wire is connected between the machine and the load neutrals, so as to form a four-wire three-phase system, find the current in each line wire and in the neutral.

32-16. Use the same generator coils and generated emfs as in Prob. 30-16, but connected in mesh. Find the equations of the phase and line emfs and the current circulating in the mesh-connected winding with no load on the generator.

33-16. Draw to scale all component waves and the resultant wave for the emf $e = 800 \sin \alpha - 200 \sin 3\alpha$.

34-16. Given the following current wave from a 60-cycle a-c circuit:

Angle	i	Angle	i	Angle	i
0°	0	60°	12.0	120°	13.0
15°	2.0	75°	13.0	135°	11.0
30°	6.0	90°	14.0	150°	7.0
45°	10.0	105°	15.0	165°	3.0
				180°	0

(a) Plot the current wave. (b) Pass the wave through a coil having a resistance of 10 ohms and an inductance of 0.03 henry, and plot the waves of resistance drop, reactance drop, and impressed emf.

35-16. Find the effective value of the wave $e = 500 + 2{,}000 \sin 371t + 400 \sin (1{,}131t + 40°)$.

36-16. Find the effective value of the wave $e = 400 \sin (377t + 60°) + 100 \sin (1{,}131t + 60°)$.

37-16. Find the form factor of the wave of Prob. 36-16.

38-16. Find analytically the half-wave average and the effective values of the wave $e = 1{,}000 \sin \alpha + 500 \sin (3\alpha - 30°) + 500 \sin (5\alpha - 90°)$.

39-16. Find the half-wave average, the effective value, the crest factor, and the form factor for the wave of Prob. 4-16.

40-16. A wave of current represented by the equation, $i = 500 \sin 157t + 60 \sin (471t - 20°)$, is flowing through a circuit having a resistance of 20 ohms and a reactance of 20 ohms at fundamental frequency, the reactance being inductive. (a) Derive the equation for the impressed emf wave (b) What active and reactive powers are consumed? (c) What is the power factor?

41-16. An emf represented by the wave, $e = 200 \sin 377t + 100 \sin (1{,}131t + 90°)$, is impressed across a coil having a resistance of 25 ohms and a reactance of 30 ohms at 60 cycles. (a) What is the equation of the current wave? (b) What active and reactive power are consumed? (c) What is the power factor of the circuit?

42-16. A resistance of 10 ohms is connected in parallel with the circuit of Prob. 26-16. Find the current in this resistance, the total current, the total active power, and the power factor.

43-16. A capacitor having a capacitance of 75.6 μf is connected in parallel with the circuit of Prob. 42-16. Find the current in this capacitor, the total current taken by the three branches, the total active power, and the total power factor.

44-16. Three circuits connected in parallel have the following constants at 50 cycles: $\dot{Z}_1 = 10 + j10$, $\dot{Z}_2 = 10 + j0$, and $\dot{Z}_3 = 0 - j10$. The current flowing through Z_2 has a wave form, $i_2 = 141 \sin 314t + 47.1 \sin (942t + 60°)$. (*a*) Find the current in each of the impedances and the total current. (*b*) Find the equation for the impressed emf. (*c*) Find the effective value of this emf. (*d*) Find the total active and reactive power and the power factor.

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